Real-analytic actions of lattices

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1 Introduction

This paper introduces a new approach to the study of lattice actions on compact manifolds.

The group $\mathrm{SL}(n, \mathbf{Z})$ acts by real-analytic diffeomorphisms on the sphere S^{n-1} via the action on the space of infinite rays based at $0 \in \mathbf{R}^n$. We believe this action is minimal in the following sense.

Conjecture I: *Any smooth action of a finite-index subgroup of $SL(n, \mathbf{Z})$, $n > 2$, on a compact r-manifold factors through a finite group action if $r < n - 1$.*

This is an analogue of a special case of one of the central conjectures in the Zimmer program (see [Zi3, Zi5]), where the minimal action is the standard action of $\mathrm{SL}(n, \mathbf{Z})$ on the n-torus:

Conjecture II (Zimmer): *Any smooth, volume-preserving action of a finite-index subgroup of $SL(n, \mathbf{Z})$, $n > 2$, on a compact r-manifold factors through a finite group action if $r < n$.*

Actually the above versions of Conjectures I and II are special cases of general conjectures. In the general versions of Conjectures I and II, the lattice $\mathrm{SL}(n, \mathbf{Z})$ is replaced by an an arbitrary irreducible lattice Γ in a semisimple[1] Lie group G of $\mathbf{R}$-rank at least 2, and the integer n is replaced

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[1]By a *semisimple* group G we mean connected semisimple with finite center and no nontrivial compact factors. So we can (up to finite center) assume G is an algebraic group defined over $\mathbf{Q}$.

by a suitable integer $h(G)$ which depends only on G and which is easily computed. More precisely $h(G) = \min(\alpha(G), \beta(G))$ where :

- $\alpha(G)$ is the smallest integer α for which the Lie algebra of G admits a nontrivial representation into $\mathfrak{sl}(\alpha, \mathbf{R})$, and
- $\beta(G)$ is the least integer β such that

$$\beta(\beta+1)/2 \geq \min\{\dim_{\mathbf{C}} G' | G' \text{ is a simple factor of } G\}.$$

The conditions $r < h(G) - 1$ (for Conjecture I) and $r < h(G)$ (for Conjecture II) serve to rule out the known algebraic actions (see [Zi3, Zi5]).

These conjectures may be viewed as part of a program to generalize the Margulis Superrigidity Theorem to a nonlinear context (another aspect of this program may be found in [Hu, KL]). The conjectures have proven to be difficult, even in low dimensions.

Weinberger [We] has shown how to obtain interesting special cases of Conjectures I and II for lattices Γ which contain sufficiently large finite subgroups F. The idea is to use the theory of compact transformation groups to show that F cannot act effectively on M, and then to use the Margulis finiteness theorem (see Lemma 3.1 below) to show that any Γ-action on M must be finite.

In this paper we describe an approach to the general versions of Conjectures I and II for the case of real-analytic actions. Our results apply even when Γ is torsion-free, and go some way toward settling the conjectures in low dimensions.

The techniques used in this paper differ from most other work on Conjecture II, where actions preserving volume as well as extra geometric structures such as a connection or pseudo-Riemannian metric are studied (see [Fe1, Fe2, Zi2, Zi3, Zi4, Zi5]). The latter work uses ergodic-theoretic methods, while our methods are topological. The only other papers based on topological methods seem to be [Gh, We, Wi, FS1]. The work of Weinberger [We] and Witte [Wi] was important motivation for the present work, and ideas from [Wi] play a role in several of our arguments. Although our work was done without knowledge of [Gh], some of our methods are related to Ghys's.

The arguments of [Gh] and [Wi] exploit the existence of unipotent subgroups to obtain results for nonuniform (i.e. non-cocompact) lattices. One virtue of our method is that it also applies to a wide class of uniform lattices, where nontrivial unipotent subgroups do not exist.

Zimmer [Zi5] showed, based on work of Thurston [Th], that the general version of Conjecture I is true if Γ has a global fixed point. The methods of this paper give information in the presence of a fixed point for a single noncentral element of Γ. In many situations topological methods can be used to show that such a fixed point exists.

The starting point for our arguments is the observation that real-analytic diffeomorphisms which commute with a fixed real-analytic diffeomorphism $f : M \to M$ must automatically leave invariant the lowest-dimensional "stratum" of the real-analytic variety $F = \mathrm{Fix}(f)$, and this stratum should be a closed, real-analytic manifold. This allows for general inductive procedures using centralizers (see §4 and §7). To establish finiteness of actions by this method, one must show that F is nonempty; we will do this in low-dimensional cases.

Our results will hold for a wide class of uniform and nonuniform irreducible lattices Γ in semisimple groups G of $\mathbf{R}$-rank at least 2, described as follows. We say that Γ is 1-*big* if either:

- $\mathbf{Q}$-rank$(\Gamma) \geq 2$, or
- The centralizer of some infinite-order element of Γ has a subgroup isomorphic to an irreducible lattice in some semisimple Lie group of $\mathbf{R}$-rank at least 2.

For $k \geq 1$, we say that Γ is $(k+1)$-*big* if the centralizer of some infinite-order element of Γ has a subgroup isomorphic to a k-big lattice.

Examples (see §2 below):

1. Any finite-index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ is k-big for $n \geq 2k+1$.
2. More generally let Γ be a lattice in any simple Lie group whose root system is not of type D_4. If $\mathbf{Q}$-rank$(\Gamma) \geq 2r$ then Γ is r-big. See Proposition 2.1.
3. Let K be a finite real extension of $\mathbf{Q}$, and let Φ be a nondegenerate quadratic form of type (p, q) with coefficients in K. Let Γ be the group of K-integral unimodular matrices preserving Φ. In many cases Γ is isomorphic to an r-big lattice; for example this always holds whenever Φ is diagonal and $(p, q) \geq (3r+3, 2r+2)$. Furthermore the lattice is uniform if Φ does not represent 0 over K. See §2 for more examples.

Our first result concerns actions on S^1. The general version of Conjecture I for S^1 says that any smooth action $\Gamma \to \mathrm{Diff}(S^1)$ of an irreducible, higher-rank lattice Γ must factor through a finite group. This particular conjecture has received a good deal of attention. Witte [Wi] proved it for topological actions of lattices of $\mathbf{Q}$-rank at least 2. For convenience we incorporate Witte's result in the statement of the following theorem, which is new for the case of uniform lattices.

Theorem 1.1 (Actions on S^1). *Let Γ be a 1-big lattice. Then any real-analytic action of Γ on S^1 factors through a finite group action.*

For general real-analytic actions on surfaces we have the following result.

Theorem 1.2 (Actions on surfaces, I). *Let Γ be a 2-big lattice. Then any real-analytic action of Γ on a compact surface other than the torus or Klein bottle must factor through a finite group action.*

The nonuniform case of Theorem 1.2 overlaps with [Gh]. In particular Theorem 4.5 of [Gh] states that every nilpotent group of real-analytic diffeomorphisms of the 2-sphere is metabelian. This implies, for example, that for $k \geq 4$, no finite-index subgroup of $\mathrm{SL}(k, \mathbf{Z})$ admits an infinite real-analytic actions on S^2 (note that Theorem 1.2 implies this only for $k \geq 5$). In [Gh], Ghys also discusses the possibility of extending his theorem to compact surfaces of negative Euler characteristic.

