# A strong Schottky Lemma for nonpositively curved singular spaces

*To John Stallings on his sixty-fifth birthday*

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January 22, 2001

## 1 Introduction

The classical Schottky Lemma (due to Poincare, Klein, Schottky) gives a criterion for a pair of isometries $g, h$ of hyperbolic space to have powers $g^m, h^n$ which generate a free group. This criterion was generalized by Tits to pairs of elements in linear groups in his proof of the Tits alternative.

In this paper we give a criterion (Theorem 1.1) for pairs of isometries of a nonpositively curved metric space (in the sense of Alexandrov) to generate a free group *without having to take powers.* This criterion holds only in singular spaces, for example in Euclidean buildings; in fact our criterion takes a particularly simple form in that case (Corollary 1.2).

The original motivation for our criterion was to prove that the four dimensional Burau representation is faithful. While linearity of braid groups is now known, this question is still open; it is well-known to be related to detecting the unknot with the Jones polynomial . It was shown in [4] that the faithfulness question is equivalent to proving that a specific pair of elements in $\mathrm{GL}_3(\mathbf{Z}[t, t^{-1}])$ generate a free group. In §4 we show that these elements

*Supported in part by the NSF and by the Alfred P. Sloan Foundation.
†Supported by GIF-grant G-454-213.06/95

are the image of the free group $F_2$ under a representation $\rho_{0,0}$ lying in a 2-parameter family of representations

$$\rho_{\alpha,\beta} : F_2 \to \mathrm{GL}_3(\mathbf{Q}(t))$$

for $\alpha, \beta \in \mathbf{Q}$. Our criterion applies to give faithfulness of $\rho_{\alpha,\beta}$ for all $\alpha, \beta$ except, unfortunately, for $\alpha, \beta \in \{0, 1\}$.

**The criterion.** For definitions of the terms we use we refer the reader to §2. A complete, nonpositively curved (i.e. CAT(0)) metric space $X$ is said to have *no fake angles* if there is no pair of geodesics which issue from the same point, have zero angle at that point, yet are disjoint except at that point. This is a very weak condition, and is satisfied for example by any piecewise Euclidean CAT(0) simplicial complex with finitely many isometry types of cells (Proposition 2.1).

**Theorem 1.1 (Strong Schottky).** *Let $X$ be a complete CAT(0)-space with no fake zero angles. Let $g_1, g_2$ be isometries of $X$ of hyperbolic type with axes $A_1, A_2$ respectively. Assume that either one of the following holds:*

1. *$S = A_1 \cap A_2$ is a bounded segment. Let $s_-, s_+$ be its endpoints and let $A_1^{\pm}, A_2^{\pm}$ be the infinite rays of $A_1, A_2$ with the origins $s_-, s_+$. Assume that the angles between $A_1^+, A_2^+$ and between $A_1^-, A_2^-$ are equal to $\pi$, and that the translation lengths of $g_1, g_2$ are strictly greater than the length of $S$.*

2. *$A_1, A_2$ are disjoint, and there is a geodesic $B$ connecting $a_1 \in A_1$ to $a_2 \in A_2$ with all four angles between $B$ and $A_1, A_2$ at $a_1, a_2$ equal to $\pi$.*

*Then the group generated by $g_1, g_2$ is free.*

**The case of buildings.** Let $X$ be a thick affine building. For example, let $X = \mathcal{T}_\nu$ be the building associated to $\mathrm{GL}_n(K), n \geq 2$ where $K$ is a discretely-valued field with valuation $\nu$. The link $link(x)$ of a point $x \in X$ is a spherical building. Two chambers in the link are called *opposite* if their distance is maximal in the link. Any chamber in the link has a unique opposite chamber. We say that two chambers in $A$ are opposite if their intersections with the link of a common intersection point are opposite.

