

# Problems on automorphism groups of nonpositively curved polyhedral complexes and their lattices

*To Bob Zimmer on his 60<sup>th</sup> birthday*

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Some background</b>	<b>3</b>
2.1	Polyhedral complexes . . . . .	4
2.2	Nonpositive curvature . . . . .	4
2.3	Complexes of groups . . . . .	6
<b>3</b>	<b>Examples of polyhedral complexes and their lattices</b>	<b>8</b>
3.1	Euclidean buildings . . . . .	9
3.1.1	Classical Euclidean buildings . . . . .	9
3.1.2	Nonclassical Euclidean buildings . . . . .	12
3.2	Products of trees . . . . .	13
3.3	Hyperbolic buildings . . . . .	13
3.4	Right-angled buildings . . . . .	14
3.5	Kac–Moody buildings . . . . .	15
3.6	Davis–Moussong complexes . . . . .	15
3.7	$(k, L)$ –complexes . . . . .	16
3.8	CAT(0) cubical complexes . . . . .	16
3.9	Systolic complexes . . . . .	17
<b>4</b>	<b>Properties of <math>X</math> and <math>\text{Aut}(X)</math></b>	<b>18</b>
4.1	When does local data determine $X$ ? . . . . .	18
4.2	Nondiscreteness of $\text{Aut}(X)$ . . . . .	20
4.3	Simplicity and nonlinearity . . . . .	21

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<b>5</b>	<b>Comparisons with linear groups</b>	<b>22</b>
5.1	Some linear-type properties . . . . .	22
5.2	Rigidity . . . . .	24
5.3	Geometry of the word metric . . . . .	24
5.4	Dynamics . . . . .	25
<b>6</b>	<b>Lattices in <math>\text{Aut}(\mathbf{X})</math></b>	<b>26</b>
6.1	Existence and classification theorems . . . . .	26
6.2	Commensurability . . . . .	26
6.3	Finiteness properties of lattices . . . . .	28
6.4	Covolumes . . . . .	28
6.5	Towers . . . . .	29
6.6	Biautomaticity of lattices . . . . .	31

## 1 Introduction

The goal of this paper is to present a number of problems about automorphism groups of nonpositively curved polyhedral complexes and their lattices. This topic lies at the juncture of two slightly different cultures. In geometric group theory, universal covers of 2-complexes are studied as geometric and topological models of their fundamental groups, and an important way of understanding groups is to construct “nice” actions on cell complexes, such as cubical complexes. From a different perspective, automorphism groups of connected, simply-connected, locally-finite simplicial complexes may be viewed as locally compact topological groups, to which we can hope to extend the theory of algebraic groups and their discrete subgroups. In the classification of locally compact topological groups, these automorphism groups are natural next examples to study after algebraic groups. In this paper we pose some problems meant to highlight possible directions for future research.

Let  $G$  be a locally-compact topological group with left-invariant Haar measure  $\mu$ . A *lattice* (resp. uniform lattice) in  $G$  is a discrete subgroup  $\Gamma < G$  with  $\mu(\Gamma \backslash G) < \infty$  (resp.  $\Gamma \backslash G$  compact). The classical study of Lie groups and their lattices was extended to algebraic groups  $G$  over nonarchimedean local fields  $K$  by Ihara, Bruhat–Tits, Serre and many others. This was done by realizing  $G$  as a group of automorphisms of the Bruhat–Tits (Euclidean) building  $X_G$ , which is a  $\text{rank}_K(G)$ -dimensional, nonpositively curved (in an appropriate sense) simplicial complex. More recently, Kac–Moody groups  $G$  have been studied by considering the action of  $G$  on the associated (twin) Tits buildings (see, for example, Carbone–Garland [CG] and Rémy–Ronan [RR]).

The simplest example in the algebraic case is  $G = \text{SL}(n, K)$ , where one can take  $K = \mathbf{Q}_p$  (where  $\text{char}(K) = 0$ ) or  $K = \mathbf{F}_p((t))$  (where  $\text{char}(K) = p > 0$ ). When  $n = 2$ , i.e.  $\text{rank}_K(G) = 1$ , the building  $X_G$  is the regular simplicial tree of degree  $p + 1$ . One can then extend this point of view to study the full group of simplicial automorphisms of a locally-finite tree as a locally-compact topological group, and investigate the properties of the lattices it contains. This leads to the remarkably rich theory of “tree lattices”, to which we refer the reader to the book of Bass–Lubotzky [BL] as the standard reference.

One would like to build an analogous theory in dimensions 2 and higher, with groups like  $\mathrm{SL}(n, \mathbf{Q}_p)$  and  $\mathrm{SL}(n, \mathbf{F}_p((t)))$ , for  $n \geq 3$ , being the “classical examples”. The increase in dimension makes life much harder, and greatly increases the variety of phenomena that occur.

Now, let  $X$  be a locally-finite, connected, simply-connected simplicial complex. The group  $G = \mathrm{Aut}(X)$  of simplicial automorphisms of  $X$  naturally has the structure of a locally-compact topological group, where a decreasing neighborhood basis of the identity consists of automorphisms of  $X$  which are the identity on bigger and bigger balls. With the right normalization of the Haar measure  $\mu$ , due to Serre [Se], there is a useful combinatorial formula for the covolume of a discrete subgroup  $\Gamma < G$ :

$$\mu(\Gamma \backslash G) = \sum_{v \in A} \frac{1}{|\Gamma_v|}$$

where the sum is taken over vertices  $v$  in a fundamental domain  $A \subseteq X$  for the  $\Gamma$ -action, and  $|\Gamma_v|$  is the order of the  $\Gamma$ -stabilizer of  $v$ . A discrete subgroup  $\Gamma$  is a *lattice* if and only if this sum converges, and  $\Gamma$  is a *uniform lattice* if and only if the fundamental domain  $A$  is compact.

In this paper we concentrate on the case when  $\dim(X) = 2$ . Most questions also make sense in higher dimensions, where even less is understood. When  $X$  is a product of trees much is known (see, for example, Burger–Mozes [BM]). However, the availability of projections to trees makes this a special (but deep) theory; we henceforth assume also that  $X$  is not a product. There are several themes we wish to explore, many informed by the classical (algebraic) case and the theory of tree lattices in [BL]. We also hope that classical cases may be re-understood from a new, more geometric point of view. Part of our inspiration for this paper came from Lubotzky’s beautiful paper [Lu], where he discusses the theory of tree lattices in relation to the classical (real and  $p$ -adic) cases.

This paper is not meant to be encyclopedic. It is presenting a list of problems from a specific and biased point of view. An important criterion in our choice of problem is that it presents some new phenomenon, or requires some new technique or viewpoint in order to solve it. After some background in Section 2, we describe the main examples of polyhedral complexes and their lattices in Section 3. We have grouped problems on the structure of the complex  $X$  itself together with basic group-theoretic and topological properties of  $\mathrm{Aut}(X)$  in Section 4. Section 5 focusses on whether important properties of linear groups and their lattices hold in this setting, while Section 6 discusses group-theoretic properties of lattices in  $\mathrm{Aut}(X)$  themselves.

We would like to thank Noel Brady and John Crisp for permission to use Figure 2, and Laurent Saloff-Coste for helpful discussions.

## 2 Some background

This preliminary material is mostly drawn from Bridson–Haefliger [BH]. We give the key definitions for polyhedral complexes in Section 2.1. (Examples of polyhedral complexes are described in Section 3 below.) Conditions for a polyhedral complex  $X$  to have nonpositive curvature, and some the consequences for  $X$ , are recalled in Section 2.2. The theory of

complexes of groups, which is used to construct both polyhedral complexes and their lattices, is sketched in Section 2.3.

## 2.1 Polyhedral complexes

Polyhedral complexes may be viewed as generalizations of (geometric realizations of) simplicial complexes. The quotient of a simplicial complex by a group acting by simplicial automorphisms is not necessarily simplicial, and so we work in this larger category. Roughly speaking, a polyhedral complex is obtained by gluing together polyhedra from some constant curvature space by isometries along faces.

More formally, let  $\mathbf{X}^n$  be  $S^n$ ,  $\mathbf{R}^n$  or  $\mathbf{H}^n$ , endowed with Riemannian metrics of constant curvature 1, 0 and  $-1$  respectively. A *polyhedral complex*  $X$  is a finite-dimensional CW-complex such that:

1. each open cell of dimension  $n$  is isometric to the interior of a compact convex polyhedron in  $\mathbf{X}^n$ ; and
2. for each cell  $\sigma$  of  $X$ , the restriction of the attaching map to each open codimension one face of  $\sigma$  is an isometry onto an open cell of  $X$ .

A polyhedral complex is said to be (piecewise) *spherical*, *Euclidean* or *hyperbolic* if  $\mathbf{X}^n$  is  $S^n$ ,  $\mathbf{R}^n$  or  $\mathbf{H}^n$  respectively. Polyhedral complexes are usually not thought of as embedded in any space. A 2-dimensional polyhedral complex is called a *polygonal complex*.

Given a polyhedral complex  $X$ , we write  $G = \text{Aut}(X)$  for the group of automorphisms, or cellular isometries, of  $X$ . A subgroup  $H \leq G$  is said to act *without inversions* on  $X$  if for every cell  $\sigma$  of  $X$ , the setwise stabilizer of  $\sigma$  in  $H$  is equal to its pointwise stabilizer. Note that any subgroup  $H \leq G$  acts without inversions on the barycentric subdivision of  $X$ . The quotient of a polyhedral complex by a group acting without inversions is also a polyhedral complex.

Let  $x$  be a vertex of an  $n$ -dimensional polyhedral complex  $X$ . The *link* of  $x$ , written  $\text{Lk}(x, X)$ , is the spherical polyhedral complex obtained by intersecting  $X$  with an  $n$ -sphere of sufficiently small radius centered at  $x$ . For example, if  $X$  has dimension 2, then  $\text{Lk}(x, X)$  may be identified with the graph having vertices the 1-cells of  $X$  containing  $x$  and edges the 2-cells of  $X$  containing  $x$ ; two vertices in the link are joined by an edge in the link if the corresponding 1-cells in  $X$  are contained in a common 2-cell. The link may also be thought of as the space of directions, or of germs of geodesics, at the vertex  $x$ . By rescaling so that for each  $x$  the  $n$ -sphere around  $x$  has radius say 1, we induce a metric on each link, and we may then speak of isometry classes of links of  $X$ .

## 2.2 Nonpositive curvature

In this section, we recall conditions under which the metrics on the cells of  $X$ , a Euclidean or hyperbolic polyhedral complex, may be pieced together to obtain a global metric which is respectively CAT(0) or CAT( $-1$ ). Some of the consequences for  $X$  are then described.

Any polyhedral complex  $X$  has an intrinsic pseudometric  $d$ , where for  $x, y \in X$ , the value of  $d(x, y)$  is the infimum of lengths of paths  $\Sigma$  from  $x$  to  $y$  in  $X$ , such that the restriction

of  $\Sigma$  to each cell of  $X$  is geodesic. If  $X$  is locally finite, the Hopf–Rinow Theorem implies that  $d$  is in fact a metric, and that  $(X, d)$  is a complete geodesic space. More generally, Bridson [BH] showed that if  $X$  has only finitely many isometry types of cells, then  $(X, d)$  is a complete geodesic metric space.

Now assume  $X$  is a Euclidean (respectively, hyperbolic) polyhedral complex such that  $(X, d)$  is a complete geodesic space. By the Cartan–Hadamard Theorem, if  $X$  is locally CAT(0) (respectively, locally CAT(−1)), then the universal cover  $\tilde{X}$  is CAT(0) (respectively, CAT(−1)). Thus to see whether a simply-connected  $X$  has a global metric of nonpositive curvature, we need only check a neighborhood of each point  $x \in X$ .

If  $\dim(X) = n$  and  $x$  is in the interior of an  $n$ -cell of  $X$ , then a neighborhood of  $x$  is isometric to a neighborhood in Euclidean (respectively, hyperbolic)  $n$ -space. If  $x$  is not a vertex but is in the intersection of two  $n$ -cells, then it is not hard to see that a neighborhood of  $x$  is also CAT(0) (respectively, CAT(−1)). Hence the nonpositive curvature of  $X$  depends only on neighborhoods of vertices, that is, the links.

There are two special cases in which it is easy to check whether neighborhoods of vertices are CAT(0) or CAT(−1). These are when  $\dim(X) = 2$ , and when  $X$  is a cubical complex (defined below, and discussed in Section 3.8).

**Theorem 1** (Gromov Link Condition). *A 2-dimensional Euclidean (respectively, hyperbolic) polyhedral complex  $X$  is locally CAT(0) (respectively, CAT(−1)) if and only if for every vertex  $x$  of  $X$ , every injective loop in the graph  $\text{Lk}(x, X)$  has length at least  $2\pi$ .*

Let  $I^n = [0, 1]^n$  be the cube in  $\mathbf{R}^n$  with edge lengths 1. A *cubical complex* is a Euclidean polyhedral complex with all  $n$ -cells isometric to  $I^n$ . Let  $L$  be a simplicial complex. We say  $L$  is a *flag complex* if whenever  $L$  contains the 1-skeleton of a simplex, it contains the simplex (“no empty triangles”).

**Theorem 2** (Gromov). *A finite-dimensional cubical complex  $X$  is locally CAT(0) if and only if the link  $L$  of each vertex of  $X$  is a flag simplicial complex.*

In general, let  $X$  be a polyhedral complex of piecewise constant curvature  $\kappa$  (so  $\kappa = 0$  for  $X$  Euclidean, and  $\kappa = -1$  for  $X$  hyperbolic).

**Theorem 3** (Gromov). *If  $X$  is a polyhedral complex of piecewise constant curvature  $\kappa$ , and  $X$  has finitely many isometry types of cells, then  $X$  is locally CAT( $\kappa$ ) if and only if for all vertices  $x$  of  $X$ , the link  $\text{Lk}(x, X)$  is a CAT(1) space.*

The condition that a metric space be nonpositively curved has a number of implications, described for example in [BH]. We highlight the following results:

- Any CAT(0) space  $X$  is contractible.
- Let  $X$  be complete CAT(0) space. If a group  $\Gamma$  acts by isometries on  $X$  with a bounded orbit, then  $\Gamma$  has a fixed point in  $X$ .

In particular, suppose  $X$  is a locally finite CAT(0) polyhedral complex and  $\Gamma < \text{Aut}(X)$  is a finite group acting on  $X$ . Then  $\Gamma$  is contained in the stabilizer of some vertex of  $X$ .

### 2.3 Complexes of groups

The theory of complexes of groups, due to Gersten–Stallings [St] and Haefliger [Hae, BH], generalizes Bass–Serre theory to higher dimensions. It may be used to construct both polyhedral complexes and lattices in their automorphism groups. We give here only the main ideas and some examples, and refer the reader to [BH] for further details.

