THE RATIONAL COHOMOLOGY OF THE MAPPING CLASS GROUP VANISHES IN ITS VIRTUAL COHOMOLOGICAL DIMENSION

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Abstract. Let Mod_g be the mapping class group of a genus $g \geq 2$ surface. The group Mod_g has virtual cohomological dimension $4g-5$. In this note we use a theorem of Broaddus and the combinatorics of chord diagrams to prove that $H^{4g-5}(\mathrm{Mod}_g;\mathbb{Q})=0$.

1. Introduction

Let Mod_g be the mapping class group of a closed, oriented, genus $g \geq 2$ surface, and let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces. It is well-known that for each $i \geq 0$,

$$H^i(\mathrm{Mod}_g;\mathbb{Q}) \cong H^i(\mathcal{M}_g;\mathbb{Q}).$$

It is a fundamental open problem to determine the maximal i for which these vector spaces are nonzero. Harer [Ha] proved that the *virtual cohomological dimension* $\mathrm{vcd}(\mathrm{Mod}_g)$ equals $4g-5$. More precisely, he proved that $H^{4g-5}(\mathrm{Mod}_g;\mathrm{St}_g \otimes \mathbb{Q}) \neq 0$ for a certain Mod_g-module St_g (see below for details) and that $H^i(\mathrm{Mod}_g; V \otimes \mathbb{Q}) = 0$ for all $i > 4g-5$ and all Mod_g-modules V. Thus the first step of the problem above is to determine whether $H^{4g-5}(\mathrm{Mod}_g;\mathbb{Q}) \neq 0$. The purpose of this note is to answer this question.

Let $\mathrm{Mod}_{g,*}$ (resp. $\mathrm{Mod}_{g,1}$) denote the mapping class group of the genus g surface with one marked point (resp. one boundary component).

Theorem 1. *For any $g \geq 2$,*

$$H^{4g-5}(\mathrm{Mod}_g;\mathbb{Q}) = H^{4g-5}(\mathcal{M}_g;\mathbb{Q}) = 0.$$

Further, the rational cohomology of $\mathrm{Mod}_{g,*}$ (resp. the integral cohomology of $\mathrm{Mod}_{g,1}$) vanishes in its virtual cohomological dimension.

This theorem was announced some years ago by Harer, but he has informed us that his proof will not appear. We recently learned that Morita–Sakasai–Suzuki [MSS] have independently found a proof of Theorem 1 using a completely different method. They apply a theorem of Kontsevich on graph homology to their computation of a generating set for a certain symplectic Lie algebra. Our proof combines some results about the combinatorics of chord diagrams with the work of Broaddus [Br] on the Steinberg module of Mod_g. We thank Allen Hatcher and Takuya Sakasai for their comments on an earlier version of this paper, and John Harer for informing us about the paper [MSS] and his own work.

Theorem 1 is consistent with the well-studied analogy between mapping class groups and arithmetic groups. For example, Theorem 1.3 of Lee–Szczarba [LS] states that the rational cohomology of $\mathrm{SL}(n,\mathbb{Z})$ vanishes in its cohomological dimension.

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2. Background

We begin by briefly summarizing previous results that make our computation possible; for details see Broaddus [Br].

Teichmüller space and its boundary. Let S_g be a connected, closed orientable surface of genus $g \geq 2$. Let $\mathcal{C}_g$ be the *curve complex* of S_g defined by Harvey [Harv], i.e. the flag complex whose k-simplices are the $(k+1)$-tuples of distinct free homotopy classes of simple closed curves in S_g that can be realized disjointly. Harer [Ha] proved that $\mathcal{C}_g$ is homotopy equivalent to a wedge of spheres $\bigvee_{i=1}^{\infty} S^{2g-2}$.