An isometry $f \in \mathrm{Isom}(X)$ of hyperbolic type is said to be *generic* if none of its (parallel) axes are contained in any wall of any apartment of $X$. Note that $f$ is generic if and only if it has a unique invariant apartment $P_f$. The axis of $f$ is contained in the union of two (equivalence classes of) sectors $P_f^+, P_f^-$ of $P_f$ which are invariant by translation by $f$, respectively $f^{-1}$. A generic isometry $f$ determines, for any fixed choice of basepoint $x \in P_f$, a pair of chambers in $link(x)$. We say that generic $f, g \in \mathrm{Isom}(X)$ are *opposite* if $P_f \cap P_g = x \in X$ and if each of the chambers determined by $f$ is opposite in $link(x)$ to each of the chambers determined by $g$.

As a straightforward consequence of Theorem 1.1 we have the following.

**Corollary 1.2 (Strong Schottky for buildings).** *Let $X$ be a (thick) Euclidean building, and let $f, g \in Isom(X)$. If $f, g$ are opposite then they generate a free subgroup of $Isom(X)$.*

In §4 we will apply Corollary 1.2 to a 2-parameter family of pairs of elements in the affine building associated to $\mathrm{GL}_3(\mathbf{Q}(t))$.

# 2 CAT(0) preliminaries

## 2.1 Definitions

Let $X$ be a geodesic metric space. The *comparison triangle* for a geodesic triangle $\Delta$ in $X$ is the Euclidean triangle $\Delta'$ with the same side lengths as $\Delta$; this triangle is unique up to isometry. We say that $X$ is $CAT(0)$, or *nonpositively curved*, if for any geodesic triangle $\Delta$ in $X$ and any two points $x, y$ on $\Delta$, the distance between $x$ and $y$ in $X$ is less than or equal to the Euclidean distance between the corresponding points $x', y'$ on the comparison triangle $\Delta'$ ( [3], II.1.1). The CAT(0) condition implies the uniqueness of geodesics and the geodesicity of local geodesics. In the following we assume that $X$ is a complete CAT(0) space.

Let $\varepsilon > 0$ and let $\sigma_1, \sigma_2 : [0, \varepsilon] \to X$ be two unit speed geodesics with $\sigma_1(0) = \sigma_2(0) =: x$. For $s, t \in (0, \varepsilon)$ let $\gamma(s,t)$ be the angle at $x'$ of the comparison triangle $x'\sigma_1(s)'\sigma_2(t)'$. Then $\gamma(s,t)$ is monotonically decreasing as $s, t$ decrease and hence $\angle(\sigma_1, \sigma_2) := \lim_{s,t\to 0} \gamma(s,t)$ exists and is called the *angle* subtended by $\sigma_1$ and $\sigma_2$ ([1], I.(3.8)). A metric expression for the angle is given by the "cosine theorem"([1], I.(3.11)

$$\cos(\angle(\sigma_1, \sigma_2)) = \lim_{s,t\to 0} \frac{s^2 + t^2 - d^2(\sigma_1(s), \sigma_2(t))}{2st}.$$

It follows from this formula that if the angle is strictly less than $\pi/2$, and $s/t$ is sufficiently small, then $d(\sigma_1(s), \sigma_2(t)) < t$, in particular $\sigma_1(0) = \sigma_2(0)$ does not minimize the distance from $\sigma_2(t)$ to the segment $\sigma_1([0, \varepsilon])$. We need this for the properties of the projection map below.

The sum of the angles of a geodesic triangle is less than or equal to $\pi$ ( [1], I.5.2). It follows immediately from this property that if $I, J$ are geodesic segments issuing from the same point and the subtended angle equals $\pi$ then the concatenation of these segments is also a geodesic segment. If $\sigma$ is a geodesic segment (possibly infinite) then for any $x \in X$ there is ([1], I.5.6) a unique point $p_\sigma(x) \in \sigma$ such that $d(x, p_\sigma(x)) = d(x, \sigma)$. The map $p_\sigma$ is called the *projection* onto $\sigma$. It follows from the cosine formula and the remarks above that for each $x \in X$, the angles of $[x, p_\sigma(x)]$ with $\sigma$ both are greater than or equal to $\pi/2$.