Throughout this section, if  $Y$  is a polyhedral complex, then  $Y'$  will denote the first barycentric subdivision of  $Y$ . This is a simplicial complex with vertices  $V(Y')$  and edges  $E(Y')$ . Each  $a \in E(Y')$  corresponds to cells  $\tau \subset \sigma$  of  $Y$ , and so may be oriented from  $i(a) = \sigma$  to  $t(a) = \tau$ . Two edges  $a$  and  $b$  of  $Y'$  are *composable* if  $i(a) = t(b)$ , in which case there exists an edge  $c = ab$  of  $Y'$  such that  $i(c) = i(b)$ ,  $t(c) = t(a)$  and  $a$ ,  $b$  and  $c$  form the boundary of a 2–simplex in  $Y'$ .

A *complex of groups*  $G(Y) = (G_\sigma, \psi_a, g_{a,b})$  over a polyhedral complex  $Y$  is given by:

1. a group  $G_\sigma$  for each  $\sigma \in V(Y')$ , called the *local group* at  $\sigma$ ;
2. a monomorphism  $\psi_a: G_{i(a)} \rightarrow G_{t(a)}$  for each  $a \in E(Y')$ ; and
3. for each pair of composable edges  $a, b$  in  $Y'$ , an element  $g_{a,b} \in G_{t(a)}$ , such that

$$\text{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where  $\text{Ad}(g_{a,b})$  is conjugation by  $g_{a,b}$  in  $G_{t(a)}$ , and for each triple of composable edges  $a, b, c$  the following cocycle condition holds

$$\psi_a(g_{b,c}) g_{a,bc} = g_{a,b} g_{ab,c}$$

If all  $g_{a,b}$  are trivial, the complex of groups is *simple*. To date, most applications have used only simple complexes of groups. In the case  $Y$  is 2–dimensional, the local groups of a complex of groups over  $Y$  are often referred to as face, edge and vertex groups.

**Example:** Let  $P$  be a regular right-angled hyperbolic  $p$ –gon,  $p \geq 5$ , and let  $q$  be a positive integer  $\geq 2$ . Let  $G(P)$  be the following *polygon of groups* over  $P$ . The face group is trivial, and each edge group is the cyclic group  $\mathbf{Z}/q\mathbf{Z}$ . The vertex groups are the direct products of adjacent edge groups. All monomorphisms are natural inclusions, and all  $g_{a,b}$  are trivial.

Let  $G$  be a group acting without inversions on a polyhedral complex  $X$ . The action of  $G$  induces a complex of groups, as follows. Let  $Y = G \backslash X$  with  $p: X \rightarrow Y$  the natural projection. For each  $\sigma \in V(Y')$ , choose  $\tilde{\sigma} \in V(X')$  such that  $p(\tilde{\sigma}) = \sigma$ . The local group  $G_\sigma$  is the stabilizer of  $\tilde{\sigma}$  in  $G$ , and the  $\psi_a$  and  $g_{a,b}$  are defined using further choices. The resulting complex of groups  $G(Y)$  is unique (up to isomorphism).

Let  $G(Y)$  be a complex of groups. The *fundamental group of the complex of groups*  $\pi_1(G(Y))$  is defined so that if  $Y$  is simply connected, then  $\pi_1(G(Y))$  is isomorphic to the direct limit of the family of groups  $G_\sigma$  and monomorphisms  $\psi_a$ .

A complex of groups is *developable* if it is isomorphic to a complex of groups associated, as just described, to an action (without inversions) of a group on a polyhedral complex. If  $G(Y)$  is developable, then it has a *universal cover*  $\widetilde{G(Y)}$ . This is a simply-connected

polyhedral complex, equipped with an action of  $\pi_1(G(Y))$ , so that the complex of groups induced by the action of the fundamental group on the universal cover is isomorphic to  $G(Y)$ .

Unlike graphs of groups, complexes of groups are not in general developable:

**Example:** (K. Brown) Let  $G(Y)$  be the triangle of groups with trivial face group and edge groups infinite cyclic, generated by say  $a$ ,  $b$ , and  $c$ . Each vertex group is isomorphic to the Baumslag–Solitar group  $BS(1,2) = \langle x, y \mid xyx^{-1} = y^2 \rangle$ , where the generators  $x$  and  $y$  are identified with the generators of the adjacent edge groups. The fundamental group of  $G(Y)$  then has presentation  $\langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle$ . It is an exercise that this is the trivial group. Thus  $G(Y)$  is not developable.

We now describe a local condition for developability. Let  $Y$  be a connected polyhedral complex and let  $\sigma \in V(Y')$ . The *star* of  $\sigma$ , written  $\text{St}(\sigma)$ , is the union of the interiors of the simplices in  $Y'$  which meet  $\sigma$ . If  $G(Y)$  is a complex of groups over  $Y$  then, even if  $G(Y)$  is not developable, each  $\sigma \in V(Y')$  has a *local development*. That is, we may associate to  $\sigma$  an action of  $G_\sigma$  on the star  $\text{St}(\tilde{\sigma})$  of a vertex  $\tilde{\sigma}$  in some simplicial complex, such that  $\text{St}(\sigma)$  is the quotient of  $\text{St}(\tilde{\sigma})$  by the action of  $G_\sigma$ . To determine the local development, its link may be computed in combinatorial fashion.

**Example:** Suppose  $G(Y)$  is a simple polygon of groups, with  $G_\sigma = V$  a vertex group, with adjacent edge groups  $E_1$  and  $E_2$ , and with face group  $F$ . We identify the groups  $E_1$ ,  $E_2$  and  $F$  with their images in  $V$ . The link  $L$  of the local development at  $\sigma$  is then a bipartite graph. The two sets of vertices of  $L$  correspond to the cosets of  $E_1$  and  $E_2$  respectively in  $V$ , and the edges of  $L$  correspond to cosets of  $F$  in  $V$ . The number of edges between vertices  $g_1E_1$  and  $g_2E_2$  is equal to the number of cosets of  $F$  in the intersection  $g_1E_1 \cap g_2E_2$ . In the polygon of groups  $G(P)$  given above, the link of the local development at each vertex of  $P$  will be the complete bipartite graph  $K_{q,q}$ .

If  $G(Y)$  is developable, then for each  $\sigma \in V(Y')$ , the local development  $\text{St}(\tilde{\sigma})$  is isomorphic to the star of each lift  $\tilde{\sigma}$  of  $\sigma$  in the universal cover  $\widetilde{G(Y)}$ . The local development has a metric structure induced by that of the polyhedral complex  $Y$ . We say that a complex of groups  $G(Y)$  is *nonpositively curved* if for all  $\sigma \in V(Y')$ , the star  $\text{St}(\tilde{\sigma})$  is locally CAT(0) in this induced metric. The importance of this condition is given by:

**Theorem 4** (Stallings [St], Haefliger [Hae, BH]). *A nonpositively curved complex of groups is developable.*

**Example:** The polygon of groups  $G(P)$  above is nonpositively curved and thus developable. The links are  $K_{q,q}$  with edge lengths  $\frac{\pi}{2}$ , and so Gromov’s Link Condition (Theorem 1 above) is satisfied.

Let  $G(Y)$  be a developable complex of groups, with universal cover a locally finite polyhedral complex  $X$ , and fundamental group  $\Gamma$ . We say that  $G(Y)$  is *faithful* if the action of  $\Gamma$  on  $X$  is faithful. If so,  $\Gamma$  may be regarded as a subgroup of  $\text{Aut}(X)$ . Moreover,  $\Gamma$  is

discrete if and only if all local groups of  $G(Y)$  are finite, and  $\Gamma$  is a uniform lattice if and only if  $Y$  is compact.

**Example:** Let  $G(P)$  be the (developable) polygon of groups above, with fundamental group say  $\Gamma$  and universal cover say  $X$ . Then  $G(P)$  is faithful since its face group is trivial. As all the local groups are finite, and  $P$  is compact,  $\Gamma$  may be identified with a uniform lattice in  $\text{Aut}(X)$ .

### 3 Examples of polyhedral complexes and their lattices

In this section we present the main examples of locally finite polyhedral complexes  $X$  and their lattices. For each case, we give the key definitions, and sketch known constructions of  $X$  and of lattices in  $\text{Aut}(X)$ . There is some overlap between examples, which we describe. We will also indicate the distinctive flavor of each class. While results on existence of  $X$  are recalled here, we defer questions of uniqueness of  $X$ , given certain local data, to Section 4.1 below. Existence of lattices is also discussed further, in Section 6.1.

Many of the examples we discuss are buildings, which form an important class of non-positively curved polyhedral complexes. Roughly, buildings may be thought of as highly symmetric complexes, which contain many flats, and often have algebraic structure. Classical buildings are those associated to groups such as  $SL(n, \mathbf{Q}_p)$ , and play a similar role for these groups to that of symmetric spaces for real Lie groups. The basic references for buildings are Ronan [Ron2] and Brown [Br]. A much more comprehensive treatment by Abramenko–Brown [AB] is to appear shortly. These works adopt a combinatorial approach. For our purposes we present a more topological definition, from [HP2].

Recall that a *Coxeter group* is a group  $W$  with a finite generating set  $S$  and presentation of the form

$$W = \langle s \in S \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where  $s_i, s_j \in S$ ,  $m_{ii} = 1$ , and if  $i \neq j$  then  $m_{ij}$  is an integer  $\geq 2$  or  $m_{ij} = \infty$ , meaning that there is no relation between  $s_i$  and  $s_j$ . The pair  $(W, S)$ , or  $(W, I)$  where  $I$  is the finite indexing set of  $S$ , is called a *Coxeter system*. A spherical, Euclidean or hyperbolic *Coxeter polytope* of dimension  $n$  is an  $n$ -dimensional compact convex polyhedron  $P$  in the appropriate space, with every dihedral angle of the form  $\pi/m$  for some integer  $m \geq 2$  (not necessarily the same  $m$  for each angle). The group  $W$  generated by reflections in the codimension one faces of a Coxeter polytope  $P$  is a Coxeter group, and its action generates a tessellation of the space by copies of  $P$ .

**Definition:** Let  $P$  be an  $n$ -dimensional spherical, Euclidean or hyperbolic Coxeter polytope. Let  $W = (W, S)$  be the Coxeter group generated by the set of reflections  $S$  in the codimension one faces of  $P$ . A *spherical, Euclidean or hyperbolic building* of type  $(W, S)$  is a polyhedral complex  $X$  equipped with a maximal family of subcomplexes, called *apartments*, each polyhedrally isometric to the tessellation of respectively  $S^n$ ,  $\mathbf{R}^n$  or  $\mathbf{H}^n$  by the images of  $P$  under  $W$  (called *chambers*), such that:

1. any two chambers of  $X$  are contained in a common apartment; and

2. for any two apartments  $A$  and  $A'$ , there exists a polyhedral isometry from  $A$  onto  $A'$  which fixes  $A \cap A'$ .

The links of vertices of  $n$ -dimensional buildings are spherical buildings of dimension  $n - 1$ , with the induced apartment and chamber structure. Using this, and Theorem 3 above, it follows that Euclidean (respectively, hyperbolic) buildings are CAT(0) (respectively, CAT(-1)). Since buildings are such important examples in the theory, we will spend some time describing them in detail.

### 3.1 Euclidean buildings

Euclidean buildings are also sometimes known as *affine buildings*, or *buildings of affine type*. A simplicial tree  $X$  is a 1-dimensional Euclidean building of type  $(W, S)$ , where  $W$  is the infinite dihedral group, acting on the real line with fundamental domain  $P$  an interval. The chambers of  $X$  are the edges of the tree, and the apartments  $X$  are the geodesic lines in the tree. Since the product of two buildings is also a building, it follows that products of trees are higher-dimensional (reducible) Euclidean buildings (see Section 3.2 below). In this section we consider Euclidean buildings  $X$  of dimension  $n \geq 2$  which are not products.

#### 3.1.1 Classical Euclidean buildings

Classical Euclidean buildings are those Euclidean buildings which are associated to algebraic groups, as we now outline. We first construct the building for  $G = \mathrm{SL}(n, K)$  where  $K$  is a nonarchimedean local field, in terms of lattices in  $K^n$  and then in terms of  $BN$ -pairs (defined below). We then indicate how the latter construction generalizes to other algebraic groups. Our treatment is based upon [Br].

Let  $K$  be a field. We recall that a *discrete valuation* on  $K$  is a surjective homomorphism  $v: K^* \rightarrow \mathbf{Z}$ , where  $K^*$  is the multiplicative group of nonzero elements of  $K$ , such that

$$v(x + y) \geq \min\{v(x), v(y)\}$$

for all  $x, y \in K^*$  with  $x + y \neq 0$ . We set  $v(0) = +\infty$ , so that  $v$  is defined and the above inequality holds for all of  $K$ . A discrete valuation induces an absolute value  $|x| = e^{-v(x)}$  on  $K$ , which satisfies the nonarchimedean inequality

$$|x + y| \leq \max\{|x|, |y|\}$$

A metric on  $K$  is obtained by setting  $d(x, y) = |x - y|$ . The set  $\mathcal{O} = \{x \in K \mid |x| \leq 1\}$  is a subring of  $K$  called the *ring of integers*. The ring  $\mathcal{O}$  is compact and open in the metric topology induced by  $v$ . Pick an element  $\pi \in K$  with  $v(\pi) = 1$ , called a *uniformizer*. Every  $x \in K^*$  is then uniquely expressible in the form  $x = \pi^n u$  where  $n \in \mathbf{Z}$  and  $u$  is a unit of  $\mathcal{O}^*$  (so  $v(u) = 0$ ). The ideal  $\pi\mathcal{O}$  generated by  $\pi$  is a maximal ideal, since every element of  $\mathcal{O}$  not in  $\pi\mathcal{O}$  is a unit. Hence  $k = \mathcal{O}/\pi\mathcal{O}$  is a field, called the *residue field*.

**Examples:**

1. For any prime  $p$  the  $p$ -adic valuation  $v$  on the field of rationals  $\mathbf{Q}$  is defined by  $v(x) = n$ , where  $x = p^n a/b$  and  $a$  and  $b$  are integers not divisible by  $p$ . The field of  $p$ -adics  $K = \mathbf{Q}_p$  is the completion of  $\mathbf{Q}$  with respect to the metric induced by  $v$ , and the valuation  $v$  extends to  $\mathbf{Q}_p$  by continuity. The ring of integers is the ring of  $p$ -adic integers  $\mathbf{Z}_p$ , and we may take  $\pi = p$  as uniformizer. The residue field of  $\mathbf{Q}_p$  is then the finite field  $k = \mathbf{F}_p$ .
2. Let  $q$  be a power of a prime  $p$ . The field  $K = \mathbf{F}_q((t))$  of formal Laurent series with coefficients in the finite field  $\mathbf{F}_q$  has valuation  $v$  given by

$$v\left(\sum_{j=-m}^{\infty} a_j t^j\right) = -m,$$

a uniformizer is  $t$ , and the ring of integers is the ring of formal power series  $\mathbf{F}_q[[t]]$ . The residue field is  $k = \mathbf{F}_q$ .

A *local nonarchimedean field* is a field  $K$  which is complete with respect to the metric induced by a discrete valuation, and whose residue field is finite. Examples are  $K = \mathbf{Q}_p$ , which has  $\text{char}(K) = 0$ , and  $K = \mathbf{F}_q((t))$ , which has  $\text{char}(K) = p > 0$ . In fact, all local nonarchimedean fields arise as finite extensions of these examples.