Our proof of Theorem 1.2 fails for surfaces of Euler characteristic 0 since we need some element which acts with a fixed point. For area-preserving actions we can find such an element in Γ by using a theorem of Franks (see §6), so can prove the theorem for all surfaces.

Theorem 1.3 (Actions on surfaces, II). *Let Γ be a 2-big lattice. Then any real-analytic, area-preserving action of Γ on a compact surface must factor through a finite group action.*

The following theorem implies that Conjecture I for real-analytic actions is true in dimension 3 for many manifolds-with-boundary.

Theorem 1.4 (Actions on 3-manifolds with boundary). *Let Γ be a 2-big lattice, and let M be a compact 3-manifold whose boundary contains a surface of nonzero Euler characteristic. Then any real-analytic action $\Gamma \to \mathit{Diff}(M)$ factors through a finite group.*

In §8 we prove a general result, Theorem 8.1, for lattice actions on closed 3-manifolds. Here we mention a corollary.

Corollary 1.5 (Actions on homology 3-spheres). *Let Γ be any finite-index subgroup of $SL(n, \mathbf{Z}), n \geq 7$ which contains a nonidentity diagonal* *matrix γ with at least five 1's on the diagonal. Then any real-analytic, volume-preserving action of Γ on a $\mathbf{Z}/2\mathbf{Z}$-homology 3-sphere factors through a finite action.*

Note that in the above corollary, while Γ is assumed to have a finite subgroup, the result still applies in cases where all finite subgroups of Γ themselves admit effective actions on M. We also remark that many 3-manifolds are $\mathbf{Z}/2\mathbf{Z}$-homology 3-spheres.

As mentioned above, in order to apply our technique in full generality one needs to rule out actions of the lattice Γ which are *almost free*, by which we mean no noncentral element of Γ has a fixed point. This may be viewed as a natural analogue of the classical problem of determining when a finite group can act freely.

Conjecture III (Freeness Conjecture): *Let Γ be an irreducible, nonuniform lattice in a semisimple Lie group G. Then Γ does not admit a smooth, almost free action on any compact r-manifold if r is less than the dimension of every simple factor of G.*

Note that the Freeness Conjecture posits a much stronger conclusion than Conjecture I, since *dimension* of the simple factors of G replaces *rank* as the lower bound for the dimension in which nontrivial actions may occur. There is a corresponding conjecture for uniform lattices which is more technical to state.

If the Freeness Conjecture for nonuniform lattices holds, then Conjecture I holds within a reasonable range of dimensions.

Theorem 1.6 (Implication of Freeness Conjecture). *If the Freeness Conjecture is true, then any real-analytic action of any finite-index subgroup of $SL(n, \mathbf{Z}), n \geq 3$ on a closed r-manifold is finite for $r \leq (\frac{3n+7}{2})^{2/3} - 3$.*

There is also a version (Theorem 9.4 below) for a much more general class of lattices.

The existence of free actions of higher-rank lattices on compact manifolds seems surprising. However there are examples to show that the dimension assumption in Conjecture III is best possible. We discuss such examples in §9.

Finally, we would like to mention that several of the ideas in this paper apply to groups Γ which are not lattices in Lie groups. Examples include mapping class groups of surfaces and Thompson's finitely-presented, torsion-free, infinite simple group. These applications will be given in [FS2].

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2 Examples of r-big lattices

In this section we give some concrete examples of uniform and nonuniform lattices which are r-big. The reader who is content with understanding our results for finite-index subgroups of $\mathrm{SL}(n, Z)$ for appropriate n may skip this section without loss.

2.1 Nonuniform lattices

Any finite-index subgroup Γ of $\mathrm{SL}(5, \mathbf{Z})$ contains a product $A \times B$, where B is a finite-index subgroup of $\mathrm{SL}(3, \mathbf{Z})$ (sitting inside $\mathrm{SL}(5, \mathbf{Z})$ in block form) and A is a finite-index subgroup of $\mathrm{SL}(2, \mathbf{Z})$ (sitting in $\mathrm{SL}(5, \mathbf{Z})$ as a complementary block). Choosing any infinite order $a \in A$, we see that B lies in the centralizer $C_\Gamma(a)$ of a in Γ. Hence Γ is 1-big.

More generally, any finite-index subgroup of $\mathrm{SL}(n, \mathbf{Z}), n \geq 5$ contains a product of finite-index subgroups of $\mathrm{SL}(n-2, \mathbf{Z})$ and $\mathrm{SL}(2, \mathbf{Z})$. So by induction we have:

For all $n > 2r$, any finite-index subgroup of $SL(n, \mathbf{Z})$ is r-big.

Even more generally we have the following result.

Proposition 2.1. *Let Γ be a lattice in any simple Lie group G whose root system is not of type D_4. If $\mathbf{Q}$-rank$(\Gamma) \geq 2r$ then Γ is r-big.*

Not that the condition that the root system is not of type D_4 automatically holds when $r > 3$.

Proof. For $r = 1$ the proposition follows from the definition of 1-big, so we assume $r > 1$.

We first consider the special case where Γ is the group of $\mathbf{Z}$-points of an algebraic group G defined over $\mathbf{Q}$. Since G is simple the $\mathbf{Q}$-root system Φ of G is irreducible.

By going through the list of Dynkin diagrams (see, e.g., §11.4 of [Hum]) with at least $2r$ vertices, $r > 1$, one sees that for every such diagram except for that of type D_4, one may "erase a vertex v" of the diagram to obtain a a graph with 2 components: one with a single vertex and another which is a Dynkin diagram with at least $2r-2$ vertices. Let G_1 and G_2 be the root subgroups corresponding to these two components of the Dynkin diagram. Then G_1 is noncompact, G_2 has $\mathbf{Q}$-rank at least $2r-2$, and G_1 commutes with G_2.

Now $\Gamma_i = \Gamma \cap G_i$ is an arithmetic lattice in G_i for $i = 1, 2$, since by a theorem of Borel-Harish-Chandra [BH] the $\mathbf{Z}$-points of an algebraic group defined over $\mathbf{Q}$ form a lattice in the $\mathbf{R}$-points. So Γ contains an element $\gamma \in G_1$ which is centralized by a lattice $\Lambda \subset G_2$ of $\mathbf{Q}$-rank at least $2r-2$. The special case of the proposition now follows by induction on r.

In the general case, it follows from the Arithmeticity Theorem of Margulis ([Zi1], Theorem 6.1.2) that Γ is commensurate with the group of $\mathbf{Z}$-points of a semisimple algebraic group H defined over $\mathbf{Q}$. Furthermore, H is isomorphic to the product of G with a compact group. An argument similar to the one above applied to H then gives the proposition. $\diamond$

2.2 Uniform lattices

There are also many uniform lattices which are r-big. We begin with a very concrete example.