### 2.2 Fake zero angles

We say that a complete CAT(0) space $X$ *has fake zero angles* if there are two geodesics issuing from the same point, are disjoint except at that point, and if the angle subtended by them at that point is zero.

**Proposition 2.1.** *A piecewise Euclidean CAT(0) complex $X$ with finitely many isometry types of cells has no fake zero angles.*

**Proof.** The assumptions imply that the path metric on $X$ is geodesic and complete. The angles can be defined in terms of link distance [2]. Namely, let $X$ be a piecewise Euclidean complex, $x \in X$. The *link* $\mathrm{Lk}_x\, A$ of the

Euclidean cell $A$ is the set of unit tangent vectors $\xi$ at $x$ such that a nontrivial line segment with initial direction $\xi$ is contained in $A$. We define the link $link(x)$ of $x \in X$ by $link(x) = \cup_{A \ni x} \operatorname{Lk}_x A$, where the union is taken over all closed cells containing $x$. Angles in $\operatorname{Lk}_x A$ induce a natural length metric $d_x$ on $link(x)$ which turns it into a piecewise spherical complex. The angle between $\xi, \eta \in link(x)$ is then defined by $\angle_x(\xi, \eta) = \min(d_x(\xi, \eta), \pi)$.

Any two segments $\sigma_1, \sigma_2$ in $X$ with the same endpoint $x$ have the natural projection image in the link of $x$ and $\angle_x(\sigma_1, \sigma_2)$ equals the angle between these two projections. Now the assertion of the lemma is clear since if the segments are disjoint, apart from the origin, then their images in the link are distinct and hence the link distance is nonzero. $\diamond$

**Example (V. Berestovskii):** Take $R^2$ with the positive $x$-axis removed and glue in the region $\{(x, y) : x \geq 0,\ y \leq x^2\}$ along the obvious isometry of the boundary. The result is a CAT(0) space with fake zero angles.

## 3 Proof of Theorem 1.1

We will need the following well-known lemma.

**Lemma 3.1 (Ping-Pong Lemma).** *Let $\Gamma$ be a group of permutations on a set $X$, let $g_1, g_2$ be elements of $\Gamma$ of order at least 3. If $X_1, X_2$ are disjoint subsets of $X$ and if, for all $n \neq 0$, $i \neq j$, $g_i^n X_j \subset X_i$ then $g_1, g_2$ freely generate the free group $F_2$.*

The proof of Theorem 1.1 divides into two cases, depending on which of the two hypotheses is assumed.

**Assuming (1):** Let $s$ be the midpoint of $S$. Let $D_1, D_2$ be the fundamental domains for the action of $g_1, g_2$ on $A_1, A_2$, chosen as open segments on $A_1, A_2$ with the center $s$. Let $p_1, p_2$ be the geodesic projection maps of $X$ onto $A_1, A_2$ respectively. Set $X_1 = p_1^{-1}(A_1 - D_1)$ and $X_2 = p_2^{-1}(A_2 - D_2)$. To apply the Ping-Pong Lemma we need to show that $X_1 \cap X_2$ is empty. Suppose, to the contrary, it is not and let $x \in X_1 \cap X_2$. So $p_1(x) \in A_1, p_2(x) \in A_2$. Suppose that $p_1(x) \in A_1^+, p_2(x) \in A_2^-$, the other cases being similar. Consider the geodesic triangle $xp_1(x)p_2(x)$. The sum of its angles is at most $\pi$ and we would like to get the contradiction with this. Suppose first that the angle at $x$ is nonzero. By the angles assumption we get that $[p_1(x), p_2(x)]$ is the concatenation of $[p_1(x), s_-], [s_-, s_+], [s_+, p_2(x)]$. By the property of the projection map, the angle of the triangle at $p_1(x), p_2(x)$ is at least $\pi/2$. Hence the sum of the angles is strictly greater than $\pi$.