We now fix  $K$  to be a local nonarchimedean field,  $\mathcal{O}$  its ring of integers,  $\pi$  a uniformizer, and  $k$  its residue field. The *Euclidean building* associated to the group  $G = \text{SL}(n, K)$  is the geometric realization  $|\Delta|$  of the abstract simplicial complex  $\Delta$  which we now describe.

Let  $V$  be the vector space  $K^n$ . A *lattice* in  $V$  is an  $\mathcal{O}$ -submodule  $L \subset V$  of the form  $L = \mathcal{O}v_1 \oplus \cdots \oplus \mathcal{O}v_n$  for some basis  $\{v_1, \dots, v_n\}$  of  $V$ . If  $L'$  is another lattice, then we may choose a basis  $\{v_1, \dots, v_n\}$  for  $L$  such that  $L'$  admits the basis  $\{\lambda_1 v_1, \dots, \lambda_n v_n\}$  for some  $\lambda_i \in K^*$ . The  $\lambda_i$  may be taken to be powers of  $\pi$ . Two lattices  $L$  and  $L'$  are *equivalent* if  $L = \lambda L'$  for some  $\lambda \in K^*$ . We write  $[L]$  for the equivalence class of  $L$ , and  $[v_1, \dots, v_n]$  for the equivalence class of the lattice with basis  $\{v_1, \dots, v_n\}$ .

The abstract simplicial complex  $\Delta$  is defined to have vertices the set of equivalence classes of lattices in  $V$ . To describe the higher-dimensional simplices of  $\Delta$ , we introduce the following incidence relation. (An *incidence relation* is a relation which is reflexive and symmetric.) Two equivalence classes of lattices  $\Lambda$  and  $\Lambda'$  are *incident* if they have representatives  $L$  and  $L'$  such that

$$\pi L \subset L' \subset L$$

This relation is symmetric, since  $\pi L' \in \Lambda'$  and  $\pi L \in \Lambda$  satisfy

$$\pi L' \subset \pi L \subset L'$$

The simplices of  $\Delta$  are then defined to be the finite sets of pairwise incident equivalence classes of lattices in  $V$ .

By the definition of incidence, every top-dimensional simplex of  $\Delta$  has vertex set

$$[v_1, \dots, v_i, \pi v_{i+1}, \dots, \pi v_n] \text{ for } i = 1, \dots, n,$$

for some basis  $\{v_1, \dots, v_n\}$  of  $V$ . Hence  $\Delta$  is a simplicial complex of dimension  $n - 1$ . The geometric realization  $X = |\Delta|$  is thus a Euclidean polyhedral complex of dimension  $n - 1$ . We note that  $n - 1$  is equal to the  $K$ -rank of  $G = \mathrm{SL}(n, K)$ .

We now construct a simplicial complex isomorphic to  $\Delta$ , using certain subgroups  $B$  and  $N$  of  $G = \mathrm{SL}(n, K)$ . For now, we state without proof that, with the correct Euclidean metrization,  $X = |\Delta|$  is indeed a Euclidean building, with chambers its  $(n - 1)$ -cells, and that the vertex set of an apartment of  $X$  is the set of equivalence classes

$$[\pi^{m_1}v_1, \dots, \pi^{m_n}v_n]$$

where the  $m_i$  are integers  $\geq 0$ , and  $\{v_1, \dots, v_n\}$  is a basis for  $V$ .

Observe that the group  $G = \mathrm{SL}(n, K)$  acts on the set of lattices in  $V$ . This action preserves equivalence of lattices and the incidence relation, so  $G$  acts without inversions on  $X$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $V$ . We define the *fundamental chamber* of  $X$  to be the simplex with vertices

$$[e_1, \dots, e_i, \pi e_{i+1}, \dots, \pi e_n], \text{ for } i = 1, \dots, n,$$

and the *fundamental apartment* of  $X$  to be the subcomplex with vertex set

$$[\pi^{m_1}e_1, \dots, \pi^{m_n}e_n], \text{ where } m_i \geq 0$$

Define  $B$  to be the stabilizer in  $G$  of the fundamental chamber, and  $N$  to be the stabilizer in  $G$  of the fundamental apartment. There is a surjection  $\mathrm{SL}(n, \mathcal{O}) \rightarrow \mathrm{SL}(n, k)$  induced by the surjection  $\mathcal{O} \rightarrow k$ . It is not hard to verify that  $B$  is the inverse image in  $\mathrm{SL}(n, \mathcal{O})$  of the upper triangular subgroup of  $\mathrm{SL}(n, k)$ , and that  $N$  is the monomial subgroup of  $\mathrm{SL}(n, K)$  (that is, the set of matrices with exactly one nonzero entry in each row and each column). We say that a subgroup of  $G$  is *special* if it contains a coset of  $B$ .

Now, from the set of cosets in  $G$  of special subgroups, we form a partially ordered set, ordered by opposite inclusion. There is an abstract simplicial complex  $\Delta(G, B)$  associated to this poset. The vertices of  $\Delta(G, B)$  are cosets of special subgroups, and the simplices of  $\Delta(G, B)$  correspond to chains of opposite inclusions. Using the action of  $G$  on  $\Delta$  and the construction of  $\Delta(G, B)$ , it is not hard to see that  $\Delta(G, B)$  is isomorphic to (the barycentric subdivision of)  $\Delta$ .

We now generalize the construction of  $\Delta(G, B)$  to algebraic groups besides  $G = \mathrm{SL}(n, K)$ . Let  $G$  be an absolutely almost simple, simply connected linear algebraic group defined over  $K$ . Examples other than  $\mathrm{SL}(n, K)$  include  $\mathrm{Sp}(2n, K)$ ,  $\mathrm{SO}(n, K)$ , and  $\mathrm{SU}(n, K)$ . All such groups  $G$  have a Euclidean  $BN$ -pair, which we now define. A  $BN$ -pair is a pair of subgroups  $B$  and  $N$  of  $G$ , such that:

- $B$  and  $N$  generate  $G$ ;
- the subgroup  $T = B \cap N$  is normal in  $N$ ; and
- the quotient  $W = N/T$  admits a set of generators  $S$  satisfying certain (technical) axioms, which ensure that  $(W, S)$  is a Coxeter system.

A  $BN$ -pair is *Euclidean* if the group  $W$  is infinite. The letter  $B$  stands for the Borel subgroup,  $T$  for the torus,  $N$  for the normalizer of the torus, and  $W$  for the Weyl group.

For  $G = \mathrm{SL}(n, K)$ , the  $B$  and  $N$  defined above, as  $G$ -stabilizers of the fundamental chamber and fundamental apartment of  $\Delta$ , are a  $BN$ -pair. Their intersection  $T$  is the diagonal subgroup of  $\mathrm{SL}(n, \mathcal{O})$ . The group  $W$  acts on the fundamental apartment of  $\Delta$  with quotient the fundamental chamber, and is in fact isomorphic to the Coxeter group generated by reflections in the codimension one faces of a Euclidean  $(n - 1)$ -simplex (with certain dihedral angles).

For any group  $G$  with a Euclidean  $BN$ -pair, one may construct the simplicial complex  $\Delta(G, B)$  from the poset of cosets of special subgroups, as described above. The geometric realization of  $\Delta(G, B)$  is a Euclidean polyhedral complex, of dimension equal to the  $K$ -rank of  $G$ . To prove that the geometric realization of  $\Delta(G, B)$  is a building, one uses the axioms for a  $BN$ -pair, results about Coxeter groups, and the Bruhat–Tits decomposition of  $G$ .

For classical Euclidean buildings  $X$ , there is a close relationship between the algebraic group  $G$  to which this building is associated, and the group  $\mathrm{Aut}(X)$ , so long as  $\dim(X) \geq 2$ .

**Theorem 5** (Tits [Ti3]). *Let  $G$  be an absolutely almost simple, simply-connected linear algebraic group defined over a nonarchimedean local field  $K$ . Let  $X$  be the Euclidean building for  $G$ . If  $\mathrm{rank}_K(G) \geq 2$ , then  $G$  has finite index in  $\mathrm{Aut}(X)$  when  $\mathrm{char}(K) = 0$ , and is cocompact in  $\mathrm{Aut}(X)$  when  $\mathrm{char}(K) = p > 0$ .*

Thus the lattice theory of  $\mathrm{Aut}(X)$  is very similar to that of  $G$ . Existence and construction of lattices in groups  $G$  as in Theorem 5 are well-understood. If  $\mathrm{char}(K) = 0$  then  $G$  does not have a nonuniform lattice (Tamagawa [Ta]), but does admit a uniform lattice, constructed by arithmetic means (Borel–Harder [BHar]). If  $\mathrm{char}(K) = p > 0$  then  $G$  has an arithmetic nonuniform lattice, and an arithmetic uniform lattice if and only if  $G = \mathrm{SL}(n, K)$  (Borel–Harder [BHar]). In real rank at least 2 (for example, if  $G = \mathrm{SL}(n, K)$ , for  $n \geq 3$ ) every lattice of  $G$  is arithmetic (Margulis [Ma]).

### 3.1.2 Nonclassical Euclidean buildings

Nonclassical Euclidean buildings are those Euclidean buildings (see Definition 3) which are not the building for any algebraic group  $G$  over a nonarchimedean local field. Tits constructed uncountably many isometry classes of nonclassical Euclidean buildings [Ti4]. Nonclassical buildings may also be constructed as universal covers of finite complexes, a method developed by Ballmann–Brin [BB1], and examples of this kind were obtained by Barré [Ba] as well. Ronan [Ron1] used a construction similar to the inductive construction of Ballmann–Brin, described in Section 3.7 below, to construct 2-dimensional nonclassical Euclidean buildings.

Very few lattices are known for nonclassical buildings. In [CMSZ], exotic lattices which act simply transitively on the vertices of various classical and nonclassical Euclidean buildings (of type  $\tilde{A}_2$ ) are constructed by combinatorial methods.

### 3.2 Products of trees

Let  $T_1$  and  $T_2$  be locally finite simplicial trees. The product space  $T_1 \times T_2$  is a polygonal complex, where each 2-cell is a square (edge  $\times$  edge), and the link at each vertex is a complete bipartite graph. Products of more than two trees may also be studied.

The group  $G = \text{Aut}(T_1 \times T_2)$  is isomorphic to  $\text{Aut}(T_1) \times \text{Aut}(T_2)$  (with a semidirect product with  $\mathbf{Z}/2\mathbf{Z}$  if  $T_1 = T_2$ ). Thus any subgroup of  $G$  may be projected to the factors. Because of this availability of projections, the theory of lattices for products of trees is a special (but deep) theory. See, for example, the work of Burger–Mozes [BM]. Many of the problems listed below may be posed in this context, but we omit questions specific to this case.

### 3.3 Hyperbolic buildings

The simplest example of a hyperbolic building is *Bourdon’s building*  $I_{p,q}$ , defined and studied in [B1]. Here  $p$  and  $q$  are integers,  $p \geq 5$  and  $q \geq 2$ . The building  $I_{p,q}$  is the (unique) hyperbolic polygonal complex such that each 2-cell (chamber) is isometric to a regular right-angled hyperbolic  $p$ -gon  $P$ , and the link at each vertex is the complete bipartite graph  $K_{q,q}$ . The apartments of  $I_{p,q}$  are hyperbolic planes tessellated by copies of  $P$ . Bourdon’s building is  $\text{CAT}(-1)$ , and may be regarded as a hyperbolic version of the product of two  $q$ -regular trees, since it has the same links. However,  $I_{p,q}$  is not globally a product space. The example of a polygon of groups  $G(P)$  given in Section 2.3 above has universal cover  $I_{p,q}$ , and the fundamental group  $\Gamma$  of this polygon of groups is a uniform lattice in  $\text{Aut}(I_{p,q})$ .

Bourdon’s building is a *Fuchsian building*, that is, a hyperbolic building of dimension two. More general Fuchsian buildings have all chambers hyperbolic  $k$ -gons,  $k \geq 3$ , with each vertex angle of the form  $\pi/m$ , for some integer  $m \geq 2$  (depending on the vertex). The link at each vertex with angle  $\pi/m$  is a one-dimensional spherical building  $L$  which is a generalized  $m$ -gon, that is, a graph with diameter  $m$  and girth  $2m$ . For example, a complete bipartite graph is a generalized 2-gon.

Unlike Euclidean buildings, hyperbolic buildings do not exist in arbitrary dimension. This is because there is a bound ( $n \leq 29$ ), due to Vinberg [Vi], on the dimension  $n$  of a compact convex hyperbolic Coxeter polytope. Gaboriau–Paulin [GP] broadened the definition of building given above (Definition 3) to allow hyperbolic buildings with noncompact chambers, in which case there are examples in any dimension, with chambers for example ideal hyperbolic simplexes.

Various constructions of hyperbolic buildings are known. In low dimensions, right-angled buildings (see Section 3.4) may be equipped with the structure of a hyperbolic building. In particular, Bourdon’s building is a right-angled building. Certain hyperbolic buildings arise as Kac–Moody buildings (see Section 3.5 below), and some Davis–Moussong complexes may also be metrized as hyperbolic buildings (see Section 3.6). Vdovina constructed some Fuchsian buildings as universal covers of finite complexes [Vd]. Fuchsian buildings were constructed as universal covers of polygons of groups by Bourdon [B1, B2] and by Gaboriau–Paulin [GP]. Haglund–Paulin [HP2] have constructed 3-dimensional hyperbolic buildings using “tree-like” decompositions of the corresponding Coxeter systems.

Many of these constructions of hyperbolic buildings  $X$  also yield lattices in  $\text{Aut}(X)$ .

When a hyperbolic building  $X$  is a Kac–Moody building then a few lattices in  $\text{Aut}(X)$  are known from Kac–Moody theory (see for example [Rem1]), and when  $X$  is a Davis–Moussong complex for a Coxeter group  $W$  then  $W$  may be regarded as a uniform lattice in  $\text{Aut}(X)$ . If  $X$  is the universal cover of a finite complex, the fundamental group of that complex is a uniform lattice in  $\text{Aut}(X)$ . As described in Section 2.3, if  $X$  is the universal cover of a finite complex of finite groups, such as a polygon of finite groups, then the fundamental group of the complex of groups is a uniform lattice in  $\text{Aut}(X)$ . More elaborate complexes of groups were used by Thomas to construct both uniform and nonuniform lattices for certain Fuchsian buildings in [Th3]. In [B2] Bourdon was able to “lift” lattices for affine buildings to uniform and nonuniform lattices for certain Fuchsian buildings.

### 3.4 Right-angled buildings

Recall that  $(W, I)$  is a *right-angled* Coxeter system if all the  $m_{ij}$  with  $i \neq j$  equal 2 or  $\infty$ . A building  $X$  of type  $(W, I)$  is then a *right-angled building*. Products of trees are examples of right-angled buildings, with associated Coxeter group the direct product of two infinite dihedral groups.