There exists a constant $\delta > 0$ such that any two closed geodesics on a hyperbolic surface of length $\leq \delta$ are disjoint (the *Margulis constant* for hyperbolic surfaces). Let $\mathcal{T}_g^{\text{thick}}$ be the Teichmüller space of marked hyperbolic surfaces diffeomorphic to S_g having no closed geodesic of length $< \delta$. It is known that $\mathcal{T}_g^{\text{thick}}$ is a $(6g-6)$-dimensional manifold with corners. Ivanov [Iv] proved that $\mathcal{T}_g^{\text{thick}}$ is contractible and that its boundary $\partial\mathcal{T}_g^{\text{thick}}$ is homotopy equivalent to $\mathcal{C}_g$. Briefly, for each simplex σ of $\mathcal{C}_g$, let $\mathcal{T}_\sigma$ be the subset of $\partial\mathcal{T}_g^{\text{thick}}$ consisting of surfaces where each curve in σ has length δ. Each T_σ is contractible, and $\mathcal{T}_\sigma \cap \mathcal{T}_{\sigma'} = \emptyset$ unless $\sigma \cup \sigma'$ is a simplex of $\mathcal{C}_g$, in which case $T_\sigma \cap \mathcal{T}_{\sigma'} = \mathcal{T}_{\sigma\cup\sigma'}$.

Duality in the mapping class group. The mapping class group Mod_g acts properly discontinuously on $\mathcal{T}_g^{\text{thick}}$ with finite stabilizers. Defining $\mathcal{M}_g^{\text{thick}} = \mathcal{T}_g^{\text{thick}}/\text{Mod}_g$, it follows that $H^*(\text{Mod}_g;\mathbb{Q}) \cong H^*(\mathcal{M}_g^{\text{thick}};\mathbb{Q})$. Mumford's compactness criterion states that $\mathcal{M}_g^{\text{thick}}$ is compact. Combining this with the previous two paragraphs, the work of Bieri–Eckmann [BE, Theorem 6.2] shows that $\text{vcd}(\text{Mod}_g) = 4g-5$ and that

$$H^{4g-5}(\text{Mod}_g;\mathbb{Q}) \cong H_0(\text{Mod}_g; H_{2g-2}(\mathcal{C}_g;\mathbb{Q})). \tag{1}$$

In fact, we can say more. Let St_g denote the *Steinberg module*, i.e. the Mod_g-module $H_{2g-2}(\mathcal{C}_g;\mathbb{Z})$. Then $\text{St}_g \otimes \mathbb{Q}$ is the rational *dualizing module* for Mod_g, meaning that

$$H^{4g-5-k}(\text{Mod}_g; M \otimes \mathbb{Q}) \cong H_k(\text{Mod}_g; M \otimes \text{St}_g \otimes \mathbb{Q})$$

for any k and any M. Moreover St_g is also the dualizing module for $\text{Mod}_{g,*}$ and $\text{Mod}_{g,1}$, which act on St_g via the natural surjections $\text{Mod}_{g,*} \to \text{Mod}_g$ and $\text{Mod}_{g,1} \to \text{Mod}_g$ [Ha]. This implies that for $\nu = \text{vcd}(\text{Mod}_{g,*}) = 4g-3$ we have $H^{\nu-k}(\text{Mod}_{g,*}; M \otimes \mathbb{Q}) \cong H_k(\text{Mod}_{g,*}; M \otimes \text{St}_g \otimes \mathbb{Q})$. For $\text{Mod}_{g,1}$ we obtain a similar result with $\nu = \text{cd}(\text{Mod}_{g,1}) = 4g-2$, except that since $\text{Mod}_{g,1}$ is torsion-free the result holds integrally: $H^{\nu-k}(\text{Mod}_{g,1}; M) \cong H_k(\text{Mod}_{g,1}; M \otimes \text{St}_g)$.