Proposition 2.2. *Let $K = \mathbf{Q}(\sqrt{2})$, and let $\Gamma_{p,q}$ be the group of K-integral unimodular matrices preserving the form*

$$\Phi = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} \sqrt{2} x_i^2$$

Then for $p+q > 4$, the group $\Gamma_{p,q}$ is an irreducible, uniform lattice in a semisimple Lie group containing $SO(p,q)$ as a factor. Furthermore, $\Gamma_{p,q}$ is r-big if $(p,q) \geq (r+2, r+3)$.

Proof.

Since $p+q > 4$, it follows from the theory of arithmetic groups (see for example [Ma] p.296), that Γ is an irreducible lattice in a semisimple Lie group containing $\mathrm{SO}(p,q)$ as a factor.

Now $\Gamma_{p,q}$ is uniform if and only if Φ does not represent 0 over K (see , e.g., [Ma], p.296). Let $\overline{x}$ denote the Galois conjugate of x, and let

$$\overline{\Phi} = \sum_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{p+q} \sqrt{2} x_i^2$$

Then for any $x_1, \dots, x_{p+q} \in K$ such that $\Phi(x_1, \dots, x_{p+q}) = 0$, we have $\overline{\Phi}(\overline{x_1}, \dots, \overline{x_n}) = 0$, so that each $x_i = 0$. Hence Φ does not represent 0 over K, so $\Gamma_{p,q}$ is uniform.

It remains to be shown that $\Gamma_{p,q}$ is r-big when $p \geq r+2$ and $q \geq r+3$. We prove this by induction on r as follows. We can write Φ as a direct sum $\Phi = \Phi_1 \oplus \Phi_2$ of forms of type $(1,1)$ and $(p-1, q-1)$, respectively. The subgroup of Γ consisting of those elements which leave invariant the two subspaces corresponding to Φ_1 and Φ_2, and which preserve each form separately, is isomorphic to the product $\Gamma_{1,1} \times \Gamma_{p-1,q-1}$.

Notice that $\Gamma_{1,1}$ contains the infinite-order element

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

Clearly $\Gamma_{p-1,q-1}$ centralizes γ. Since γ has infinite order, this proves both the base case $r = 1$ (since $\Gamma_{p-1,q-1}$ is an irreducible lattice in higher-rank semisimple Lie group) and the induction step. $\diamond$

Of course the above example can be greatly generalized. For example, for a nondegenerate quadratic form

$$\Phi = \sum_{i=1}^{p} a_i x_i^2 - \sum_{i=p+1}^{p+q} b_j x_i^2$$

with $a_i, b_j > 0$ elements of a finite extension $K \subset \mathbf{R}$ of $\mathbf{Q}$, one simply needs a condition to allow one to "pick off" a form Φ_1 of type $(1,1)$ to write $\Phi = \Phi_1 \oplus \Phi_2$, and to have an infinite-order, unimodular, integral matrix preserving Φ_1. This will happen when there is an infinite-order unit in the quadratic extension $K[\alpha]$, where α is a root of $a_i x_i^2 - b_j x_j^2 = 0$.

If one is willing to increase p and q slightly, it is possible to drop all number-theoretic conditions on the coefficients of Φ. The point is that, since $3 + 2 > 4$, the group of unimodular, K-integral matrices preserving a nondegenrate quadratic form of type $(3,2)$ is actually a lattice in a higher-rank semisimple Lie group; in particular it has an infinite-order element. One may then apply the same argument as in the proof of Proposition 2.2, picking off a form of type $(3,2)$ at each step of the induction, to obtain the following result.

Proposition 2.3. *Let $K \subset \mathbf{R}$ be a finite extension of $\mathbf{Q}$. Let Φ be a non-degenerate, diagonal quadratic form of type (p,q). Let Γ be the group of unimodular, K-integral matrices preserving Φ. Then Γ is an r-big lattice whenever $(p,q) \geq (3r+3, 2r+2)$.*

There are also many examples when K is an imaginary extension. In fact the above method works more easily in this case.

3 Preliminaries

3.1 Real-analytic manifolds and actions

Recall that a *real-analytic structure* on a compact n-manifold M is defined by an atlas in which the overlap maps are real-analytic diffeomorphisms of open subsets of $\mathbf{R}^n$.

It is a classical result of Whitney that every smooth manifold has a real-analytic structure. By *real-analytic manifold* we mean a smooth manifold equipped with a fixed real-analytic structure. A *real-analytic action* of a group Γ on a real-analytic manifold M is an action of Γ on M by diffeomorphisms which are real-analytic with respect to a fixed real-analytic structure.

3.2 Almost simple groups

We say that a group Γ is *almost simple* if every normal subgroup of Γ is either finite and central, or has finite index in Γ. A group Γ is *non-elementary* if it is not virtually cyclic. The Margulis Finiteness Theorem (see, e.g., [Zi1], Theorem 8.1), states that any irreducible lattice in a semisimple Lie group of $\mathbf{R}$-rank at least 2 is almost simple.

3.3 Basic lemmas on group actions

The following lemma is immediate from the definitions.

Lemma 3.1. *Let $\psi : \Gamma \to H$ be a homomorphism with Γ an almost simple group. If $\psi(g) = 1$ for some noncentral element $g \in \Gamma$, then ψ factors through a homomorphism to a finite group.*

The following lemma is true even in the case of actions by diffeomorphisms (see [Zi5]). We include a proof here since it is simpler in the real-analytic case.

Lemma 3.2 (Finite orbits). *Let Γ be a group and let n be a positive integer. Suppose that any representation of any finite-index subgroup of Γ into $GL(n, \mathbf{R})$ has finite image. Suppose that Γ acts by real-analytic diffeomorphisms on a connected n-manifold M, and that Γ has some finite orbit in M. Then the action of Γ on M is finite.*

Proof. First note that the hypothesis implies that Γ admits no homomorphism onto $\mathbf{Z}$, hence every homomorphism from Γ to a torsion-free nilpotent group is trivial.

By passing to a finite-index subgroup if necessary, we may assume that the action has a global fixed point $x \in M$. The action is smooth and so Γ acts on the tangent space $T_x M$ by automorphisms $\rho : \Gamma \to \mathrm{GL}(n, \mathbf{R})$. By hypothesis, after passing to a finite-index subgroup of Γ if necessary, we may assume ρ is trivial. This implies that, for each $i > 1$, the natural representation of Γ into the automorphism group of the vector space of i-jets at x has torsion-free nilpotent image. Hence Γ acts trivially on the i-jets for every i. By real-analyticity, it now follows that the action of Γ on M is trivial. $\diamond$

4 Canonical stratifications and an inductive step

In this section we use some of the theory of stratifications to prove a theorem which plays the role of an inductive step in our principal results.

Definition (Analytic set): Let M be a compact, real-analytic n-manifold. A subset $V \subset M$ is called an *analytic set* in M if

1. V is closed, and
2. every point $v \in V$ has a neighborhood U_v in M such that $U_v \cap V$ is the set of zeros of some finite collection of real-analytic functions on U_v.

The fixed-point set F of a real-analytic diffeomorphism $f : M \to M$ is an analytic set, as it is defined as the solution space of the equation $f(x) = x$. This is the reason why real-analytic sets are important in this paper.

We will be using the theory of analytic sets via the following lemma.