Bourdon’s building  $I_{p,q}$ , discussed in Section 3.3 above, is another basic example of a right-angled building. The Coxeter group  $W$  here is generated by reflections in the sides of a regular right-angled hyperbolic  $p$ -gon. Right-angled Coxeter polytopes exist only in dimensions  $n \leq 4$ , and this bound is sharp (Potyagailo–Vinberg [PV]). Thus right-angled buildings may be metrized as hyperbolic buildings (with compact chambers) only in dimensions  $\leq 4$ .

We may broaden the definition of building given above (Definition 3) to allow apartments which are Davis–Moussong complexes for  $W$  (see Section 3.6 below), rather than just the manifold  $S^n$ ,  $\mathbf{R}^n$  or  $\mathbf{H}^n$  tessellated by the action of  $W$ . With this definition, right-angled buildings, equipped with a piecewise Euclidean metric, exist in arbitrary dimensions (Januszkiewicz–Świątkowski [JS1]).

The following construction of a right-angled building  $X$  and a uniform lattice in  $\text{Aut}(X)$  appears in [HP1]. It is a generalization of the polygon of groups  $G(P)$  in Section 2.3 above. Let  $(W, I)$  be a right-angled Coxeter system, and  $\{q_i\}_{i \in I}$  a set of cardinalities with  $q_i \geq 2$ . Let  $N$  be the finite nerve of  $W$ , with first barycentric subdivision  $N'$ , and let  $K$  be the cone on  $N'$ . For example, if  $W$  is generated by reflections in the sides of a right-angled hyperbolic  $p$ -gon  $P$ , then  $N$  is a circuit of  $p$  edges, and  $K$  is isomorphic to the barycentric subdivision of  $P$ . For each  $i \in I$ , let  $G_i$  be a group of order  $q_i$ . Each vertex of  $K$  has a type  $J$ , where  $J \subset I$  is such that the group  $W_J$  generated by  $\{s_i\}_{i \in J}$  is finite. Let  $G(K)$  be the complex of groups where the vertex of  $K$  with type  $J$  has local group the direct product

$$\prod_{i \in J} G_i$$

and all monomorphisms are natural inclusions. This complex of groups is developable, with universal cover a right-angled building  $X$  of type  $(W, I)$ , so that each codimension one cell containing a vertex of type  $\{i\}$  in  $X$  is a face of  $q_i$  distinct chambers. Moreover, the fundamental group of this complex of groups may be viewed as a uniform lattice in  $\text{Aut}(X)$  (if all  $q_i$  are finite).

Many other lattices for right-angled buildings (in any dimension) were obtained by promoting tree lattices, using complexes of groups, in Thomas [Th2].

### 3.5 Kac–Moody buildings

Kac–Moody groups over finite fields  $\mathbf{F}_q$  may be viewed as infinite-dimensional analogs of Lie groups. See, for example, Carbone–Garland [CG] and Rémy–Ronan [RR]. For any Kac–Moody group  $\Lambda$  there are associated (twin) buildings  $X_+$  and  $X_-$ , constructed using twin  $BN$ -pairs  $(B^+, N)$  and  $(B^-, N)$  (see Section 3.1.1 above). The group  $\Lambda$  acts diagonally on the product  $X_+ \times X_-$ , and for  $q$  large enough  $\Lambda$  is a nonuniform lattice in  $\text{Aut}(X_+ \times X_-)$  (see [Rem1]). A *Kac–Moody building* is a building which appears as one of the twin buildings for a Kac–Moody group. Kac–Moody buildings are hyperbolic buildings, but unlike classical Euclidean buildings (see Theorem 5), non-isomorphic Kac–Moody groups may have the same building (Rémy [Rem2]). One may also study the *complete Kac–Moody group*  $G$ , which is the closure of  $\Lambda$  in the automorphism group of one of its twin buildings. Very few lattices in complete Kac–Moody groups are known.

### 3.6 Davis–Moussong complexes

Given any Coxeter system  $(W, S)$ , the associated Davis–Moussong complex is a locally finite, CAT(0), piecewise Euclidean polyhedral complex on which  $W$  acts properly discontinuously and cocompactly. We describe a special case of this construction in dimension two.

Let  $L$  be a connected, finite simplicial graph with all circuits of length at least 4, and let  $k \geq 2$  be an integer. The Coxeter system corresponding to this data has a generator  $s_i$  of order two for each vertex  $v_i$  of  $L$ , and a relation  $(s_i s_j)^k = 1$  if and only if the vertices  $v_i$  and  $v_j$  are connected by an edge in  $L$ . The Coxeter group defined by this Coxeter system is denoted  $W = W(k, L)$ . If  $k = 2$ , then  $W$  is a right-angled Coxeter group.

For any such  $W = W(k, L)$ , Davis–Moussong constructed a CAT(0) piecewise Euclidean complex  $X = X(2k, L)$  (see [D, Mo]). The cells of  $X$  correspond to cosets in  $W$  of spherical subgroups of  $W$ , and in particular the 0-cells of  $X$  correspond to the elements of  $W$ , viewed as cosets of the trivial subgroup. Recall that a *spherical subgroup* of  $W$  is a subgroup  $W_T$  generated by some subset  $T \leq S$ , such that  $W_T$  is finite.

The Davis–Moussong complex may be identified with (the first barycentric subdivision of) a polygonal complex  $X$  with all links  $L$  and all 2-cells regular Euclidean  $2k$ -gons. The group  $W$  has a natural left action on  $X$  which is properly discontinuous, cellular, and simply transitive on the vertices of  $X$ . Thus  $W$  may be viewed as a uniform lattice in  $\text{Aut}(X)$ . This construction can also be carried out in higher dimensions, provided  $L$  is a CAT(1) spherical simplicial complex. In dimension 2, where  $L$  is a graph, this is equivalent to all circuits having length at least 4, by the Gromov Link Condition (Theorem 1 above). If  $W$  is right-angled, then each apartment of a right-angled building of type  $W$  is isomorphic to the Davis–Moussong complex for  $W$ .

Davis–Moussong also found easy-to-verify conditions on  $L$  such that  $X(2k, L)$  may be equipped with a CAT(−1) piecewise hyperbolic structure. In this way, some hyperbolic buildings (or rather, their first barycentric subdivisions) may be constructed as Davis–Moussong complexes, with the graph  $L$  a one-dimensional spherical building.

### 3.7 $(k, L)$ -complexes

Let  $L$  be a finite graph and  $k$  an integer  $\geq 3$ . A  $(k, L)$ -*complex* is a polygonal complex  $X$  such that the link of each vertex of  $X$  is  $L$ , and each 2-cell of  $X$  is a regular  $k$ -gon (usually but not necessarily Euclidean).

Many polygonal complexes already described are  $(k, L)$ -complexes. For example, 2-dimensional Euclidean or hyperbolic buildings, with all links the same, are  $(k, L)$ -complexes with  $L$  is a one-dimensional spherical building. The two-dimensional Davis–Moussong complexes described in Section 3.6 above are barycentric subdivisions of  $(k, L)$ -complexes with  $k \geq 4$  even. An example of a  $(k, L)$ -complex which is not a building or a Davis–Moussong complex is where  $k$  is odd and  $L$  is the Petersen graph (Figure 1).

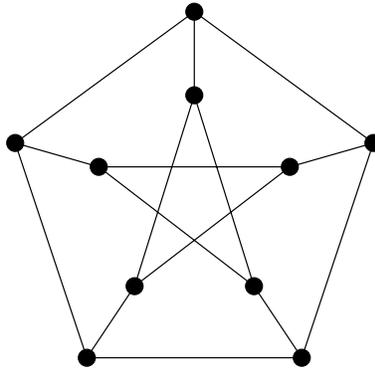


Figure 1: Petersen graph

There are simple conditions on the pair  $(k, L)$  ensuring that a  $(k, L)$ -complex satisfies Gromov’s Link Condition (Theorem 1 above) and thus has nonpositive curvature. Ballmann–Brin [BB1] showed that any  $(k, L)$ -complex where  $k$  and  $L$  satisfy these conditions may be constructed in an inductive manner, by adding  $k$ -gons to the previous stage without obstructions. This construction is discussed in more detail in Section 4.1 below. Some  $(k, L)$ -complexes may also be constructed as universal covers of triangles of groups, as done in [JLVV]. In this case the fundamental group of the triangle of groups is a uniform lattice. Constructions of uniform and nonuniform lattices as fundamental groups of complexes of groups are carried out for certain highly symmetric  $(k, L)$ -complexes, including those with Petersen graph links, in Thomas [Th4].

### 3.8 CAT(0) cubical complexes

Recall that a *cubical complex* is a Euclidean polyhedral complex with all  $n$ -cells isometric to the Euclidean  $n$ -cube, and that a cubical complex  $X$  is locally CAT(0) if and only if each vertex of  $X$  is a flag simplicial complex (Theorem 2 above). Trees and products of trees are examples of CAT(0) cubical complexes.

Groups of automorphisms of CAT(0) cubical complexes are different in many ways from groups acting on the Euclidean buildings discussed in Section 3.1 above. Examples of discrete groups that act properly on CAT(0) cube complexes include finitely generated Coxeter

groups [NRe2], many small cancellation groups [W4], one-relator groups with torsion [LW], many diagram groups, including Thompson’s group  $F$  [Far], and groups acting properly on products of trees.

In this setting, the main geometric objects of study are hyperplanes, defined as follows. Consider two edges of a CAT(0) cube complex  $X$  to be *equivalent* if they are opposite edges of some 2–cube. This relation generates an equivalence relation whose equivalence classes are the *combinatorial hyperplanes* of  $X$ . One can also define *geometric hyperplanes* of  $X$  as unions of *midplanes* of cubes, where a midplane of the cube  $C = [0, 1]^n$  is a subset of the form

$$[0, 1] \times \cdots \times [0, 1] \times \{1/2\} \times [0, 1] \times \cdots \times [0, 1].$$

Thus  $C$  has  $n$  midplanes, which intersect transversely at the barycenter of  $C$ . Given a combinatorial hyperplane  $H$ , the corresponding geometric hyperplane is the union of all midplanes meeting the barycenters of the edges of  $H$ . Each geometric hyperplane is itself a CAT(0) cubical complex, whose cubes are midplanes of  $X$ . Each geometric hyperplane separates  $X$  into two complementary components, called *halfspaces*. The properties of hyperplanes generalize the separation properties of edges in a tree. The main new feature in higher dimensions, not present in trees, is that hyperplanes can have transverse intersections. In fact, CAT(0) cubical complexes have a rich combinatorial structure arising from the incidence and nesting properties of hyperplanes.

Geometrically, the most significant subgroups in a group acting on a CAT(0) cubical complex are the codimension–1 subgroups, which typically arise as stabilizers of hyperplanes. If a group  $\Gamma$  has a finite generating set  $\mathcal{S}$ , a subgroup  $H \leq \Gamma$  is *codimension–1* provided that some neighborhood of  $H$  separates  $\text{Cayley}(G, \mathcal{S})$  into at least two “deep” complementary components, where a component is *deep* if it contains elements arbitrarily far away from  $H$ . For instance, if  $M$  is a 3–manifold with an immersed, incompressible surface  $S$ , then  $\pi_1(S)$  is a codimension–1 subgroup of  $\pi_1(M)$ .

Sageev has shown (together with a result proved independently by Gerasimov and Niblo–Roller) that a finitely generated group  $\Gamma$  has a codimension–1 subgroup if and only if  $\Gamma$  acts on a CAT(0) cube complex with no global fixed point [Sa, Ger, NRo]. The cube complex produced by Sageev’s theorem is sometimes infinite dimensional and sometimes locally infinite.

Several representation-theoretic aspects of actions on trees extend naturally to actions on CAT(0) cubical complexes. If a topological group with Property (T) acts on a CAT(0) cubical complex, then the action must have a global fixed point. On the other hand, if a topological group  $G$  acts metrically properly on a CAT(0) cubical complex  $X$  then  $G$  is a-T-menable [NRo]. In particular, if  $X$  is locally finite then any discrete subgroup  $\Gamma \leq \text{Aut}(X)$  is a-T-menable. Niblo–Reeves have also shown that if  $X$  is any CAT(0) cube complex, then every uniform lattice  $\Gamma \leq \text{Aut}(X)$  is biautomatic [NRe1].

### 3.9 Systolic complexes

In [JŚ2], Januszkiewicz–Świątkowski introduced an analog of nonpositive curvature for simplicial complexes. A simplicial complex is *systolic* if it is connected and simply connected, and every cycle consisting of fewer than 6 edges in any of its links has some two consecutive

edges contained in a 2–simplex of this link. This latter condition is called *local 6–largeness*.

A simplicial complex can be metrized in many ways, but the most natural metric, called the *standard piecewise Euclidean metric*, is given by declaring each simplex to be isometric to a regular Euclidean simplex with all side lengths equal to 1. In dimension 2, a simplicial complex is systolic exactly when the its standard piecewise Euclidean metric is CAT(0). In higher dimensions being systolic is neither stronger nor weaker than the standard piecewise Euclidean metric being CAT(0). A much more subtle question is whether a systolic complex admits any piecewise Euclidean metric that is CAT(0). No answer is known, but the answer is generally expected to be negative.

Systolic complexes do share many properties with CAT(0) spaces. For example, any finite dimensional systolic simplicial complex is contractible. As with CAT(0) cubical complexes, any group acting properly discontinuously and cocompactly on a systolic complex is biautomatic.

Systolic complexes are constructed in [JS2] as universal covers of simplices of groups, using the result that a locally 6–large complex of groups is developable. The fundamental groups of these simplices of groups are uniform lattices in the automorphism group of the universal cover.

## 4 Properties of $X$ and $\text{Aut}(X)$

The goal of this section is to understand the general structure of a polyhedral complex  $X$  and its full automorphism group  $\text{Aut}(X)$ . For instance, how much local data is required in order to uniquely determine  $X$ ? What are the basic topological and group-theoretic properties of  $\text{Aut}(X)$ ?

### 4.1 When does local data determine $X$ ?

As seen in many examples in Section 3 above, polyhedral complexes  $X$  are often constructed as universal covers of complexes of groups, and lattices in  $\text{Aut}(X)$  are often fundamental groups of complexes of groups. In each case, the local structure of the universal cover is determined by the local structure of the quotient space, together with the attached local groups of the complex of groups. Thus it is critical to know how much local data is needed in order to uniquely specify a desired polyhedral complex  $X$ . To simplify matters, we focus on the special case when  $X$  is a  $(k, L)$ –complex (see Section 3.7 above).

**Question 6.** *For a fixed  $(k, L)$ , is there a unique  $(k, L)$ –complex  $X$ ? If not, then what additional local data is needed to determine  $X$  uniquely?*

If  $L$  is the complete bipartite graph  $K_{m,n}$ , then in many cases there is a unique  $(k, L)$ –complex  $X$ . If  $k = 4$ , this complex is the product of an  $m$ –valent and an  $n$ –valent tree [W1]. If  $k > 4$  and either  $k$  is even or  $n = m$ , the unique  $(k, L)$ –complex is isomorphic to Bourdon’s building  $I_{p,q}$ , a right-angled Fuchsian building, with  $k = p$  and  $L = K_{q,q}$  ([B1, Św1];  $I_{p,q}$  is discussed in Section 3.3 above). If  $k > 4$  is odd and  $n \neq m$  then there does not exist a  $(k, L)$ –complex.