Suppose now that the angle at $x$ is zero. Then by the fake zero angles assumption, the geodesics $(x, p_1(x)], (x, p_2(x)]$ are not disjoint hence $[x, p_1(x)] \cap [x, p_2(x)] = [x, y]$ for some $y \neq x$. To get the contradiction consider the triangle $yp_1(x)p_2(x)$. If $y$ is strictly closer to $x$ than both of $p_1(x), p_2(x)$, then the $y$-angle of the triangle $yp_1(x)p_2(x)$ is nonzero and the sum of angles in this triangle is strictly greater than $\pi$ - a contradiction. If not, then say $y = p_1(x)$. Since $y$ lies on $[x, p_2(x)]$, its projection onto $A_2$ is the same as that of $x$. Hence $p_2(x) = p_2(y)$. By definition, $p_2(y)$ is the point on $A_2$ closest to $y$. But $s_+ \in A_2$ lies on the geodesic $[y, p_2(x)]$ and $s_+ \neq p_2(x)$; again, this is a contradiction.

Finally it remains to check that $g_i^n X_j \subset X_i, i \neq j, n \neq 0$. Note first that $g_i$ commutes with $p_i$. Indeed, for any $x \in X$, the point $p_i(x)$ is the unique point in $A_i$ such that $d(x, p_i(x)) = d(x, A_i)$. But

$$d(g_i x, A_i) = d(g_i x, g_i A_i) = d(x, A_i) = d(x, p_i(x)) = d(g_i x, g_i p_i(x))$$

That is, the point $g_i p_i(x)$ realizes the distance $d(g_i x, A_i)$ and thus it is the projection of $g_i x$ hence $g_i p_i(x) = p_i g_i(x)$.

Clearly $g_i^n D_i \subset A_i - D_i, n \neq 0$, whence $g_i^n(X - X_i) \subset X_i$. Finally, $X_j \subset X - X_i$, hence $g_i^n(X_j) \subset X_i$.

**Assuming (2):** Let $p_1, p_2$ be the geodesic projection maps of $X$ onto $A_1, A_2$ respectively. Let $D_1, D_2$ be the fundamental domains for $g_1, g_2$ chosen as the open segments on $A_1, A_2$ with the centers $a_1, a_2$ respectively. Set $X_1 = p_1^{-1}(A_1 - D_1)$ and $X_2 = p_2^{-1}(A_2 - D_2)$ and repeat the argument of the first case. $\diamond$

# 4 A family of examples

## 4.1 Motivation: the Burau representation

The braid group on $n$ strands, denoted $B_n$, is the group with generators $s_1, s_2, \ldots, s_{n-1}$ and relations

$$s_j s_k = s_k s_j, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for all possible $i, j, k$ with $|k-j| \geq 2$ and $i+1, j, k \leq n-1$. The *Burau representation* $s_i \mapsto \sigma_i$ is a natural representation of $B_n$ on the $n$-dimensional linear space $V = K^n$ over the field $K = \mathbf{Q}(t)$; in the standard basis $\{e_i | i = 1, \ldots, n\}$ the Burau representation is determined by

$$\sigma_i(e_j) = \begin{cases} (1-t)e_i + e_{i+1} & \text{if } j = i \\ te_i & \text{if } j = i+1 \\ e_j & \text{if } j \neq i, i+1 \end{cases}$$

The subspace $Ke$, where e$=\sum_{i=1}^n e_i$, is clearly an invariant subspace for the group action, and the group acts trivially there. Thus the quotient space $V/Ke$ has a natural action of the braid group. In the induced basis $e_1, e_2, \ldots, e_{n-1}$ the elements $\sigma_i$ act via matrices which we denote by $b_1, b_2, \ldots, b_{n-1}$. The only new aspect in this *reduced Burau representation* is that

$$b_{n-1}(e_{n-1}) = (1-t)e_{n-1} + e_n = -(\sum_{i=1}^{n-2} e_i + te_{n-1}).$$

Our interest is in the case of four strands. In this case there is the following well known result (however, the matrices are incorrectly specified in [4]).