Lemma 4.1 (Canonical submanifold). *Let M be a compact real-analytic manifold and let $V \subset M$ be a nonempty, analytic subset of M. Then there is a subset $W \subseteq V$ which has the following properties:*

- *W is a compact, real-analytic submanifold of M.*
- *W is canonical in the following sense: If $f : M \to M$ is any real-analytic diffeomorphism which leaves V invariant, then f leaves W invariant.*

- *The induced map $f : W \to W$ is a real-analytic diffeomorphism.*

The proof of Lemma 4.1 is based on the notion of singular set.

Definition (Singular set): For each integer $0 \leq q \leq n$, the set of *q-regular* points of a set V, denoted $r^q(V)$, is the set of $v \in V$ such that for some neighborhood U_v in M, the set $V \cap U_v$ is a q-dimensional real-analytic submanifold of M. The *singular set* of V, denoted $s(V)$, is the complement in V of $\bigcup_{q=0}^{m} r^q(V)$.

Observation. Note that singular sets are *canonical* in the following sense. If X is any subset of a real-analytic manifold M, and if X is invariant under a diffeomorphism $f : M \to M$, then $s(X)$ is automatically invariant under f.

The idea behind Lemma 4.1 is now easily described. Define a sequence of subsets $V = V_0 \supset V_1 \supset \cdots$ by setting $V_{i+1} = s(V_i)$. By the observation after the definition of singular set, each V_i is invariant under any diffeomorphism of M. If there is a largest j such that $V_j \neq \emptyset$, then $W = V_j$ is readily seen to satisfy the conclusion of the lemma. Naïvely one might expect that each V_i is an analytic set, in which case one could use the Noetherian property for real-analytic sets (see Corollary V.1 of [Na]) to conclude that V_i is eventually empty. Examples show, however, that the singular set of an analytic set is not necessarily analytic.

This can be remedied by using either semi-analytic sets or subanalytic sets. We describe both approaches.

Definition (Semi-analytic, subanalytic): Let M be a compact real-analytic n-manifold. A subset $V \subset M$ is called a *semi-analytic set* in M if every point $v \in V$ has a neighborhood U_v in M so that $U_v \cap V$ can be described as a finite union of connected components, each component being the set of points satisying a finite number of equalities and inequalities in a finite number of analytic functions. A subset $V \subset M$ is a *subanalytic set* if it belongs to the smallest collection of sets containing semi-analytic sets which is also closed under union, intersection, symmetric difference, and taking images under proper analytic maps.

By the *dimension* of a semi-analytic or subanalytic set A, denoted $\dim A$, we will mean the Hausdorff dimension of A.

Proof of Lemma 4.1 using subanalytic sets: Let $V_0 = V$, and define $V_{i+1} = s(V_i)$. Note that the V_i are closed sets. By the observation after the definition of singular set, each V_i is invariant under any diffeomorphism of M which leaves V invariant.

Since $V \neq \emptyset$ we know $\dim V \geq 0$. By Theorem 2.4.2 of [Ta] (see [BM] for a different proof), the singular set $s(A)$ of a subanalytic set A in M is a subanalytic set with $\dim s(A) \leq (\dim A) - 1$. Hence V_i is subanalytic for each i, and $V_r = \emptyset$ for some $1 \leq r \leq (\dim V) + 1$. Let $W = V_p$, where p is the largest number with $V_p \neq \emptyset$. Then W is compact since it is closed and M is compact, and W is a real-analytic manifold since its singular set $s(W) = s(V_p) = V_{p+1} = \emptyset$ by our choice of p. The final statement is clear. $\diamond$

Proof of Lemma 4.1 using semi-analytic sets: This proof is the same as above except that one uses the fact that the singular set $s(A)$ of a semi-analytic set A is semi-analytic with $\dim s(A) \leq (\dim A) - 1$. This fact is due to Lojasiewicz [Lo1, Lo2]. $\diamond$

We say that an action $\psi : \Gamma \to \mathrm{Diff}(M)$ *infinite* if $\psi(\Gamma)$ is infinite.

Theorem 4.2 (Inductive step). *Let Γ be a nonelementary, almost simple group. Suppose we are given an infinite, real-analytic action of Γ on a closed, connected n-manifold M. Suppose there exists:*

- *a noncentral element $\gamma \in \Gamma$ with a fixed point, and*
- *a noncentral subgroup B in $C_\Gamma(\gamma)$ with the property that any representation of any finite-index subgroup of B in $GL(n, \mathbf{R})$ has finite image.*

Then there is a nonempty, real-analytic submanifold $W \subset M$ of codimension at least 1 which is invariant under a finite-index subgroup of B. Furthermore, the action of this subgroup on W is infinite.

Proof. Let $F = \mathrm{Fix}(\gamma) \neq \emptyset$. It follows from Lemma 4.1, applied with $V = F$, that there is some set W with the following properties.

- $W \subset F$.
- W is a real-analytic submanifold of M.
- B leaves W invariant.

After replacing B by a finite-index subgroup if necessary, we may assume that

- W is connected.

We must show that W has codimension at least 1 and that the action of B on W is infinite. If W has codimension 0, then $F = M$, hence γ acts trivially on M. Lemma 3.1 then implies that the action of Γ on M is finite, contradicting the hypothesis.

Suppose the action of B on W is finite. Then B has a finite orbit in M. Since any representation of any finite-index subgroup of B in $\mathrm{GL}(n, \mathbf{R})$ has finite image, Lemma 3.2 implies that the action of B on M is finite. Again by applying Lemma 3.1 we have that the action of Γ on M is finite, a contradiction. $\diamond$

5 Actions on S^1

Proof of Theorem 1.1:

Let $\psi : \Gamma \to \mathrm{Diff}_{C^\omega}(S^1)$ be the given real-analytic action. If $\mathbf{Q}$-rank$(\Gamma) \geq 2$, the conclusion follows from Witte's theorem [Wi]. Hence, in view of the definition of 1-big, we may suppose that there is an element $\gamma \in \Gamma$ such that $\mathrm{C}_\Gamma(\gamma)$ has a subgroup Λ isomorphic to an irreducible lattice in a higher-rank semisimple group. By passing to a subgroup of Γ of index 2 if necessary, we may assume the action is orientation-preserving.

Let $f : \mathbf{R} \to \mathbf{R}$ denote any lift of $\psi(\gamma)$. Recall ([CFS], Theorem 1, p.74) that the limit

$$\lim_{n\to\infty} f^n(x)/n = \alpha$$

exists and does not depend on the choice of $x \in \mathbf{R}$. Furthermore, this *rotation number* α of $\psi(\gamma)$ is rational if and only if $\psi(\gamma)^r$ has a fixed point for some $r > 0$.

Suppose this is the case, and let $F = \mathrm{Fix}(\psi(\gamma^r)) = \mathrm{Fix}(\psi(\gamma)^r) \neq \emptyset$. Note that any representation of any finite index subgroup of Λ into $\mathrm{GL}(1, \mathbf{R})$ is trivial since Λ is an irreducible, higher-rank lattice. If $\psi(\Gamma)$ is infinite we may then apply Theorem 4.2 to obtain a compact, 0-dimensional manifold which is invariant by $\psi(\Lambda)$; that is, $\psi(\Lambda)$ has a finite orbit in S^1. But then the action of Λ is finite by Lemma 3.2. So $\psi(g) = 1$ for some element $g \in \Lambda < \Gamma$ of infinite order, so $\psi(\Gamma)$ is finite by Lemma 3.1.