On the other hand, when  $L$  is the complete graph  $K_n$  for  $n \geq 4$ , Ballmann–Brin [BB1] and Haglund [H1] independently constructed uncountably many non-isometric  $(k, L)$ –complexes.

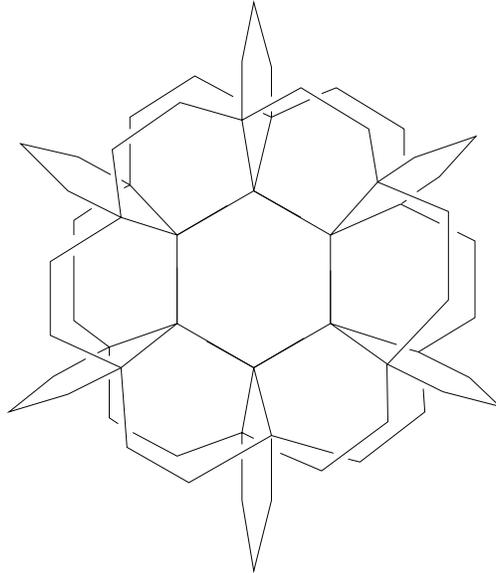


Figure 2: If the union of the twelve hexagons surrounding  $A$  is a Möbius band, then the holonomy around the boundary of  $A$  is nontrivial.

We now discuss these constructions. As mentioned in Section 3.7 above, simply connected nonpositively curved complexes can be constructed “freely” by building successive balls outward from a given cell. Provided that certain obvious local obstructions do not occur, we can glue in cells arbitrarily at each stage. Ballmann–Brin showed that for many choices of  $k$  and  $L$ , every nonpositively curved  $(k, L)$ –complex can be constructed in this manner [BB1].

In this inductive construction of a  $(k, L)$ –complex, choices may or may not arise. Let us consider the case when  $k = 6$  and  $L = K_4$ . Then each 2–cell of a  $(k, L)$ –complex  $X$  is a regular hexagon. Each 1–cell of  $X$  is contained in three distinct hexagons. Fix a 2–cell  $A$  of  $X$ , and consider the twelve surrounding 2–cells which contain one of the six 1–cells bounding  $A$ . These 2–cells are arranged locally in two sheets, whose union is a band surrounding  $A$ . However, if one follows the sheets around the boundary of  $A$ , there are two cases, depending on whether the union of the 12 hexagons is an annulus, or is the Möbius band shown in Figure 2.

To describe and analyze this phenomenon, Haglund [H3] introduced the notion of *holonomy*, which measures the twisting of the 2–cells neighboring a given 2–cell  $C$  as one traverses the boundary cycle of  $C$ . In many cases, the choices of holonomies around each 2–cell uniquely determine the isomorphism type of a nonpositively curved  $(k, L)$ –complex. The existence of holonomies depends on combinatorial properties of the graph  $L$ .

For instance, when  $n \geq 4$ , the complete graph  $L = K_n$  admits nontrivial holonomies. Roughly speaking, Ballmann–Brin and Haglund constructed uncountably many  $(k, K_n)$ –complexes by showing that, at each stage, a countable number of holonomies can be specified arbitrarily. In particular,  $K_4$  has a unique nontrivial holonomy, which is illustrated in Figure 2. The unique  $(6, K_4)$ –complex with trivial holonomies around every 2–cell is the

Cayley complex for the presentation  $\langle a, b \mid ba^2 = ab^2 \rangle$ , which defines the Geisking 3-manifold group. The unique  $(6, K_4)$ -complex with nontrivial holonomies around every 2-cell is the Cayley complex for the presentation  $\langle a, b \mid aba^2 = b^2 \rangle$ , which is  $\delta$ -hyperbolic (see [BC] for more details). On the other hand, the complete bipartite graph  $L = K_{m,n}$  admits only the trivial holonomy, which explains why there is a unique  $(k, L)$ -complex in this case.

Świątkowski [Św1] considered  $(k, L)$ -complexes  $X$  where  $L$  is a trivalent graph and  $X$  has *Platonic symmetry*, that is,  $\text{Aut}(X)$  acts transitively on the set of flags (vertex, edge, face) in  $X$ . He found elementary graph-theoretic conditions on  $L$  that imply that such an  $X$  is unique. Januszkiewicz–Leary–Valle–Vogeler [JLVV] classify Platonic  $(k, L)$ -complexes  $X$  in which  $L$  is a complete graph. Their main results are for finite complexes  $X$ .

In general holonomies are not enough to uniquely determine a  $(k, L)$ -complex. For instance, Haglund has observed that the Euclidean buildings for  $\text{SL}(3, \mathbf{Q}_p)$  and  $\text{SL}(3, \mathbf{F}_p((t)))$  are  $(3, L)$ -complexes with the same link  $L$  and the same holonomies. Yet the buildings are not isomorphic, by Theorem 5 above.

Nonclassical buildings with given local structures have been studied by Gaboriau–Paulin and Haglund–Paulin, who proved results analogous to those for  $(k, K_n)$ -complexes and  $(k, K_{m,n})$ -complexes discussed above. If  $q > 4$  is a prime power, Gaboriau–Paulin [GP] proved that for every hyperbolic Coxeter polygon  $P$  with all vertex angles  $\pi/6$ , and for every prime power  $q > 4$ , there exist uncountably many hyperbolic buildings with chambers  $P$  such that the links of vertices are all isomorphic to the building for the projective plane over the finite field  $\mathbf{F}_q$ . On the other hand, Haglund–Paulin [HP2] showed that if  $(W, I)$  is a right-angled Coxeter system and  $(q_i)_{i \in I}$  is a collection of cardinalities, then there exists a unique building  $X$  of type  $(W, I)$  such that for each  $i \in I$ , each codimension one cell containing a vertex of type  $\{i\}$  in  $X$  is a face of  $q_i$  distinct chambers. This generalizes the result that Bourdon’s building  $I_{p,q}$  is the unique  $(p, K_{q,q})$ -complex.

In many cases it is still unknown how much local data is required to uniquely specify a  $(k, L)$ -complex.

## 4.2 Nondiscreteness of $\text{Aut}(X)$

Let  $X$  be a locally finite, nonpositively curved polyhedral complex. The most basic question about the locally compact group  $G = \text{Aut}(X)$  is whether or not it is discrete. Recall that in the compact-open topology, the group  $G = \text{Aut}(X)$  is nondiscrete exactly when, for each positive integer  $n$ , there is an element  $g_n \in G$ , with  $g_n$  fixing pointwise the ball of radius  $n$  in  $X$ , and  $g_n \neq \text{Id}$ . The theory of lattices in a discrete group is trivial, hence this issue is of crucial importance. We again focus on the case of  $(k, L)$ -complexes (see Section 3.7 above).

**Question 7.** *Given a  $(k, L)$ -complex  $X$ , is  $G = \text{Aut}(X)$  discrete?*

The answer is known in certain cases, and is closely related to the notion of a flexible complex.

**Definition:** A complex  $X$  is *flexible* if there exists  $\phi \in \text{Aut}(X)$  such that  $\phi$  fixes the star of some vertex in  $X$  but  $\phi \neq \text{Id}$ .

Flexibility was introduced by Ballmann–Brin in [BB1]. If  $X$  is locally finite and not flexible, then the stabilizer of any vertex  $v \in X$  is finite, since an automorphism of  $X$  that

fixes  $v$  is uniquely determined by its action on the link of  $v$ . In particular,  $\text{Aut}(X)$  is discrete if  $X$  is not flexible. The following result is nearly immediate from the definition of flexibility.

**Theorem 8** (Discreteness criterion). *If the graph  $L$  is not flexible, then any  $(k, L)$ -complex  $X$  is not flexible, and  $\text{Aut}(X)$  is discrete.*

Theorem 8 has the following converse when  $X = X(2k, L)$  is the Davis–Moussong complex for the Coxeter group  $W = W(k, L)$ , discussed in Section 3.6 above. The result was proved independently by Haglund and Świątkowski in the case that  $X$  is 2-dimensional [H2, Św1], and was extended to arbitrary Coxeter systems by Haglund–Paulin [HP1].

**Theorem 9** (Nondiscreteness criterion). *Suppose  $L$  is a finite simplicial graph and  $k \geq 2$ . Let  $X = X(2k, L)$  be the Davis–Moussong complex for the Coxeter group  $W = W(k, L)$ . If  $L$  is flexible then  $\text{Aut}(X)$  is nondiscrete.*

The proof of Theorem 9 relies on the fact that Davis–Moussong complexes have numerous symmetries. For other  $(k, L)$ -complexes, particularly those with  $k$  odd, much less is known. It is not clear whether this reflects the limitations of our techniques, or actual differences in behavior for  $k$  odd and  $k$  even.

### 4.3 Simplicity and nonlinearity

Let  $X$  be a locally finite, nonpositively curved polyhedral complex, with locally compact automorphism group  $G = \text{Aut}(X)$ . In this section we discuss whether two basic group-theoretic properties, simplicity and (non)linearity, hold for  $G$ .

**Question 10.** *When is  $G = \text{Aut}(X)$  a simple group?*

For  $X$  a locally finite regular or biregular tree, Tits [Ti1] proved simplicity of the group  $\text{Aut}_0(X)$  of type-preserving automorphisms of  $X$  (which is finite index in the full automorphism group  $G = \text{Aut}(X)$ ). Haglund–Paulin [HP1] showed that various type-preserving automorphism groups in several higher-dimensional cases are simple. We note that the method of proof of these results lies in geometric group theory.

We say that a group  $G$  is *linear* if it has a faithful representation  $G \rightarrow \text{GL}(n, K)$  for some field  $K$ . On the question of linearity, suppose  $X$  is a classical Euclidean building, associated to the algebraic group  $G$  over a local nonarchimedean field  $K$  (see Section 3.1.1 above). Theorem 5 above says that if  $\text{char}(K) = 0$  then  $G$  is finite index in  $\text{Aut}(X)$  (and if  $\text{char}(K) = p > 0$  then  $G$  is cocompact in  $\text{Aut}(X)$ ). By inducing, we see in particular that when  $\text{char}(K) = 0$ , the group  $\text{Aut}(X)$  has a faithful linear representation over  $K$ . On the other hand, for several higher-dimensional complexes  $X$  which are not classical buildings, Haglund–Paulin [HP1] proved that the full automorphism group  $\text{Aut}(X)$  has no such faithful linear representation. For  $\dim(X) = 2$ , we pose the following question:

**Problem 11.** *Find conditions on the link  $L$  so that a  $(k, L)$ -complex  $X$  has linear automorphism group.*

Haglund [H4] has recently shown that  $\text{Aut}(X)$  is nonlinear for certain Fuchsian buildings  $X$  (see Section 3.3 above). Is it possible that linearity of  $\text{Aut}(X)$  characterizes those  $X$  which are classical Euclidean buildings, among all nonpositively curved  $X$ ?

## 5 Comparisons with linear groups

While one expects some of the phenomena and results from the theory of linear groups  $G \subset \mathrm{GL}(n, \mathbf{C})$  to hold for the group  $G = \mathrm{Aut}(X)$  and its lattices, most of the methods from that theory are unavailable in this new context. There are no eigenvalues or traces. There are no vectors to act on. It therefore seems important to attack such questions, as they will (hopefully) force us to come up with new methods and tools.

### 5.1 Some linear-type properties

One of the basic properties of linear groups  $G$  is the *Tits alternative*: any finitely generated linear group either contains a nonabelian free group or has a solvable subgroup of finite index (see [Ti2]). The following problem is well known.

**Problem 12.** *Let  $X$  be a nonpositively curved polyhedral complex. Prove that finitely generated subgroups of  $G = \mathrm{Aut}(X)$  satisfy the Tits alternative.*

When  $X$  is a  $\mathrm{CAT}(-1)$  space, uniform lattices in  $G = \mathrm{Aut}(X)$  are word-hyperbolic, and thus satisfy the Tits alternative (Gromov [Gr]). The usual ping-pong argument for the Tits alternative requires strong expanding/contracting behavior for the action of isometries of  $X$  on the visual boundary  $\partial X$ . The difficulty with Problem 12 lies in the fact that if  $X$  is just nonpositively curved, rather than negatively curved, this behavior on  $\partial X$  is not strong enough to immediately allow for the usual ping-pong argument to work.

The Iwasawa decomposition  $KAN$  of a semisimple Lie group  $G$  plays a fundamental role in the representation theory of  $G$ . Here,  $K$  is a compact subgroup,  $A$  is abelian and  $N$  is nilpotent. In the topology on  $G = \mathrm{Aut}(X)$ , where  $X$  is a locally finite polyhedral complex, the stabilizers of vertices are maximal compact subgroups.

**Question 13.** *For which  $X$  does  $G = \mathrm{Aut}(X)$  have a  $KAN$  structure?*

Answering this question might be a first step towards investigating various analytic properties of  $X$ , the group  $G = \mathrm{Aut}(X)$ , and its lattices. For instance, random walks on classical buildings have been studied using the representation theory of the associated algebraic group (see, for example, Cartwright–Woess [CW] and Parkinson [Pa]), but for more general complexes  $X$  this machinery is not available.

Kazhdan proved that simple Lie groups  $G$  have *property (T)*: the trivial representation is isolated in the unitary dual of  $G$  (see, for example, [Ma]). Ballmann–Świątkowski [BS], Żuk [Zu], and Dymara–Januszkiewicz [DJ] have proven that many  $G = \mathrm{Aut}(X)$  satisfy this important property.

**Question 14.** *For which  $X$  does  $\mathrm{Aut}(X)$  have Property (T)?*

We remark that a locally compact topological group  $G$  has property (T) if and only if any of its lattices has property (T).

One of the deepest theorems about irreducible lattices  $\Gamma$  in higher rank semisimple Lie groups is Margulis’s Normal Subgroup Theorem (see [Ma]), which states that any normal subgroup of  $\Gamma$  is finite or has finite index in  $\Gamma$ .

**Question 15.** *For which  $X$  does a normal subgroup theorem hold for  $\text{Aut}(X)$ ?*

Such a theorem has been shown for products of trees by Burger–Mozes [BM].

Recall that the *Frattini subgroup*  $\Phi(\Gamma)$  of a group  $\Gamma$  is the intersection of all maximal subgroups of  $\Gamma$ . Platonov [Pl] proved that  $\Phi(\Gamma)$  is nilpotent for every finitely generated linear group. Ivanov [I] proved a similar result for mapping class groups. I. Kapovich [K] proved that  $\Phi(\Gamma)$  is finite for subgroups of finitely generated word-hyperbolic groups.