An alternate model for St_g. Fix a finite-volume hyperbolic metric on $S_g - \{*\}$. Another model for St_g comes from the *arc complex* $\mathcal{A}_g$, the flag complex whose k-simplices are the disjoint $(k+1)$-tuples of simple geodesics on $S_g - \{*\}$ beginning and ending at the cusp $*$. Let $\mathcal{A}_g^\infty$ be the subcomplex consisting of collections of geodesics $\gamma_1, \ldots, \gamma_{k+1}$ for which $S - \bigcup \gamma_i$ has some noncontractible component. Harer proved that $\mathcal{A}_g^\infty$ is homotopy equivalent to $\mathcal{C}_g$ [Ha], and that $\mathcal{A}_g$ is contractible [Ha2] (see also [Hat]). Thus

$$\text{St}_g = H_{2g-2}(\mathcal{C}_g) \simeq H_{2g-2}(\mathcal{A}_g^\infty) \simeq H_{2g-1}(\mathcal{A}_g/\mathcal{A}_g^\infty).$$

Chord diagrams. By examining how the geodesics are arranged in a neighborhood of $*$, an $(n-1)$-simplex of $\mathcal{A}_g$ can be encoded by a n-chord diagram; see [Br, §4.1]. An *ordered n-chord diagram* is an ordered sequence $U = (u_1, \ldots, u_n)$, where u_i is an unordered pair of distinct points on S^1 (a *chord*) and $u_i \cap u_j = \emptyset$ if $i \neq j$. We will visually depict U by drawing arcs connecting the points in each u_i (see Figure 1 for examples). Two ordered chord diagrams are identified if they differ by an orientation-preserving homeomorphism of the circle.

Filling systems. An unlabeled *k-filling system* of genus g is a $(2g+k)$-chord diagram satisfying the conditions described in [Br, §4.1]: no chord should be parallel to another chord or to the boundary circle, and the chords should determine exactly $k+1$ boundary cycles. These conditions, which guarantee that these chords define a simplex of $\mathcal{A}_g - \mathcal{A}_g^\infty$, have the following simple combinatorial formulation. Given $U = (u_1, \ldots, u_n)$, consider two permutations of the $2n$ points $u_1 \cup \cdots \cup u_n$: let ω be the $2n$-cycle which takes each point to the point immediately adjacent in the clockwise direction, while τ exchanges the two points of each chord u_i and thus is a product of n transpositions. Then a $(2g+k)$-chord diagram is a k-filling system of genus g if $\tau \circ \omega$ has $k+1$ orbits, none of which have length 1 or 2. Finally, let t_i be the straight line in D^2 connecting the two points of u_i. Then we say that U is *disconnected* if the set $t_1 \cup \cdots \cup t_n \subset D^2$ is not connected.

The chord diagram chain complex. Fix a genus g, and set $n = 2g+k$. Let $\mathcal{U}_k$ be the free abelian group spanned by ordered k-filling systems of genus g modulo the following relation. For $\sigma \in S_n$ and $U = (u_1, \ldots, u_n)$, define $\sigma \cdot U = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. We impose the relation $\sigma \cdot U = (-1)^\sigma U$. The differential $\partial : \mathcal{U}_k \to \mathcal{U}_{k-1}$ is defined as follows. Consider an ordered k-filling system $U = (u_1, \ldots, u_n)$ of genus g. For $1 \leq i \leq n$, let $\partial_i U$ equal $(u_1, \ldots, \widehat{u_i}, \ldots, u_n)$ if this is an ordered $(k-1)$-filling system of genus g; otherwise, let $\partial_i U = 0$. Then

$$\partial(U) = \sum_{i=1}^{n} (-1)^{i-1} \partial_i U.$$

Broaddus's results. We will need the following theorem of Broaddus [Br]. Recall that if Γ is a group and M is a Γ-module, then the *module of coinvariants*, denoted M_Γ, is the quotient $M / \langle g \cdot m - m \,|\, g \in \Gamma, m \in M \rangle$. Let X be the 0-filling system of genus g depicted in Figure 1a.

Theorem 2 (Broaddus [Br])**.** *For each $g \geq 0$, the following hold.*

(i) $(\mathrm{St}_g)_{\mathrm{Mod}_g} \cong \mathcal{U}_0 / \partial(\mathcal{U}_1)$.

(ii) The abelian group $\mathcal{U}_0/\partial(\mathcal{U}_1)$ is spanned by the image $[X] \in \mathcal{U}_0/\partial(\mathcal{U}_1)$ of $X \in \mathcal{U}_0$.