Now suppose the rotation number α of $\psi(\gamma)$ is not rational. Then since $\psi(\gamma)$ is a C^∞ diffeomorphism, the Denjoy Theorem (see, e.g,. [CFS], Theorem 1, p.84) tells us that there is a homeomorphism ϕ of S^1 with $\phi^{-1} \circ \psi(\gamma) \circ \phi = T_\alpha$, where T_α is the rotation of the circle by angle α.

We now show that the representation $\Psi : \Gamma \to \text{Homeo}(S^1)$ defined by

$$\Psi(g)(z) = \phi^{-1} \circ \psi(g) \circ \phi(z) \quad \text{for } g \in \Gamma, z \in S^1$$

factors through a finite group, hence so does ψ. It is interesting to note that we have lost quality of the action in the sense that Ψ is only an action by homeomorphisms.

The cyclic group $\langle T_\alpha \rangle$ generated by T_α is dense in the group R of rotations of S^1 since α is irrational. Since $\Psi(\Lambda)$ commutes with $\Psi(\gamma) = \phi^{-1} \circ \psi(\gamma) \circ \phi = T_\alpha$, hence with $\langle T_\alpha \rangle$, and since $\langle T_\alpha \rangle$ is dense in R, by continuity we have that $\Psi(\Lambda)$ commutes with R. But any circle homeomorphism which commutes with every rotation must be a rotation, so $\Psi(\Lambda)$ lies in R. Hence $\Psi(\Lambda)$ is finite since the commutator subgroup of Λ has finite index in Λ by Margulis finiteness. Hence there is an infinite-order element $g \in \Lambda < \Gamma$ with $\psi(g) = 1$, and we are done by Lemma 3.1. $\diamond$

6 Actions on surfaces

The proof of Theorem 1.2 depends on the following lemma.

Lemma 6.1 (Powers have fixed points). *Let S be a closed surface with $\chi(S) \neq 0$. Then any diffeomorphism $f : S \to S$ has the property that f^r has a fixed point for some $r > 0$.*

Proof. We may assume, by passing to a power of f if necessary, that f induces the identity on $H_2(S; \mathbf{C})$. This is trivial if S is nonorientable since $H_2(S; \mathbf{C}) = 0$ in this case, and follows for orientable S since f^2 is orientation-preserving.

Let $d = \dim(H_1(S; \mathbf{C}))$ and let $A = f_* : H_1(S; \mathbf{C}) \to H_1(S; \mathbf{C})$. We claim that there is a sequence $n_k \to \infty$ such that either $\text{trace}(A^{n_k}) \to \infty$ or $\text{trace}(A^{n_k}) \to d$ as $k \to \infty$. To prove this, let $\lambda_1, \ldots, \lambda_d$ denote the characteristic roots of A, and write $\lambda_i = r_i \alpha_i$ where $r_i > 0$ and $|\alpha_i| = 1$. Choose a sequence n_k such that $\alpha_i^{n_k} \to 1$ for each i as $k \to \infty$. For big enough k we then have that $\text{Re}\, \lambda_i^{n_k} > 0$. Hence if $r_j > 1$ for some j, then $\text{trace}(A^{n_k}) \to \infty$. Otherwise $r_i = 1$ for each i, and so $\lambda_i^{n_k} \to 1$ for each i, in which case $\text{trace}(A^{n_k}) \to d$.

It now follows that the Lefschetz number of f^{n_k} either approaches ∞ or approaches $\chi(S) \neq 0$ as $k \to \infty$. The lemma now follows from the Lefschetz Fixed Point Theorem. $\diamond$

Proof of Theorem 1.2:

Suppose the real-analytic action $\psi : \Gamma \to \mathrm{Diff}_{C^\omega}(S)$ is infinite. By definition of 2-big, there exists an infinite-order element $\gamma \in \Gamma$ whose centralizer contains a subgroup Λ isomorphic to a 1-big lattice. By Lemma 6.1 we have that γ^r has a fixed point in S for some $r > 0$. By Margulis superrigidity, any representation of any finite-index subgroup of Λ into $\mathrm{GL}(2, \mathbf{R})$ has finite image. Applying Theorem 4.2 to γ^r with $B = \Lambda$ gives an infinite, real-analytic action of Λ on a closed manifold V of dimension one or zero. If $\dim V = 0$ this is clearly impossible. If $\dim V = 1$, we have a contradiction to Theorem 1.1 since Λ is a 1-big lattice. $\diamond$

Proof of Theorem 1.3:

Let $\psi : \Gamma \to \mathrm{Diff}_{C^\omega}(S)$ be a real-analytic, area-preserving action, where Γ is a 2-big lattice. In light of Theorem 1.2 and its proof, we may assume S is the torus or the Klein bottle, and it suffices to show that, given any element $\gamma \in \Gamma$, then some positive power of γ has a fixed point. Note that, unlike in the case of higher genus surfaces, there are diffeomorphisms (even isotopically trivial isometries) γ of T^2 for which no power of γ has a fixed point. Hence we will need to use global properties of Γ to find an element with a fixed point.

First consider the case where $S = T^2$. Let $D = \mathrm{Diff}_{C^\infty}(T^2)$ and let V be the subgroup of D consisting of area-preserving diffeomorphisms. Let D^0 and V^0 denote the components of D and V containing the identity. We claim that

$$D^0 \cap V = V^0$$

To prove this claim, let $f : T^2 \to T^2$ be a diffeomorphism preserving a smooth area form ω, and suppose f is isotopic to the identity via an isotopy f_t with $f_0 = f$ and $f_1 = Id$. Let ω_t be the family of area forms defined by $\omega_t = f_t^* \omega$. Note that

$$\int_{T^2} \omega_t = \int_{T^2} \omega$$

for each $0 \leq t \leq 1$. We may now apply Theorem 2 of [Mo], which gives a family of diffeomorphisms $\phi_t : S \to S$ with $\phi_t^* \omega_t = \omega_0 = \omega$ and $\phi_0 = \phi_1 = Id$ since $\omega_0 = \omega_1 = \omega$. Let $h_t = f_t \circ \phi_t$. Then $h_t^* \omega = \omega$, so h_t is an isotopy through area-preserving diffeomorphisms from $h_0 = f$ to $h_1 = Id.$, and the claim is proved.

By [Ep] we have that $\pi_0(\mathrm{Diff}_{C^\infty}(T^2)) = \mathrm{GL}(2, \mathbf{Z})$. By Margulis superrigidity, any homomorphism $\Gamma \to \mathrm{GL}(2, \mathbf{Z})$ has finite image. Hence we may, after passing to a finite-index subgroup of Γ if necessary, assume that $\psi(\Gamma) \subset D^0$. Since by hypothesis $\psi(\Gamma) \subset V$, we have $\psi(\Gamma) \subset D^0 \cap V = V^0$.