**Problem 16.** *Compute the Frattini subgroup  $\Phi(\Gamma)$  for finitely generated subgroups  $\Gamma < \text{Aut}(X)$ .*

Part of the fascination of lattices in  $\text{Aut}(X)$  is that they exhibit a mixture of rank one and higher rank behavior. Ballmann–Eberlein (see [E]) defined an invariant  $\text{rank}(\Gamma)$ , called the *rank* of  $\Gamma$ , which is defined for any finitely generated group  $\Gamma$  as follows. Let  $\Gamma_i$  denote the set of elements  $g \in \Gamma$  so that the centralizer of  $g$  contains  $\mathbf{Z}^d$  for some  $d \leq i$  as a finite index subgroup. Let  $r(\Gamma)$  be defined to be the smallest  $i$  so that  $\Gamma$  is a finite union of translates

$$\Gamma = g_1\Gamma_i \cup \cdots \cup g_n\Gamma_i$$

for some  $g_j \in \Gamma$ . Then define  $\text{rank}(\Gamma)$  to be the maximum of  $r(\Gamma')$ , where  $\Gamma'$  runs over all finite index subgroups of  $\Gamma$ .

Work of Prasad–Raghunathan shows that this notion of rank agrees with the classical one for arithmetic lattices. Ballmann–Eberlein [BE] proved that the rank of the fundamental group of a complete, finite volume, nonpositively curved manifold  $M$  equals the geometric rank of the universal cover of  $M$ . Since centralizers of infinite order elements in word-hyperbolic groups  $\Gamma$  are virtually cyclic, it is clear that  $\text{rank}(\Gamma) = 1$  in these cases. Thus for nonpositively curved, connected, simply-connected 2-complexes  $X$ , lattices in  $\text{Aut}(X)$  can have rank one and also rank two (the latter, for example, when  $X$  is a classical Euclidean building, discussed in Section 3.1.1 above).

**Problem 17.** *Compute  $\text{rank}(\Gamma)$  for lattices  $\Gamma < \text{Aut}(X)$ .*

A basic property of any finitely generated linear group is that it is residually finite. In contrast, there are lattices  $\Gamma$  in  $G = \text{Aut}(X)$  that are not residually finite. Indeed, Burger–Mozes [BM] have constructed, in the case when  $X$  is a product of simplicial trees, lattices which are simple groups. Wise had earlier constructed lattices in such  $X$  that are not residually finite [W2].

**Problem 18.** *Construct a lattice  $\Gamma$  in  $G = \text{Aut}(X)$  which is a simple group, and where  $X$  is not a product of trees.*

For residual finiteness, a key case is Bourdon’s building  $I_{p,q}$  (see Section 3.3 above) whose 2-cells are right-angled hyperbolic  $p$ -gons. Wise [W3] has shown that fundamental groups of polygons of finite groups, where the polygon has at least 6 sides, are residually finite. Thus there are residually finite uniform lattices for  $I_{p,q}$ ,  $p \geq 6$ , but the question is completely open for  $p = 5$ , that is, for pentagons.

**Question 19.** *Which lattices  $\Gamma < G = \text{Aut}(X)$  are residually finite?*

## 5.2 Rigidity

Automorphism groups  $G$  of nonpositively curved polyhedral complexes  $X$ , and lattices  $\Gamma < G$ , are natural places in which to study various rigidity phenomena, extending what we know in the classical, algebraic cases. A first basic problem is to prove strong (Mostow) rigidity. In other words, one wants to understand the extent to which a lattice  $\Gamma$  in  $G$  determines  $G$ .

**Problem 20** (Strong rigidity). *Let  $X_1$  and  $X_2$  be nonpositively curved polyhedral complexes, and let  $\Gamma_i$  be a lattice in  $G_i = \text{Aut}(X_i)$ ,  $i = 1, 2$ . Find conditions on the  $X_i$  which guarantee that any abstract group homomorphism  $\phi: \Gamma_1 \rightarrow \Gamma_2$  extends to an isomorphism  $G_1 \rightarrow G_2$ . Further, determine when any two copies of  $\Gamma_i$  in  $G_i$  are conjugate in  $G_i$ .*

A harder, more general problem is to prove quasi-isometric rigidity.

**Problem 21** (Quasi-isometric rigidity). *Compute the quasi-isometry groups of nonpositively curved polyhedral complexes  $X$ . Prove quasi-isometric rigidity theorems for these complexes; that is, find conditions on  $X$  for which:*

1. *Any quasi-isometry of  $X$  is a bounded distance from an isometry (automorphism), and*
2. *Any finitely-generated group quasi-isometric to  $X$  is a cocompact lattice in  $\text{Aut}(X)$ .*

A standard trick due to Cannon–Cooper shows that (1) implies (2). It is also immediate from Mostow’s original argument that (1) implies strong rigidity. Quasi-isometric rigidity was proven in the case of Euclidean buildings by Kleiner–Leeb [KL]. Bourdon–Pajot [BP] proved quasi-isometric rigidity for Bourdon’s building  $I_{p,q}$ , and Xie [X] generalized this to Fuchsian buildings (see Section 3.3). One would expect that higher-dimensional buildings would be more rigid, and indeed they seem to be harder to construct, so they might be a good place to look for rigidity phenomena.

Another kind of rigidity problem is the following:

**Problem 22.** *Suppose  $X_1$  and  $X_2$  are locally finite, connected, simply-connected 2-complexes, such that for  $i = 1, 2$ , the group  $\text{Aut}(X_i)$  acts cocompactly on  $X_i$ . If  $\text{Aut}(X_1)$  is isomorphic to  $\text{Aut}(X_2)$ , is  $X_1$  isometric to  $X_2$ ?*

A variety of other rigidity phenomena from Riemannian geometry have natural analogs in this context. Examples include rank rigidity, hyperbolic rank rigidity, minimal entropy rigidity, and marked length spectrum rigidity. A rank rigidity theorem for nonpositively 2-complexes was proven by Ballmann–Brin in [BB2].

## 5.3 Geometry of the word metric

One of the few results about the geometry of the word metric for nonuniform lattices is the theorem of Lubotzky–Mozes–Raghunathan [LMR], which we now discuss.

Let  $G$  be a semisimple Lie group over  $\mathbf{R}$  (respectively, over a nonarchimedean local field  $K$ ), and let  $X$  be the associated symmetric space (respectively, Euclidean building). Thus  $X$  is a nonpositively curved Riemannian manifold (respectively, simplicial complex) on which  $G$  acts by isometries. Let  $\Gamma$  be a lattice in  $G$ . If  $K$  is nonarchimedean and  $G$  has rank one

over  $K$  then nonuniform lattices in  $G$  are not finitely generated. On the other hand, when  $G$  is either a real Lie group or a nonarchimedean group with  $K$ -rank at least 2, then all lattices  $\Gamma$  in  $G$  are finitely generated. In this section, we consider only the finitely-generated case, and we endow  $\Gamma$  with the word metric for a finite generating set.

If  $\Gamma$  is uniform, then the natural map  $\psi: \Gamma \rightarrow X$  sending  $\Gamma$  to any of its orbits is a quasi-isometry. When  $\Gamma$  is nonuniform, the orbit map is never a quasi-isometry, since the quotient  $\Gamma \backslash X$  is noncompact. When  $G$  has real rank one, the map  $\psi$  is not even a quasi-isometric embedding, as can be seen by considering any nonuniform lattice acting on real hyperbolic space. In this case, the maximal parabolic subgroups of  $\Gamma$  are exponentially distorted in  $X$ .

The theorem of Lubotzky–Mozes–Ragunathan [LMR] states that, when  $G$  has real rank (respectively,  $K$ -rank) at least 2, then  $\psi$  is indeed a quasi-isometric embedding. Each of the known proofs of this result is heavily algebraic, depending on the structure of matrix groups. Thus the following problem presents an interesting challenge, even in terms of giving a geometric proof in the (nonarchimedean) algebraic case.

**Question 23.** *Let  $\Gamma$  be a nonuniform lattice in the automorphism group of a nonpositively curved polyhedral complex  $X$ . If  $\Gamma$  is finitely generated, is the natural map  $\psi: \Gamma \rightarrow X$ , sending  $\Gamma$  to any of its orbits, a quasi-isometric embedding?*

When  $X$  is a product of trees,  $\psi$  need not be a quasi-isometric embedding. When  $X$  is not a product of trees, is  $\psi$  always a quasi-isometric embedding?

## 5.4 Dynamics

Let  $G$  be (any) locally compact topological group, equipped with Haar measure, and let  $\Gamma$  be a lattice in  $G$ . Then  $G$  acts on the left on  $G/\Gamma$ , preserving the finite measure on  $G/\Gamma$  induced by the Haar measure on  $G$ . We thus obtain an action of every closed subgroup  $H < G$  on  $G/\Gamma$ . It is a basic question understand these dynamical systems, in particular to determine when the action of  $H$  on  $G/\Gamma$  is *ergodic*; that is, when every  $H$ -invariant set has zero or full measure. When  $G$  is a semisimple Lie group with no compact factors, and  $\Gamma$  is an irreducible lattice in  $G$ , *Moore's Ergodicity Theorem* (see [Zi]) states that the  $H$ -action on  $G/\Gamma$  is ergodic if and only if  $H$  is noncompact.

Now let  $X$  be a simply connected, locally finite polyhedral complex of nonpositive curvature. Equip  $G = \text{Aut}(X)$  with left-invariant Haar measure, and let  $\Gamma$  be a lattice in  $G$ .

**Problem 24.** *Determine which closed subgroups of  $G = \text{Aut}(X)$  act ergodically on  $G/\Gamma$ .*

One reason we consider Problem 24 to be worthwhile is that the usual method of proving Moore's Ergodicity Theorem uses the unitary representation theory of  $G$ . We thus believe that, apart from being interesting in its own right, attempts to solve Problem 24 will require us either to find new approaches to Moore's theorem, or to develop the unitary representation theory of  $G = \text{Aut}(X)$ .

## 6 Lattices in $\text{Aut}(X)$

In this section we consider properties of the lattices in  $\text{Aut}(X)$  themselves. Some lattice properties have already been mentioned in Section 5 above, on comparisons with linear groups. Here, we discuss topics where new phenomena, contrasting with classical cases, have already been observed, and where the known techniques of proof are combinatorial or geometric in flavor.

### 6.1 Existence and classification theorems

Given a locally compact group  $G$ , the most basic question in the lattice theory of  $G$  is whether  $G$  admits a uniform or nonuniform lattice.

For algebraic groups, the existence of both uniform and nonuniform lattices was settled by Borel and others, using arithmetic constructions (see the final paragraph of Section 3.1.1 above). For automorphism groups of trees, precise conditions are known for the existence of both uniform lattices (Bass–Kulkarni [BK]) and nonuniform lattices (Bass–Carbone–Rosenberg, in [BL]). In Section 3 above, for each example  $X$  of a polyhedral complex, we described known constructions of lattices in  $G = \text{Aut}(X)$ . These constructions are non-arithmetic, for  $X$  not a classical building. The following question is still largely open.

**Question 25.** *When does  $G = \text{Aut}(X)$  admit a uniform lattice? A nonuniform lattice?*

A special case of this question is:

**Question 26.** *For which positive integers  $k \geq 3$  and finite simplicial graphs  $L$  does the automorphism group of a  $(k, L)$ -complex  $X$  admit lattices?*

Once one establishes the existence of lattices in a given  $G = \text{Aut}(X)$ , the next problem is to classify all such lattices. We discuss commensurability of lattices in Section 6.2 below. An even more fundamental question is:

**Problem 27.** *Classify lattices in  $G = \text{Aut}(X)$  up to conjugacy.*

We note that in the case of real Lie groups, classification theorems are difficult. For  $\text{SO}(3, 1)$ , for example, the classification is precisely the classification of all finite volume, complete hyperbolic orbifolds. On the other hand, for higher rank real (and  $p$ -adic) semisimple Lie groups, Margulis’s arithmeticity theorem (see [Ma]) states that all lattices are arithmetic, and arithmetic lattices can in some sense be classified (although this is also not easy). So, even solving Problem 27 in any special case, for example for specific hyperbolic buildings, would be of great interest.

### 6.2 Commensurability

One of the basic problems about a locally compact topological group  $G$  is to classify its lattices up to commensurability. Recall that two lattices  $\Gamma_1, \Gamma_2 \leq G$  are *commensurable in  $G$*  if there exists  $g \in G$  so that  $g\Gamma_1g^{-1} \cap \Gamma_2$  has finite index in both  $g\Gamma_1g^{-1}$  and  $\Gamma_2$ . Since covolume is multiplicative in index, two commensurable lattices have covolumes that are commensurable real numbers, that is, they have a rational ratio.

**Problem 28.** *Classify lattices in  $G = \text{Aut}(X)$  up to commensurability. As a subproblem, find commensurability invariants of lattices.*

If  $G$  is an algebraic group of rank at least two over a nonarchimedean local field  $K$ , then there exist noncommensurable arithmetic lattices in  $G$ . If  $G$  is a rank one simple real Lie group, then lattices are again not all commensurable, as there exist both arithmetic and nonarithmetic lattices.

For  $G = \text{Aut}(X)$ , commensurability of uniform lattices is strikingly different. When  $X$  is a locally finite tree, Leighton proved in [Lei] that all torsion-free uniform lattices in  $G = \text{Aut}(X)$  are commensurable. The torsion-free hypothesis was removed by Bass–Kulkarni in [BK], establishing that there is at most one commensurability class of uniform lattices in the tree case. Haglund [H4] has shown the same result for many Fuchsian buildings (see Section 3.3). He has also found a sufficient condition for a uniform lattice in the automorphism group of a Davis–Moussong complex  $X$  (see Section 3.6) to be commensurable to the corresponding Coxeter group  $W$ . As a specific instance of Problem 28, we have:

**Problem 29.** *Suppose  $X$  is a  $(k, L)$ -complex. Find conditions on  $L$  such that all uniform lattices in  $\text{Aut}(X)$  are commensurable, and find examples of such  $L$ .*

For nonuniform lattices in  $G = \text{Aut}(X)$ , the situation seems much wilder. Even in the tree case, there seems to be a great deal of flexibility in the construction of nonuniform lattices. For instance, Farb–Hruska [FH] have shown that when  $X$  is the biregular tree there are uncountably many commensurability classes of nonuniform lattices in  $G = \text{Aut}(X)$  with any given covolume  $v > 0$ . To prove this result, they construct several new commensurability invariants, and then evaluate them on lattices constructed using graphs of groups.

A similar result holds when  $X$  is a right-angled building (see Section 3.4), by work of Thomas [Th2]. Lattices in right-angled hyperbolic buildings, such as Bourdon’s building  $I_{p,q}$ , are known to exhibit higher-rank phenomena, such as quasi-isometric rigidity (see [BP] and Section 5.2 above). In contrast, Thomas’ theorem indicates a similarity of these lattices with tree lattices. In fact, Thomas proves this theorem by constructing a functor that takes tree lattices to lattices in right-angled buildings. This functor preserves many features of the lattice.