**Proposition 4.1 ([4], Theorem 3.19).** *The reduced representation of $B_4$ is faithful iff it is faithful on the free group generated by $a = s_3 s_1^{-1}$ and $b = s_2(s_3 s_1^{-1})s_2^{-1}$.*

Let $f$ (resp. $k$) be the image of $a$ (resp. $b$) in $\mathrm{GL}_3(K)$ under the reduced Burau representation. It is not difficult to see that both $f$ and $k$ are diagonalizable. In fact, by conjugating the Burau representation, and changing $t$

to $-t$, we may take $f$ to be the diagonal matrix with entries $1, t^{-1}, t$, and $k$ becomes the matrix $k = sfs^{-1}$ where

$$s = (1-t)^{-2} \begin{pmatrix} -(1+t) & 1+t^2 & -t(1+t^2) \\ 1 & -t & t \\ 1 & -1 & t^2 \end{pmatrix}$$

We consider the action of $\mathrm{GL}_3(K)$ on the Bruhat-Tits building $\mathcal{T}_\nu$ for the field $K$ with the discrete valuation $\nu = \nu_\infty$ at infinity. We have two elements of $\mathrm{GL}_3(K)$ which are acting so that they each stabilize an entire apartment of $\mathcal{T}$; these apartments $A_u$ and $A_v$ on general principles will meet in a convex subset of the building.

However, by analogy with the case of actions on trees, we might expect that if the intersection of $P_f$ and $P_k$ is sufficiently small with respect to the translation distances of $f$ and $k$ then the group generated by $f$ and $k$ is free. Since $f$ is semisimple, it is easy to see that it acts by translation on its apartment, and we will see that no axis for $f$ is parallel to any wall in the apartment; similarly for $k$. We can determine the intersection $P_f \cap P_k$.

**Lemma 4.2 ($P_f \cap P_k$ is a point).** *The intersection of $P_f$ and $P_k$ consists of precisely one lattice class.*

**Proof.** Let $\nu$ denote the discrete valuation, with valuation ring $\mathcal{O}$ and uniformizer $\pi$. The lattices which represent the lattice classes in $P_f$ are

$$L_{a_1,a_2,a_3} = \mathcal{O}\pi^{a_1}e_1 + \mathcal{O}\pi^{a_2}e_2 + \mathcal{O}\pi^{a_3}e_3.$$

Since $k = sfs^{-1}$, the lattices in $P_k$ are precisely the lattice classes $[sL]$ for $[L]$ in $P_f$, the standard apartment. Thus a lattice class of $L_{-a_1,-a_2,-a_3}$ belongs to the intersection if there are integers $a_1, a_2, a_3$ and $b_1, b_2, b_3$ so that $[sL_{b_1,b_2,b_3}] = [L_{-a_1,-a_2,-a_3}]$. In other words, there is some matrix $m$ in $\mathrm{GL}_3(\mathcal{O})$ so that

$$s \begin{pmatrix} \pi^{b_1} & 0 & 0 \\ 0 & \pi^{b_2} & 0 \\ 0 & 0 & \pi^{b_3} \end{pmatrix} = \begin{pmatrix} \pi^{-a_1} & 0 & 0 \\ 0 & \pi^{-a_2} & 0 \\ 0 & 0 & \pi^{-a_3} \end{pmatrix} m$$

This gives rise to the conditions: $\pi^{a_i} s_{i,j} \pi^{b_j} \in \mathcal{O}$ for all entries $s_{i,j}, 1 \leq i,j \leq 3$ of $s$. This implies

$$a_i + b_j + \nu(s_{i,j}) \geq 0$$

Since $m$ is invertible, $\nu(\det(m)) = 0$. This implies

$$a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + \nu det(s) = 0.$$

We can convert many of the above inequalities into equalities by the following argument. Direct calculation gives $\nu(\det(s)) = 2$. Hence