By Margulis's finiteness theorem we know that the commutator subgroup of Γ has finite index in Γ. Hence some power γ^r is a commutator in Γ. As $\psi(\Gamma) \in V^0$, we have

$$\psi(\gamma^r) \in \psi([\Gamma, \Gamma]) = [\psi(\Gamma), \psi(\Gamma)] \subset [V^0, V^0]$$

Now for any $f \in V^0$, say f preserves the volume μ, the *mean translation vector* $\tau(f)$ is defined to be

$$\tau(f) = \int_\Omega (\tilde{f}(x) - x)\mu$$

where $\tilde{f}$ is a lift of f to $\mathbf{R}^2$ and Ω is a fundamental domain. By a change of variables it is easy to see that $\tau : V^0 \to \mathbf{R}^2$ is a homomorphism, hence vanishes on $[V^0, V^0]$. So $\tau(\psi(\gamma^r)) = 0$. Corollary 3.3. of [Fr] states that any $f \in V^0$ with $\tau(f) = 0$ has a fixed point, so we are done.

If S is the Klein bottle, the action of Γ on S gives rise to an action of a $\mathbf{Z}/2\mathbf{Z}$ central extension $\hat{\Gamma}$ of Γ on the orientation cover T^2 of S. Note that :

- Any representation of $\hat{\Gamma}$ into $\mathrm{GL}(2, \mathbf{Z})$ has finite image, and
- The commutator subgroup of $\hat{\Gamma}$ has finite index in $\hat{\Gamma}$.

These properties follow immediately from the corresponding properties of Γ. Pick an element $\hat{\gamma} \in \hat{\Gamma}$ which projects to $\gamma \in \Gamma$. The above proof then shows that $\hat{\gamma}^r$ has a fixed point in T^2 for some $r > 0$. Hence γ^r has a fixed point in S. $\diamond$

7 Volume-preserving actions

For real-analytic actions which in addition preserve volume, we have a variant of Theorem 4.2 which gives an invariant submanifold of codimension at least *two*. We do not know if the action on this submanifold is necessarily volume-preserving.

Theorem 7.1 (Volume-preserving actions). *Let Γ be an almost simple group. Suppose we are given an infinite, volume-preserving, real-analytic action of Γ on a closed, connected n-manifold M. Suppose that Γ contains commuting subgroups A and B with the following properties:*

- *There exists an element $\gamma \in A$, noncentral in Γ, having a fixed point in M.*

- *A is isomorphic to an irreducible lattice of $\mathbf{Q}$-rank ≥ 2.*

- *B is noncentral in Γ.*

- *Any representation of any finite-index subgroup of B in $GL(n, \mathbf{R})$ has finite image.*

Then there is a nonempty, connected, real-analytic submanifold $W \subset M$ of codimension at least 2 which is invariant under a finite-index subgroup of B. Furthermore, the action of this subgroup on W is infinite.

The proof of Theorem 7.1 uses the following lemma, which is an immediate consequence of Corollary V.1 of [Na].

Lemma 7.2 (Noetherian property for analytic sets). *Let M be a compact real-analytic manifold. Any family of real-analytic subsets of M has a minimal element (with respect to inclusion).*

$\diamond$

We will also need the following easy topological fact.

Lemma 7.3. *Let X, Y be compact, codimension 0 submanifolds of a closed, connected manifold M. Suppose that ∂X and ∂Y are connected and that $\partial X \cap \partial Y = \emptyset$. Then either $X \cap Y = \emptyset$, $X \subset \operatorname{int} Y$, $Y \subset \operatorname{int} X$, or $\operatorname{int} X \cup \operatorname{int} Y = M$.*

$\diamond$

The following somewhat more surprising lemma illustrates the way in which disjointness of codimension-1 submanifolds affects their homological behavior.

Lemma 7.4. *Let W_1 and W_2 be disjoint, connected, codimension-1 submanifolds of a closed, orientable n-manifold M. If $[W_1] = [W_2] \in H_{n-1}(M; \mathbf{Z}/2\mathbf{Z})$ then W_1 and W_2 are orientable, and for any choice of orientations on W_1 and W_2 we have $[W_1] = \pm[W_2] \in H_{n-1}(M; \mathbf{Z})$.*

Proof. First, since W_1 and W_2 are disjoint and $[W_1] = [W_2] \in H_{n-1}(M; \mathbf{Z}/2\mathbf{Z})$, there is a codimension-0 submanifold $N \subset M$ such that $N = W_1 \cup W_2$. Now an orientation on M restricts to an orientation on N, which in turn induces orientations on W_1 and W_2. These induced orientations on W_1, W_2 satisfy $[W_1] = -[W_2] \in H_{n-1}(M; \mathbf{Z})$. $\diamond$

Proof of Theorem 7.1: Let $F = \mathrm{Fix}(\gamma) \neq \emptyset$. Let $\mathcal{P}$ denote the set of all finite subsets of A. For every $P \in \mathcal{P}$, set

$$S_P = \bigcap_{p \in P} p \cdot F.$$

Let $\mathcal{S}$ denote the set of all nonempty sets of the form $S_P, P \in \mathcal{P}$. Note that $F = S_{\{Id.\}} \in \mathcal{S}$. Note also that each S_P is a real-analytic set, being the finite intersection of real-analytic sets. Lemma 7.2 now gives a minimal element S of $\mathcal{S}$. It follows from Lemma 4.1, applied with $V = S$, that there is some set W with the following properties.

- $W \subset S \subset F$.
- W is a real-analytic submanifold of M.
- B leaves W invariant.

The minimality of S gives in addition:

- For each $a \in A$, we have $a \cdot W = W$ or $a \cdot W \cap W = \emptyset$.

After replacing B by a finite-index subgroup if necessary, we may also assume

- W is connected.

Our first claim is that the action of B on W is infinite. If not, then B has a finite orbit in M. Since any representation of any finite-index subgroup of B in $\mathrm{GL}(n, \mathbf{R})$ has finite image, Lemma 3.2 implies that the action of B on M is finite. By Lemma 3.1, it follows that the action of Γ on M is finite, contradicting the hypothesis.

To complete the proof of the theorem, it remains to show that W has codimension at least 2.

If W has codimension 0 then γ acts trivially on W. Lemma 3.1 now implies that the action of Γ on M is finite, and we have a contradiction. We now assume that W has codimension 1. The rest of the argument is devoted to obtaining a contradiction in this case.

Case 1: W separates M. In this case W bounds a submanifold $K \subset M$ with $\mathrm{vol}(K) \leq \frac{1}{2}\mathrm{vol}(M)$. Let L denote the interior of K. We claim that, for every $a \in A$, either $a \cdot L = L$ or $a \cdot L \cap L = \emptyset$.

For any $a \in \psi(A)$ we must have either $\partial K = \partial(a \cdot K)$ or $\partial K \cap \partial(a \cdot K) = \emptyset$. In the first instance the claim is obviously true. In the second instance, we note that since a is volume-preserving, we cannot have either $a \cdot K \subset L$ or $K \subset \mathrm{int}(a \cdot K)$. Hence by Lemma 7.3 either $K \cap a \cdot K = \emptyset$ (so that $L \cap a \cdot L = \emptyset$) or $L \cup a \cdot L = M$. This last alternative implies that $\mathrm{vol}(K) > \frac{1}{2}\mathrm{vol}(M)$, a contradiction. This proves the claim.