The most important commensurability invariant of a group  $\Gamma$  inside a group  $G$  is the *commensurator*  $\text{Comm}_G(\Gamma)$  of  $\Gamma$  in  $G$ , defined by

$$\text{Comm}_G(\Gamma) := \{ g \in G \mid \Gamma \cap g\Gamma g^{-1} \text{ has finite index in both } \Gamma \text{ and } g\Gamma g^{-1} \}.$$

Margulis proved that a lattice  $\Gamma$  in a semisimple Lie group  $G$  is arithmetic if and only if  $\text{Comm}_G(\Gamma)$  is dense in  $G$  (see [Zi]). Lubotzky proposed this density property as a definition of “arithmeticity” when  $G = \text{Aut}(X)$ .

**Problem 30.** *For lattices  $\Gamma$  in  $G = \text{Aut}(X)$ , compute  $\text{Comm}_G(\Gamma)$ . Determine whether or not  $\text{Comm}_G(\Gamma)$  is dense in  $G$ .*

When  $X$  is a tree, density of commensurators of uniform lattices was proved by Bass–Kulkarni [BK] and Liu [Liu]. Haglund established density of commensurators of uniform lattices for many Davis–Moussong complexes in [H2], and Haglund [H5] and independently

Barnhill–Thomas [BT] have recently shown the same result for right-angled buildings. For commensurators of nonuniform lattices, however, even for trees very little is known (see [BL]).

### 6.3 Finiteness properties of lattices

Uniform lattices in  $G = \text{Aut}(X)$  are always finitely generated, for obvious reasons. However, nonuniform lattices need not be finitely generated.

**Question 31.** *For which  $G = \text{Aut}(X)$  are all nonuniform lattices non-finitely generated? Do there exist  $G$  which admit both finitely generated and non-finitely generated nonuniform lattices?*

Higher rank algebraic groups, such as  $G = \text{SL}(3, \mathbf{F}_q((t)))$ , have Kazhdan’s Property (T) (see Section 5.1 above). Furthermore, Property (T) is inherited by lattices, and all countable groups with Property (T) are finitely generated. Therefore lattices in higher rank groups are all finitely generated.

On the other hand, if  $X$  is a tree, every nonuniform lattice in  $\text{Aut}(X)$  is non-finitely generated [BL]. Thomas’ functor mentioned in Section 6.2 above implies that many nonuniform lattices in right-angled hyperbolic buildings are non-finitely generated as well.

**Conjecture 32.** *Let  $\Gamma$  be a nonuniform lattice in  $G = \text{Aut}(X)$ , where  $X$  is any right-angled hyperbolic building. Then  $\Gamma$  is not finitely generated.*

We are starting to believe that finite generation of nonuniform lattices in 2-complexes is actually a miracle, and could even characterize the remarkable nonuniform lattices in algebraic groups in characteristic  $p > 0$ . Even these lattices are not finitely presentable, and so we make the following:

**Conjecture 33.** *If  $\Gamma$  is a nonuniform lattice in  $G = \text{Aut}(X)$ , where  $X$  is a locally finite polyhedral complex, then  $\Gamma$  is not finitely presentable.*

### 6.4 Covolumes

One of the more striking ways in which the study of lattices in  $\text{Aut}(X)$  diverges from the case of lattices in semisimple Lie groups is the study of covolumes of lattices in a fixed  $\text{Aut}(X)$ . New phenomena are seen to occur right away, and much remains to be understood.

**Problem 34.** *Given  $G = \text{Aut}(X)$  with Haar measure  $\mu$ , describe the set of covolumes*

$$\mathcal{V}(G) := \{ \mu(\Gamma \backslash G) \mid \Gamma \text{ is a lattice in } G \}$$

Note that  $\mathcal{V}(G)$  is a set of positive real numbers.

If  $G$  is a non-compact simple real Lie group, such as  $\text{PSL}(n, \mathbf{R})$ , then the set  $\mathcal{V}(G)$  has positive lower bound (Kazhdan–Margulis, [KM]) and in most cases is discrete (see [Lu] and the references therein). If  $G$  is a higher-rank algebraic group over a nonarchimedean local field, such as  $\text{PSL}(n, \mathbf{Q}_p)$  with  $n \geq 3$ , the strong finiteness result of Borel–Prasad [BPr] implies that for any  $c > 0$ , there are only finitely many lattices in  $G$  with covolume less

than  $c$ . Hence  $\mathcal{V}(G)$  is discrete, has positive lower bound, and for any  $v \in \mathcal{V}(G)$  there are only finitely many lattices of covolume  $v$ .

The set of covolumes for tree lattices is very different. Suppose  $G$  is the group of automorphisms of a regular locally finite tree. Then, for example, Bass–Kulkarni [BK] showed that  $\mathcal{V}(G)$  contains arbitrarily small elements, by constructing a tower of uniform lattices (see Section 6.5 below). Bass–Lubotzky [BL] showed that the set of nonuniform covolumes is  $(0, \infty)$ .

A few higher-dimensional nonclassical cases have been studied. In [Th2] and [Th3], Thomas considered covolumes for, respectively, right-angled buildings and certain Fuchsian buildings (see Sections 3.4 and 3.3 respectively). In both these settings,  $\mathcal{V}(G)$  shares properties, such as nondiscreteness, with covolumes of tree lattices, even though such buildings also have some rigidity properties typical of classical cases (see Section 5.2 above). Little is known about covolumes for  $X$  not a building. In [Th4], the class of  $(k, L)$ -complexes  $X$  of Platonic symmetry (introduced by Świątkowski [Św1]; see Section 3.7) is considered. A sample result is that if  $k \geq 4$  is even, and  $L$  is the Petersen graph, then  $\mathcal{V}(G)$  is nondiscrete. Many cases are completely open.

From a different point of view, Prasad [Pr] gave a computable formula for the covolumes of lattices  $\Gamma$  in algebraic groups  $G$  over nonarchimedean local fields. This formula is in terms of discriminants of field extensions and numbers of roots. If  $\Gamma$  is viewed instead as a lattice in  $\text{Aut}(X)$ , where  $X$  is the building associated to the algebraic group  $G$ , we have also Serre’s more geometrically-flavored formula for the covolume of  $\Gamma$ , stated in the introduction.

**Question 35.** *Can Serre’s geometric formula for covolumes tell us anything new about lattices in classical cases?*

More generally, using Serre’s geometric formula, in [Th1] Thomas established a computable number-theoretic restriction on the set of covolumes of uniform lattices, for all locally finite  $X$  with  $G = \text{Aut}(X)$  acting cocompactly, in all dimensions.

**Problem 36.** *Suppose  $v > 0$  satisfies the restriction of [Th1]. Construct a uniform lattice in  $G$  of covolume  $v$ , or show that such a lattice does not exist. Also, find the cardinality of the set of uniform lattices of covolume  $v$ . For nonuniform lattices, the same questions for any  $v > 0$ .*

This problem was solved for right-angled buildings (see Section 3.4) in [Th2].

The properties of the set of volumes of hyperbolic three-manifolds are well-understood (see [Thu]), and one could investigate whether similar properties hold for volumes of lattices in  $\text{Aut}(X)$ . For instance, for every nonuniform lattice  $\Gamma$  in  $\text{SO}(3, 1)$ , there is a sequence of uniform lattices with covolumes converging to that of  $\Gamma$ , obtained by Dehn surgery. This gives a surjective homomorphism from  $\Gamma$  to each of these uniform lattices. It is not known whether any nonuniform lattices in  $\text{Aut}(X)$  surject onto uniform lattices.

## 6.5 Towers

The study of towers of lattices is closely related to covolumes (Section 6.4 above). A *tower* of lattices in a locally compact group  $G$  is an infinite strictly ascending sequence

$$\Gamma_1 < \Gamma_2 < \cdots < \Gamma_n < \cdots < G$$

where each  $\Gamma_n$  is a lattice in  $G$ .

**Question 37.** *Does  $G = \text{Aut}(X)$  admit a tower of (uniform or nonuniform) lattices?*

If  $G$  admits a tower, then the covolumes of lattices in this tower tend to zero, hence the set  $\mathcal{V}(G)$  of covolumes does not have positive lower bound. It follows that in classical (algebraic) cases,  $G$  does not admit any towers, by the Kazhdan–Margulis Theorem in Section 6.4 above.

The first examples of towers of tree lattices are due to Bass–Kulkarni [BK]. Generalizing these constructions, Rosenberg [Ros] proved that if  $X$  is a tree such that  $\text{Aut}(X)$  is nondiscrete and admits a uniform lattice, then  $\text{Aut}(X)$  admits a tower of uniform lattices. Carbone–Rosenberg [CR] considered nonuniform towers, showing that, with one exception, if  $\text{Aut}(X)$  admits a nonuniform lattice then it admits a tower of nonuniform lattices.

In higher dimensions, for  $X$  a right-angled building (see Section 3.4) Thomas [Th2] constructed a tower of uniform and of nonuniform lattices. Other higher-dimensional cases are open. In particular, it is not known whether the automorphism groups of any Fuchsian buildings which are not right-angled (see Section 3.3) admit towers.

A finer version of Question 37 is the following:

**Question 38.** *Does  $G$  admit a tower of homogeneous lattices, that is, lattices acting transitively on cells of maximum dimension in  $X$ ?*

For  $X = T_{p,q}$  the  $(p, q)$ -biregular tree, if  $p$  or  $q$  is composite there is a homogeneous tower in  $G = \text{Aut}(X)$  (Bass–Kulkarni [BK]). When  $X$  is the 3-regular tree, a deep theorem of Goldschmidt [Go] implies that  $G$  does not admit such a tower, since  $G$  contains only finitely many conjugacy classes of edge-transitive lattices. The Goldschmidt–Sims conjecture (see [Gl]), which remains open, is that if  $p$  and  $q$  are both prime, then there are only finitely many conjugacy classes of homogeneous lattices in  $\text{Aut}(T_{p,q})$ . If  $X$  is the product of two trees of prime valence, Glasner [Gl] has shown that there are only finitely many conjugacy classes of (irreducible) homogeneous lattices in  $G = \text{Aut}(X)$ . For all other higher-dimensional  $X$ , the question is open.

**Question 39.** *Does  $G$  admit maximal lattices?*

In the algebraic setting, lattices of minimal covolume are known in many cases (see [Lu] and its references), and so these lattices are maximal. Examples of maximal lattices in  $G = \text{Aut}(X)$  are some of the edge-transitive lattices for  $X$  the 3-regular tree, classified by Goldschmidt [Go].

A coarse version of the question of towers is:

**Question 40** (Lubotzky). *Let  $\Gamma$  be a uniform lattice in  $G = \text{Aut}(X)$ . Define*

$$u_\Gamma(n) = \#\{\Gamma' \mid \Gamma' \text{ is a lattice containing } \Gamma, \text{ and } [\Gamma' : \Gamma] = n\}$$

*By similar arguments to [BK],  $u_\Gamma(n)$  is finite. What are the asymptotics of  $u_\Gamma(n)$ ?*

The case  $X$  a tree was treated by Lim [Lim]. If  $X$  is the building associated to a higher-rank algebraic group, then for any  $\Gamma$ , we have  $u_\Gamma(n) = 0$  for  $n \gg 0$ , since  $\mathcal{V}(G)$  has positive

lower bound. In contrast, if  $\text{Aut}(X)$  admits a tower of lattices (for example if  $X$  is a right-angled building), there is a  $\Gamma$  with  $u_\Gamma(n) > 0$  for arbitrarily large  $n$ . Lim–Thomas [LT], by counting coverings of complexes of groups, found an upper bound on  $u_\Gamma(n)$  for very general  $X$ , and a lower bound for certain right-angled buildings  $X$ . It would be interesting to sharpen these bounds for particular cases.

## 6.6 Biautomaticity of lattices

The theory of automatic and biautomatic groups is closely related to nonpositive curvature. All word hyperbolic groups are biautomatic [ECHLPT]. Yet it is not known whether an arbitrary group acting properly, cocompactly, and isometrically on a  $\text{CAT}(0)$  space is biautomatic, or even automatic. Indeed, the following special case is open:

**Question 41.** *Suppose a group  $\Gamma$  acts properly, cocompactly and isometrically on a  $\text{CAT}(0)$  piecewise Euclidean 2-complex. Is  $\Gamma$  biautomatic? Is  $\Gamma$  automatic?*

Biautomaticity is known in several cases for groups acting on complexes built out of restricted shapes of cells. Gersten–Short established biautomaticity for uniform lattices in  $\text{CAT}(0)$  2-complexes of type  $\tilde{A}_1 \times \tilde{A}_1$ ,  $\tilde{A}_2$ ,  $\tilde{B}_2$ , and  $\tilde{G}_2$  in [GS1, GS2]. In particular, Gersten–Short’s work includes  $\text{CAT}(0)$  square complexes, 2-dimensional systolic complexes, and 2-dimensional Euclidean buildings.

Several special cases of Gersten–Short’s theorem have been extended. For instance Świątkowski proved that any uniform lattice in a Euclidean building is biautomatic [Św2]. Niblo–Reeves [NR1] proved biautomaticity of all uniform lattices acting on  $\text{CAT}(0)$  cubical complexes. In particular, this result includes all finitely generated right-angled Coxeter groups and right-angled Artin groups. Systolic groups, that is, uniform lattices acting on arbitrary systolic simplicial complexes, are also biautomatic by work of Januszkiewicz–Świątkowski [JS1].

Gersten–Short’s work applies only to 2-complexes with a single shape of 2-cell. Levitt has generalized Gersten–Short’s theorem to prove biautomaticity of any uniform lattice acting on a  $\text{CAT}(0)$  triangle-square complex, that is, a 2-complex each of whose 2-cells is either a square or an equilateral triangle [Lev].

Epstein proved that all nonuniform lattices in  $\text{SO}(n, 1)$  are biautomatic [ECHLPT, 11.4.1]. Rebbechi [Reb] showed more generally that a relatively hyperbolic group is biautomatic if its peripheral subgroups are biautomatic. Finitely generated virtually abelian groups are biautomatic by [ECHLPT, §4.2]. It follows from work of Hruska–Kleiner [HK] that any uniform lattice acting on a  $\text{CAT}(0)$  space with isolated flats is biautomatic.

By a theorem of Brink–Howlett, all finitely generated Coxeter groups are automatic [BHo]. Biautomaticity has been considerably harder to establish, and remains unknown for arbitrary Coxeter groups. Biautomatic structures exist when the Coxeter group is affine, that is, virtually abelian, and also when the Coxeter group has no affine parabolic subgroup of rank at least three by a result of Caprace–Mühlherr [CM]. Coxeter groups whose Davis–Moussong complex has isolated flats are also biautomatic by [HK]. The Coxeter groups with isolated flats have been classified by Caprace [Cap].

Let  $W$  be a Coxeter group, and let  $X$  be a building of type  $W$ . Świątkowski [Św2] has shown that any uniform lattice  $\Gamma$  in  $G = \text{Aut}(X)$  is automatic. If  $W$  has a geodesic

biautomatic structure, he shows that  $\Gamma$  is biautomatic as well. Together with Caprace's work mentioned above it follows that if  $W$  is a Coxeter group with isolated flats, then  $\Gamma$  is biautomatic [Cap]. This consequence can be seen in two ways: using the fact that  $W$  is biautomatic, or alternately using the fact, established by Caprace, that  $\Gamma$  is relatively hyperbolic with respect to uniform lattices in Euclidean buildings.