$$2 = (a_2 + b_1) + (a_1 + b_2) + (a_3 + b_3) \geq 2 + 0 + 0$$

and therefore $a_2 + b_1 = 2$, $a_1 + b_2 = 0$, and $a_3 + b_3 = 0$. It follows similarly that $a_1 + b_1 = 1$, $a_2 + b_2 = 1$, $a_3 + b_1 = 2$, $a_1 + b_3 = -1$, $a_2 + b_3 = 0$, $a_3 + b_2 = 1$. From this we can immediately solve the nine equations in six variables to get the solutions $a_1 = -1 - c, a_2 = -c, a_3 = -c, b_1 = 2 + c, b_2 = 1 + c, b_3 = c$ and so this determines exactly one lattice class solution to the intersection of the two apartments. $\diamond$

## 4.2 A 2-parameter family of representations

Based on the example afforded by the Burau representation, we consider the transformations $f$ and its conjugate $k = sfs^{-1}$, where

$$s = (1-t)^{-2} \begin{pmatrix} -(1+t) & 1+t^2 & -t(1+t^2) \\ 1 & -t & t+\beta t^2 \\ 1 & -1+\alpha t & t^2 \end{pmatrix}$$

for $\alpha$ and $\beta$ any rational numbers. We can think of this as giving a 2-parameter family of representations

$$\rho_{\alpha,\beta} : F_2 \to \mathrm{GL}_3(\mathbf{Q}(t))$$

for $\alpha, \beta \in \mathbf{Q}$. Consider any fixed parameters, giving a pair $f, k$. As in the proof of Lemma 4.2, the invariant apartments of $f$ and $k$ meet at exactly one point.

The lattices which represent the lattice classes in the standard apartment $A$ are $L_{a_1,a_2,a_3} = \mathcal{O}\pi^{a_1}e_1 + \mathcal{O}\pi^{a_2}e_2 + \mathcal{O}\pi^{a_3}e_3$, $\pi$ represents the uniformizer $t^{-1}$. This apartment is stabilized by $f$. In this standard basis, $e_1, e_2, e_3$, $f$ is represented by a diagonal matrix with diagonal entries $1, t^{-1}, t$. The apartment stabilized by $k = sfs^{-1}$ is $sA$. We consider the lattice class $x$ of the lattice $L_{-1,0,0}$ in the standard apartment as the special vertex which is the common cone point of our sectors $P_f^+, P_f^-, P_k^+, P_k^-$. This vertex is also $sL_{-2,-1,0}$ by the calculation above. The walls of the sectors $P_f^+$ and $P_f^-$ are then easily seen to be the subcomplex represented by the lattice classes of $M_n = L_{-1,n,0}$ $n \in Z$, and $N_n = L_{-1,0,n}$ $n \in Z$. The translation $f$ takes $[L_{-1,0,0}]$ to $[L_{-1,1,-1}]$ with axis bisecting the walls of the sector. We can apply $s$ to this configuration to obtain a similar description for the axis and sectors for the element $k$.

We shall calculate the link of the vertex $x$ and represent it in terms of the spherical building of $GL_3(\mathbf{Q})$, where the rational field $\mathbf{Q}$ is the residue field of the field $\mathbf{Q}(t)$ with respect to the valuation at infinity. A $q$-simplex in the building is represented by a chain of lattices $L_0 \subset L_1 \subset \cdots \subset L_q$ with $\pi L_q \subset L_0$ and $L_{r+1}/L_r \cong \mathbf{Q}$, $r \geq 0$. The link of the vertex $[L]$ is the simplicial complex whose vertices are lattice classes $[L^{'}]$ so that $\pi L \subset L^{'} \subset L$. By taking chains of such classes we obtain a simplex in the spherical building of $GL_3(\mathbf{Q})$, viewed as flags in the 3-dimensional $\mathbf{Q}$-vector space, $L/\pi L$.