Thus $\mathcal{L} = \{a \cdot L : a \in A\}$ is a family of mutually disjoint sets, all having the same (positive) volume. Since $\mathrm{vol}(M) < \infty$ the set $\mathcal{L}$ is finite. Hence there is a finite-index subgroup A_1 of A which leaves L invariant; in particular A_1 leaves W invariant.

But $\gamma^r \in A_1$ for some $r > 0$, and γ^r fixes W pointwise. Since A_1 is isomorphic to an irreducible higher-rank lattice, it follows from Lemma 3.1 that the action of A_1 on W is finite. Hence some finite-index subgroup A_2 of A_1 acts trivially on W. Pick any $w \in W$, so $A_2 \cdot w = w$. Since W has codimension-1 in M, the action of A_2 on the tangent space $T_w M$ is trivial on the codimension one subspace $T_w W \subset T_w M$, hence factors through an action of a solvable group. Since A_2 is an irreducible, higher-rank lattice the action of Γ on $T_w M$ must be finite. Hence A_2 has a finite-index subgroup A_3 which acts trivially on $T_w M$. Arguing as in Lemma 3.2, we conclude that the action of A_3 on all the higher jets at w is trivial, and hence that A_3 acts trivially on M. By Lemma 3.1 the action of Γ on M is finite.

Case 2: W doesn't separate M. We need only show that the collection of disjoint sets $\mathcal{W} = \{a \cdot W : a \in A\}$ is finite; the proof can then be completed as in Case 1.

If W is nonorientable it follows from Lemma 7.4 that if two elements of $\mathcal{W}$ are disjoint, then they represent distinct elements of $H_{n-1}(M; \mathbf{Z}/2\mathbf{Z})$. Hence $\mathcal{W}$ is finite in this case since $H_{n-1}(M; \mathbf{Z}/2\mathbf{Z})$ is finite as M is compact.

Now assume W is orientable. Since for each $a \in A$ the manifolds $a \cdot W$ and W either coincide or are disjoint, and since $H_{n-1}(M; \mathbf{Z}/2\mathbf{Z})$ is finite as M is compact, we see from Lemma 7.4 that the totality of submanifolds $\{a \cdot W : a \in A\}$ represent at most finitely many classes in $H_{n-1}(M; \mathbf{Z})$. Hence, after replacing A by a finite-index subgroup if necessary, we may assume that the elements of $\mathcal{W}$ all represent the same homology class $[W] \in H_{n-1}(M; \mathbf{Z})$.

Since W is nonseparating in M, the class $[W]$ is nonzero. The Poincaré dual of $[W]$ thus gives an infinite cyclic covering $\widetilde{M}$ of M. The manifold $\widetilde{M}$

has 2 ends, call them E^+ and E^-.

Since every element of $\mathcal{W}$ represents the homology class W, it admits a lift to $\widetilde{M}$. Let $\widetilde{\mathcal{W}}$ denote the set of all lifts to $\widetilde{M}$ of elements of $\mathcal{W}$. Each element of $\widetilde{\mathcal{W}}$ separates E^+ from E^-.

This separation property of each lift allows us to put an order relation on $\widetilde{\mathcal{W}}$ as follows. Given elements $Z_1, Z_2 \in \widetilde{\mathcal{W}}$, we declare $Z_1 < Z_2$ if Z_2 lies in the component of $\widetilde{M} \setminus Z_1$ containing E^+.

Since every element $a \in A$ fixes the homology class $[W]$, we have that a lifts to a diffeomorphism of $\widetilde{M}$. Let Λ denote the group of all lifts to $\widetilde{M}$ of elements of A. We have an exact sequence

$$1 \to \mathbf{Z} \to \Lambda \to A \to 1$$

The group Λ permutes the elements of $\widetilde{\mathcal{W}}$. Now Λ has a subgroup Λ_1 of index at most 2 whose action on $\widetilde{\mathcal{W}}$ is order-preserving. Let K denote the kernel of this action. Then the group Λ_1/K acts effectively in an order-preserving way on $\widetilde{\mathcal{W}}$, so (see Lemma 2.2 of [Wi]) is a right-orderable group. By Margulis finiteness, either Λ_1/K is a central extension of an irreducible lattice of $\mathbf{Q}$-rank at least 2 (since A_1 is such a lattice), or K projects to a finite-index subgroup of A. The first case cannot occur since Witte shows (see [Wi], Main Theorem 4.3) that no central extension of an irreducible lattice of $\mathbf{Q}$-rank at least 2 is right orderable. If the second case occurs then the action of A on $\mathcal{W}$ is finite. Since A acts transitively on $\mathcal{W}$ it then follows that $\mathcal{W}$ is finite. $\diamond$

8 Actions on 3-manifolds

We begin with a general result about lattice actions on 3-manifolds.

Theorem 8.1 (General 3-manifolds result). *Let M be any closed 3-manifold. Let Γ be an almost simple group acting on M by real-analytic, volume-preserving diffeomorphisms. Suppose Γ contains two commuting groups A, B such that*

- *A isomorphic to an irreducible lattice of $\mathbf{Q}$-rank at least 2,*
- *some noncentral element of A has a fixed-point,*
- *B is isomorphic to a 1-big lattice, and*

- *any representation of any finite-index subgroup of B into $GL(3, \mathbf{R})$ has finite image.*

Then the action of Γ on M factors through a finite group action.

Proof of Theorem 8.1. Suppose the real-analytic action $\psi : \Gamma \to \mathrm{Diff}_{C^\omega}(M)$ is infinite. Applying Theorem 7.1 gives an infinite, real-analytic action of B on a closed manifold V of dimension 1 or 0. If $\dim V = 0$ this is clearly impossible. If $\dim V = 1$, we have a contradiction to Theorem 1.1 since B is a 1-big lattice. $\diamond$

Proof of Corollary 1.5. Let $\psi : \Gamma \to \mathrm{Diff}_{C^\omega}(M)$ be the given action. Let G_1 be the upper-left block of $\mathrm{SL}(n, \mathbf{Z})$ isomorphic to $\mathrm{SL}(4, \mathbf{Z})$, and let G_2 denote the complementary $\mathrm{SL}(n-4, \mathbf{Z})$ block. The hypothesis on the element γ implies that, after a conjugation, we may assume that γ belongs to G_2 and is noncentral in G_2. Now set $A = \Gamma \cap G_2$ and $B = \Gamma \cap G_1$. We claim that there is a noncentral element of A which has a fixed point. The conclusion then follows immediately from Theorem 8.1.