## References

- [AB] P. Abramenko and K. Brown, *Buildings: Theory and Applications*, to appear.
- [BB2] W. Ballmann and M. Brin, *Orbihedra of nonpositive curvature*, Inst. Hautes Études Sci. Publ. Math. **82** (1995), 169–209.
- [BB1] W. Ballmann and M. Brin, *Polygonal complexes and combinatorial group theory*, Geom. Dedicata **50** (1994), 165–191.
- [BE] W. Ballmann and P. Eberlein, *Fundamental groups of manifolds of nonpositive curvature*, J. Differential Geom. **25** (1987), 1–22.
- [BŚ] W. Ballmann and J. Świątkowski, *On  $L^2$ -cohomology and property (T) for automorphism groups of polyhedral cell complexes*, Geom. Funct. Anal. **7** (1997), 615–645.
- [BT] A. Barnhill and A. Thomas, *Density of commensurators for right-angled buildings*, in preparation.
- [Ba] S. Barré, *Immeubles de Tits triangulaires exotiques*, Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), 575–603.
- [BK] H. Bass and R. Kulkarni, *Uniform tree lattices*, J. Amer. Math. Soc **3** (1990), 843–902.
- [BL] H. Bass and A. Lubotzky, *Tree Lattices*. Prog. in Math., 176, Birkhauser.
- [BHar] A. Borel and G. Harder, *Existence of discrete cocompact subgroups of reductive groups over local fields*, J. Reine Angew. Math. **298** (1978), 53–64.
- [BPr] A. Borel and G. Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math., 69, 1989, 119–171.
- [B2] M. Bourdon, *Sur les immeubles fuchsien et leur type de quasi-isométrie*, Ergodic Theory Dynam. Systems **20** (2000), 343–364.
- [B1] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. **7** (1997), 245–268.
- [BP] M. Bourdon and H. Pajot, *Rigidity of quasi-isometries for some hyperbolic buildings*, Comment. Math. Helv. **75** (2000), 701–736.
- [BC] N. Brady and J. Crisp, *CAT(0) and CAT(−1) dimensions of torsion free hyperbolic groups*, Comment. Math. Helv. **82** (2007), 61–85.
- [BH] M.R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.
- [BHo] B. Brink and R.B. Howlett, *A finiteness property and an automatic structure for Coxeter groups*, Math. Ann. **296** (1993), 179–190.
- [Br] K.S. Brown, *Buildings*, Springer-Verlag, New York, 1989.
- [BM] M. Burger and S. Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194.
- [Cap] P.-E. Caprace, *Buildings with isolated subspaces*, arXiv:math.GR/0703799.

- [CM] P.-E. Caprace and B. Mühlherr, *Reflection triangles in Coxeter groups and biautomaticity*, J. Group Theory **8** (2005), 467–489.
- [CG] L. Carbone and H. Garland, *Existence of lattices in Kac–Moody groups over finite fields*, Commun. Contemp. Math. **5** (2003), 813–867.
- [CR] L. Carbone and G. Rosenberg, *Infinite towers of tree lattices*, Math. Res. Lett. **8** (2001), 469–477.
- [CMSZ] D.I. Cartwright, A.M. Mantero, T. Steger, and A. Zappa, *Groups acting simply transitively on the vertices of a building of type  $\tilde{A}_2$* , Geom. Dedicata **47** (1993), 143–166.
- [CW] D.I. Cartwright and W. Woess, *Isotropic random walks in a building of type  $\tilde{A}_d$* , Math. Z. **247** (2004), 101–135.
- [D] M.W. Davis, *Groups generated by reflections and aspherical manifolds*, Ann. of Math. **117** (1983), 293–324.
- [DJ] J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math. **150** (2002), 579–627.
- [E] P. Eberlein, *Geometry of Nonpositively Curved Manifolds*, University of Chicago Press, Chicago, 1996.
- [ECHLPT] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.
- [FE] B. Farb and A. Eskin, *Quasi-flats and rigidity in higher rank symmetric spaces*, J. Amer. Math. Soc. **10** (1997), 653–692.
- [FH] B. Farb and G.C. Hruska, *Commensurability invariants for nonuniform tree lattices*, Israel Journal of Math. **152** (2006), 125–142.
- [Far] D.S. Farley, *Finiteness and CAT(0) properties of diagram groups*, Topology **42** (2003), 1065–1082.
- [GP] D. Gaboriau and F. Paulin, *Sur les immeubles hyperboliques*, Geom. Dedicata **88** (2001), 153–197.
- [Ger] V.N. Gerasimov, *Semi-splittings of groups and actions on cubings*, in Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1997. Eng. Trans. *Fixed-point-free actions on cubings*, Siberian Adv. Math., **8** (1998), 36–58.
- [GS2] S. Gersten and H. Short, *Small cancellation theory and automatic groups II*, Invent. Math. **105** (1991), 641–662.
- [GS1] S. Gersten and H. Short, *Small cancellation theory and automatic groups*, Invent. Math. **102** (1990), 305–334.
- [Gl] Y. Glasner, *A two-dimensional version of the Goldschmidt–Sims conjecture*, J. Algebra **269** (2003), 381–401.
- [Go] D.M. Goldschmidt, *Automorphisms of trivalent graphs*, Ann. of Math. (2) **111** (1980), 377–406.

- [Gr] M. Gromov, *Hyperbolic groups*, in Essays in group theory, Math. Sci. Res. Inst. Publ. **8**, (Springer, New York, 1987) 75-263.
- [Hae] A. Haefliger, *Complexes of groups and orbihedra*, Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky, ed), Proc. ICTP Trieste 1990, World Scientific, Singapore, 1991, 504–540.
- [H5] F. Haglund, *Finite index subgroups of graph products*, submitted.
- [H4] F. Haglund, *Commensurability and separability of quasiconvex subgroups*, *Algebr. Geom. Topol.* **6** (2006), 949–1024.
- [H3] F. Haglund, *Existence, unicité et homogénéité de certains immeubles hyperboliques* *Math. Z.* **242** (2002), 97–148.
- [H2] F. Haglund, *Réseaux de Coxeter-Davis et commensurateurs*, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 649–666.
- [H1] F. Haglund, *Les polyèdres de Gromov*, *C. R. Acad. Sci. Paris, Série I* **313** (1991), 603–606.
- [HP2] F. Haglund and F. Paulin, *Constructions arborescentes d'immeubles*, *Math. Ann.* **325** (2003), 137–164.
- [HP1] F. Haglund and F. Paulin, *Simplicité de groupes d'automorphismes d'espaces à courbure négative*, The Epstein birthday schrift, 181–248 (electronic), *Geom. Topol. Monogr.*, 1, *Geom. Topol. Publ.*, Coventry, 1998.
- [HK] G.C. Hruska and B. Kleiner, *Hadamard spaces with isolated flats*, with an appendix by M. Hindawi, G.C. Hruska, and B. Kleiner, *Geom. Topol.* **9** (2005), 1501–1538.
- [I] N.V. Ivanov, *Subgroups of Teichmüller modular groups and their Frattini subgroups*, *Funktsional. Anal. i Prilozhen.* **21** (1987), 76–77.
- [JLVV] T. Januszkiewicz, I. Leary, R. Valle, and R. Vogeler, *Simple Platonic polygonal complexes*, in preparation.
- [JŚ2] T. Januszkiewicz and J. Świątkowski, *Simplicial nonpositive curvature*, *Publ. Math. Inst. Hautes Études Sci.* **104** (2006), 1–85.
- [JŚ1] T. Januszkiewicz and J. Świątkowski, *Hyperbolic Coxeter groups of large dimension*, *Comment. Math. Helv.* **78** (2003), 555–583.
- [K] I. Kapovich, *The Frattini subgroups of subgroups of hyperbolic groups*, *J. Group Theory* **6** (2003), 115–126.
- [KM] D. Kazhdan and G. Margulis, *A proof of Selberg's hypothesis*, *Mat. Sbornik* **75** (1968), 162–168.
- [KL] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, *Inst. Hautes Études Sci. Publ. Math.* **86** (1997), 115–197.
- [LW] J. Lauer and D.T. Wise, *Cubulating one-relator groups with torsion*, in preparation.
- [Lei] F.T. Leighton, *Finite common coverings of graphs*, *J. Combin. Theory Ser. B* **33** (1982), 231–238.

- [Lev] R. Levitt, *Biautomaticity and triangle-square complexes*, in preparation.
- [Lim] S. Lim, *Counting overlattices in automorphism groups of trees*, *Geom. Dedicata* **118** (2006), 1–21.
- [LT] S. Lim and A. Thomas, *Counting overlattices for polyhedral complexes*, submitted.
- [Liu] Y.-S. Liu, *Density of the commensurability groups of uniform tree lattices*, *J. Algebra* **165** (1994), 346–359.
- [Lu] A. Lubotzky, *Tree-lattices and lattices in Lie groups*, *Combinatorial and geometric group theory* (Edinburgh, 1993), 217–232, *London Math. Soc. Lecture Note Ser.*, 204, Cambridge Univ. Press, Cambridge, 1995.
- [LMR] A. Lubotzky, S. Mozes, and M.S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, *Inst. Hautes Études Sci. Publ. Math.* **91** (2000), 5–53.
- [Ma] G.A. Margulis, *Discrete Subgroups of Semi-Simple Lie Groups*, Springer-Verlag, New York, 1991.
- [Mo] G. Moussong, *Hyperbolic Coxeter groups*, Ph.D. Thesis, Ohio State University, 1988.
- [NRe2] G.A. Niblo and L.D. Reeves, *Coxeter groups act on CAT(0) cube complexes*, *J. Group Theory* **6** (2003), 399–413.
- [NRe1] G.A. Niblo and L.D. Reeves, *The geometry of cube complexes and the complexity of their fundamental groups*, *Topology* **37** (1998), 621–633.
- [NRo] G.A. Niblo and M.A. Roller, *Groups acting on cubes and Kazhdan’s property (T)*, *Proc. Amer. Math. Soc.*, **126** (1998), 693–699.
- [Pa] J. Parkinson, *Isotropic random walks on affine buildings*, *Ann. Inst. Fourier (Grenoble)* **57** (2007), 379–419.
- [Pl] V.P. Platonov, *Frattini subgroup of linear groups and finitary approximability*, *Dokl. Akad. Nauk SSSR* **171** (1966), 798–801.
- [PV] L. Potyagailo and E. Vinberg, *On right-angled reflection groups in hyperbolic spaces*, *Comment. Math. Helv.* **80** (2005), 63–73.
- [Pr] G. Prasad, *Volumes of  $S$ -arithmetic quotients of semi-simple groups*, *Inst. Hautes Études Sci. Publ. Math.* **69** (1989), 91–114.
- [Reb] D.Y. Rebbechi, *Algorithmic properties of relatively hyperbolic groups*, Ph.D. thesis, Rutgers University, 2001. arXiv:math.GR/0302245.
- [Rem2] B. Rémy, *Immeubles de Kac–Moody hyperboliques. Isomorphismes abstraits entre groupes de même immeuble*, *Geom. Dedicata* **90** (2002), 29–44.
- [Rem1] B. Rémy, *Constructions de réseaux en théorie de Kac–Moody*, *C. R. Acad. Sci. Paris* **329** (1999), 475–478.
- [RR] B. Rémy and M. Ronan, *Topological groups of Kac–Moody type, right-angled twinings and their lattices*, *Comment. Math. Helv.* **81** (2006), 191–219.

- [Ron2] M.A. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, 7. Academic Press, Inc., Boston, MA, 1989.
- [Ron1] M.A. Ronan, *A construction of buildings with no rank 3 residues of spherical type*, Buildings and the geometry of diagrams (Como, 1984), 242–248, Lecture Notes in Math., 1181, Springer, Berlin, 1986.
- [Ros] G.E. Rosenberg, *Towers and Covolumes of Tree Lattices*, Ph.D. Thesis, Columbia University, 2001.
- [Sa] M. Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. **71** (1995), 585–617.
- [Se] J.-P. Serre, *Cohomologie des groupes discrets*, Ann. of Math. Studies, no. 70, Princeton Univ. Press, Princeton, N.J., 1971, 77–169.
- [St] J. Stallings, *Non-positively curved triangles of groups*, in Group theory from a geometrical viewpoint (Trieste, 1990), 491–503, World Sci. Publ., 1991.
- [Św2] J. Świątkowski, *Regular path systems and (bi)automatic groups*, Geom. Dedicata **118** (2006), 23–48.
- [Św1] J. Świątkowski, *Trivalent polygonal complexes of nonpositive curvature and Platonic symmetry*, Geom. Dedicata **70** (1998), 87–110.
- [Ta] T. Tamagawa, *On discrete subgroups of  $p$ -adic algebraic groups*, 1965 Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), pp. 11–17, Harper & Row, New York.
- [Th4] A. Thomas, *Lattices acting on Platonic polygonal complexes*, in preparation.
- [Th3] A. Thomas, *On the set of covolumes of lattices for Fuchsian buildings*, C. R. Acad. Sci. Paris Sér. I **344** (2007), 215–218.
- [Th2] A. Thomas, *Lattices acting on right-angled buildings*, Algebr. Geom. Topol. **6** (2006), 1215–1238.
- [Th1] A. Thomas, *Covolumes of uniform lattices acting on polyhedral complexes*, Bull. London Math. Soc. **39** (2007), 103–111.
- [Thu] W. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton Notes.
- [Ti4] J. Tits, *Endliche Spielungsgruppen, die als Weylgruppen auftreten*, Invent. Math **1977**, 283–295.
- [Ti3] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math. 386, Springer-Verlag, New York, 1974.
- [Ti2] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.
- [Ti1] J. Tits, *Sur le groupe d’automorphismes d’un arbre*, in Essays on Topology (Mémoires dédiés à Georges de Rham), Springer-Verlag (1970), 188–211.
- [Vd] A. Vdovina, *Combinatorial structure of some hyperbolic buildings*, Math Z. **241** (2002), 471–478.

- [Vi] E.B. Vinberg, *The non-existence of crystallographic reflection groups in Lobachevskii spaces of large dimension*, Trudy Moskov. Mat. Obshch. **47** (1984), 68–102.
- [W4] D.T. Wise, *Cubulating small cancellation groups*, Geom. Funct. Anal. **14** (2004), 150–214.
- [W3] D.T. Wise, *The residual finiteness of negatively curved polygons of finite groups*, Invent. Math. **149** (2002), 579–617.
- [W2] D.T. Wise, *A non-Hopfian automatic group*, J. Algebra **180** (1996), 845–847.
- [W1] D.T. Wise, *Non-positively curved square complexes, aperiodic tilings, and non-residually finite groups*, Ph.D. Thesis, Princeton University, 1996.
- [X] X. Xie, *Quasi-isometric rigidity of Fuchsian buildings*, Topology **45** (2006), 101–169.
- [Žu] A. Žuk, *La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), 453–458.
- [Zi] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, Boston, 1984.