(iii) If v is a disconnected 0-filling system of genus g, then the image of v in $\mathcal{U}_0/\partial(\mathcal{U}_1)$ is 0.

For part (i) of Theorem 2, see [Br, Proposition 3.3] together with the remark preceding [Br, Example 4.1]; for part (ii), see [Br, Theorem 4.2]; and for part (iii), see [Br, Proposition 4.5].

3. Proof of Theorem 1

For any group Γ and any Γ-module M, recall that $H_0(\Gamma; M) = M_\Gamma$. Since the actions of $\mathrm{Mod}_{g,*}$ and $\mathrm{Mod}_{g,1}$ on St_g factor through Mod_g, to prove Theorem 1 it suffices by (1) to show that $(\mathrm{St}_g)_{\mathrm{Mod}_g} = 0$. By Theorem 2(i), this is equivalent to showing that $\mathcal{U}_0/\partial(\mathcal{U}_1) = 0$.

For $v \in \mathcal{U}_0$, let $[v]$ denote the associated element of $\mathcal{U}_0/\partial(\mathcal{U}_1)$. Let $X = (x_1, \ldots, x_{2g})$ be the 0-filling system depicted in Figure 1(a). By Theorem 2(ii), it is enough to show that $[X] = 0$. Let $Y = (x_1, \ldots, x_{2g}, y)$ be the 1-filling system depicted in Figure 1(b). Observe that

$$\partial_1 Y = (x_2, \ldots, x_{2g}, y) = (x_1, \ldots, x_{2g}) = X,$$

where the second equality holds since the indicated chord diagrams differ by an orientation preserving homeomorphism of S^1. Similarly, $\partial_{2g+1} Y = X$. Also, $\partial_2 Y = 0$ (resp. $\partial_{2g} Y = 0$) by definition,

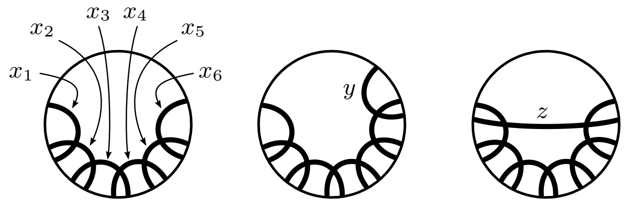

Figure 1. *(a) The oriented* 0*-filling system* $X = (x_1, \ldots, x_{2g})$*. For concreteness, we depict it for* $g = 3$*. In general,* X *has* $2g$ *chords arranged in the same pattern as the chords shown.*
(b) The 1*-filling system* $Y = (x_1, \ldots, x_{2g}, y)$*. The chord* y *intersects the chord* x_{2g}*.*
(c) The 1*-filling system* $Z = (z, x_1, \ldots, x_{2g})$*. The chord* z *intersects both* x_1 *and* x_{2g}*.*

since the chord x_1 (resp. x_{2g+1}) becomes parallel to the boundary. We thus have

$$\partial(Y) = 2X + \sum_{i=3}^{2g-1} (-1)^{i-1} \partial_i Y.$$

For $3 \le i \le 2g-1$, the 0-filling system $\partial_i Y$ is disconnected, so Theorem 2(iii) implies that $[\partial_i Y] = 0$. We conclude that $2[X] = 0$.

Now consider the 1-filling system $Z = (z, x_1, \ldots, x_{2g})$ depicted in Figure 1(c). Removing any chord from Figure 1(c) yields Figure 1(a) up to rotation, so $\partial_i Z = \pm X$ for each i. In fact, it is clear that $\partial_1 Z = X$, that $\partial_2 Z = -X$, that $\partial_3 Z = X$, and so on, with $\partial_i Z = (-1)^{i-1} X$. This shows that

$$\partial(Z) = X + X + \cdots X = (2g+1)X,$$

so $(2g+1)[X] = 0$.

Summing up, we have shown that $2[X] = (2g+1)[X] = 0$. This implies that $[X] = 0$, as desired.

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