If $\psi(\gamma)$ has a fixed point we are done, so assume it doesn't. Theorem 1 of [Mi] says that if M is a $\mathbf{Z}/2\mathbf{Z}$-homology 3-sphere, T is a fixed-point free involution of M, and $f : M \to M$ is any continuous map of odd degree, then there exists $x \in M$ such that $Tf(x) = fT(x)$. Let $\alpha \in A$ be any element with $[\alpha, \gamma]$ noncentral in A. Applying Theorem 1 of [Mi] with $f = \psi(\alpha)$ and $T = \psi(\gamma)$ gives a fixed point for $\psi([\alpha, \gamma])$. Note that f has degree one as it is a diffeomorphism. $\diamond$

Proof of Theorem 1.4 Since M is compact ∂M has finitely many connected components. Clearly this set of components is invariant under the action of Γ. So by hypothesis there is a component W of ∂M of nonzero Euler characteristic which is invariant under the action of a finite-index subgroup Γ_1 of Γ. Since Γ, hence Γ_1, is 2-big lattice, the action of Γ_1 on W factors through a finite group action. The theorem now follows by the exact same argument given in the last paragraph of Case 1 in the proof of Theorem 7.1. $\diamond$

9 Free actions

9.1 Examples of free actions

Let Λ be a uniform lattice in the semisimple Lie group G. Then $M = G/\Lambda$ is a closed manifold of dimension $\dim G$, and admits an action on the left by G, hence by any subgroup H of G. Note that this action is smooth and preserves Haar measure, hence preserves a smooth volume form on G/Λ. For $G = \mathrm{SL}(2, \mathbf{R})$ and H the diagonal subgroup (resp. upper-triangular subgroup with ones on the diagonal) this action is just the geodesic flow (resp. horocycle flow).

However we will take $H = \Gamma$ for a lattice $\Gamma < G$. For any $g \in \Lambda$, the stabilizer of the point $g\Lambda \in M$ is

$$\mathrm{stab}(g\Lambda) = \{\gamma \in \Gamma : \gamma g\Lambda = g\Lambda\}.$$

In other words the action of Γ on M is free if and only if no element of Γ is conjugate in G to any element of λ. This prompts the

Question. Let Γ be a lattice in the semisimple Lie group G. Does there exist a uniform lattice Λ in G for which no element of Γ is conjugate in G to any element of Λ?

Using arithmetic constructions, G. Prasad (personal communication) and S. Mozes [Moz] have shown that there are uniform lattices Λ, Γ in $G = \mathrm{SL}(n, \mathbf{R})$ for any $n > 1$ with the property that no element of Γ is conjugate in G to any element of Λ.

We sketch a geometric construction in $G = \mathrm{PSL}(2, \mathbf{R})$, based on the following:

Theorem 9.1 (Disjoint length spectra is generic). *Let Σ be a fixed (marked) hyperbolic surface. Then the set of (marked) hyperbolic surfaces which have length spectrum disjoint from that of Σ is a dense G_δ in the Teichmuller space Teich(Σ) of Σ.*

Proof. Let $\Sigma = \mathbf{H}^2/\Gamma$. It is a basic fact that the length spectrum of a hyperbolic surface is discrete. Let $L = \{c_1, c_2, \dots\}$ denote the length spectrum of Σ, and let $I = \{\gamma_1, \gamma_2, \dots\}$ denote the set of conjugacy classes of elements of Γ. Now for any conjugacy class γ in Γ and any real number $c > 0$, the set

$$\mathcal{C}_{\gamma,c} := \{X \in \mathrm{Teich}(\Sigma) : \ell_X(\gamma) \neq c\}$$

is an open dense subset of Teich(Σ). This is because its complement is defined by a single polynomial equation in analytic coordinates for Teich(Σ), so is a closed subset of Teich(Σ) with empty interior.

Since Teich(Σ) is diffeomorphic to $\mathbf{R}^{6g-6}$, where g is the genus of Σ, we can apply the Baire Category Theorem to show that

$$\mathcal{T} := \bigcap_{\gamma \in I, c \in C} \mathcal{C}_{\gamma,c}$$

is dense in Teich(Σ), in particular $\mathcal{T}$ is nonempty. Then any surface $\Sigma' \in \mathcal{T}$ has length spectrum disjoint from that of Σ. $\diamond$

Corollary 9.2 (Free action of surface group). *Let Γ be torsion-free lattice in $G = PSL(2, \mathbf{R})$. Then Γ admits a free, smooth, volume-preserving action on a closed 3-manifold M.*

Proof. Simply take $M = G/\Lambda$, where Λ and Γ have disjoint length spectra. $\diamond$

Applying Corollary 9.2 to a nonuniform lattice Γ shows that the dimension restriction in the Freeness Conjecture cannot be dispensed with.

9.2 Proof of Theorem 1.6

We begin with the following.

Proposition 9.3. *Let $n(1) = 3$, and for $r > 1$ define*

$$n(r) = 3 + [\sqrt{4}] + [\sqrt{5}] + \cdots + [\sqrt{r+2}]$$

If the Freeness Conjecture is true, then any real-analytic action of any finite-index subgroup Γ of $SL(N, \mathbf{Z})$ on a closed r-manifold factors through a finite group action if $N \geq n(r)$.

Proof. We proceed by induction. The case $r = 1$ is the main result of [Wi] (see Theorem 1.1 above). Let M be a closed r-manifold. Since $n(r) = n(r-1) + [\sqrt{r+2}]$ we have

$$\mathrm{SL}(N, \mathbf{Z}) \supseteq \mathrm{SL}(n(r), \mathbf{Z}) \supset \mathrm{SL}(n(r-1), \mathbf{Z}) \times \mathrm{SL}([\sqrt{r+2}], \mathbf{Z})$$

So $\Gamma \supset \Lambda_1 \times \Lambda_2$, where $\Lambda_1 = \Gamma \cap \mathrm{SL}(n(r-1), \mathbf{Z})$ and $\Lambda_2 = \Gamma \cap \mathrm{SL}([\sqrt{r+2}], \mathbf{Z})$ are finite index in $\mathrm{SL}(n(r-1), \mathbf{Z})$ and $\mathrm{SL}([\sqrt{r+2}], \mathbf{Z})$, respectively. If the

Freeness Conjecture is true, then Λ_2 has no almost free action on M, so there is a noncentral element $\gamma \in \Lambda_2$ which has a fixed point in M.

An easy induction shows that $n(r) > r+1$. Margulis superrigidity now implies that any representation of any finite index subgroup of $\mathrm{SL}(n(r-1), \mathbf{Z})$ into $\mathrm{GL}(r, \mathbf{R})$ has finite image. Hence we may apply Theorem 4.2 with $B = \Lambda_1$ together with the inductive hypothesis to obtain that the real-analytic action of Γ on M is finite. $\diamond$

To prove Theorem 1.6, note that

$$n(r) \leq 3 + \int_4^{r+3} x^{1/2} = 2/3(r+3)^{3/2} - 7/3$$

It now follows from Proposition 9.3 that any real-analytic action of any finite-index subgroup of $\mathrm{SL}(n, \mathbf{Z})$ on a closed r-manifold is finite if

$$r \leq (\frac{3n+7}{2})^{2/3} - 3.$$

$\diamond$

There is a straighforward generalization of Theorem 1.6 to a much more general class of lattices. In order to state it we define a class of lattices called r-large. We say a lattice is 1-*large* if it is 1-big. For $r > 1$ we say a lattice is $(r+1)$-*large* if it contains 2 commuting subgroups A and B, where A is isomorphic to an r-large lattice and B is isomorphic to a lattice in a semisimple Lie group all of whose simple factors have dimension at least r. A proof similar to that of Theorem 1.6 gives:

Theorem 9.4. *If the Freeness Conjecture is true, then any real-analytic action of an r-large lattice on a compact r-manifold is finite.*

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