With the labelling of the walls as above, the first chamber in the sector $P_f^+$ is represented by the lattice classes of $L = L_{-1,0,0}$ and the lattice classes of $M_1 = L_{-1,1,0}$ and $N_{-1} = L_{-1,0,-1}$ while the first chamber in $P_f^-$ is represented by the lattice classes of $L_{-1,0,0}$ and the lattice classes of $M_{-1} = L_{-1,-1,0}$ and $N_1 = L_{-1,0,1}$. We can apply $s$ to these classes to represent the other leading chambers in $P_k^+$ and $P_k^-$.

In order to obtain the precise configurations in $P_f^+$, consider the leading chamber which contains the lattice classes of $M_1$ and $N_{-1}$. We have the chain of lattices $\pi L \subset \pi N_{-1} \subset M_1 \subset L$, so in $link(x)$ we have the flag of subspaces

$$X_1 = \pi N_{-1}/\pi L \subset X_2 = M_1/\pi L,$$

which is the line $X_1$ with basis $e_3$ as a subspace of $X_2$ with basis $\{e_1, e_3\}$. The same can be done for the leading chamber of $P_f^-$ to obtain the flag

$$\pi L \subset \pi M_{-1} \subset N_1 \subset L$$

giving the line $Y_1$ with basis $e_2$ as a subspace of $Y_2$ with basis $\{e_1, e_2\}$. It is immediate that the edge $X_1 \subset X_2$ is opposite to $Y_1 \subset Y_2$.

Similarly, we have the flags $s(X_1) \subset s(X_2)$ and $s(Y_1) \subset s(Y_2)$ in $link(x)$ coming from the sectors $P_k^+$ and $P_k^-$ based at the point $[s(L_{-2,-1,0})] = t$, since $[L_{-1,0,0}] = [s(L_{-2,-1,0})]$. Consider the change of basis matrix $m$ (as described in the proof of Lemma 4.2) and reduce mod $\pi = \frac{1}{t}$ to obtain the rational matrix

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & \beta \\ 1 & \alpha & 1 \end{pmatrix}$$

to find that $s(X_1)$ has basis $(-1, \beta, 1)$ and $s(X_2)$ has basis $\{(-1, 1, 1), (-1, \beta, 1)\}$, $s(Y_1)$ has basis $(1, -1, \alpha)$ and $s(Y_2)$ has basis $\{(1, -1, \alpha), (-1, \beta, 1)\}$. Thus these edges are *opposite* to the edge $X_1 \subset X_2$ for $\alpha$ and $\beta$, rationals which are neither 0 nor 1 ; but *not opposite* if $\alpha$ or $\beta$ is 0 or 1.

To see this we can describe the local hexagons in the link. Oppositeness in this case means that the two 2-dimensional subspaces intersect along a line which is not the special line in each. We show that $X_1 \subset X_2$ is opposite to $s(X_1) \subset s(X_2)$ and $s(Y_1) \subset s(Y_2)$. For example, $X_2$ is spanned by $e_3$ and $e_1$, and $sX_2$ is spanned by $(-1, \beta, 1)$ and $(-1, 1, 1)$. The subspace $X_2 \cap s(X_2)$ does not contain either the line generated by $e_3$ or the line generated by $(-1, \beta, 1)$ iff $\beta \neq 0$. Thus we have oppositeness in this case. Similarly, we can treat the cases of $X_2$, $s(Y_2)$, and $Y_2, s(Y_2)$.

## References

[1] W. Ballmann, *Lectures on spaces of nonpositive curvature, with an appendix by Misha Brin: Ergodicity of geodesic flows.*, DMV Seminar. Bd. 25., Basel: Birkhaeuser Verlag, 1995.

[2] W. Ballmann and M. Brin, Rank rigidity of Euclidean polyhedra, *Amer. J. Math.*, 122(5):873–885, 2000.

[3] M. Bridson and A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Grundlehren der math. Wissen., Vol. 139, Springer-Verlag, 1999.

[4] J. Birman, *Braids, links, and mapping class groups*, Annals of Math. Studies, No. 82, Princeton Univ. Press, 1974.

[5] K. Brown, *Buildings*, Springer-Verlag, 1989.

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