Representation theory and homological stability

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Abstract

We introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n. A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This makes it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n+1)^{n-1}$ conjecture). The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures.

Beyond the discovery of new phenomena, the viewpoint of representation stability can be useful in solving problems outside the theory. In addition to the applications given in this paper, it is applied in [CEF] to counting problems in number theory and finite group theory. Representation stability is also used in [C] to give broad generalizations and new proofs of classical homological stability theorems for configuration spaces on oriented manifolds.

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Contents

1 Introduction

In this paper we introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n. A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This make it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n+1)^{n-1}$ conjecture). The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures.

Beyond the discovery of new phenomena, the viewpoint of representation stability can be useful in solving problems outside the theory. In addition to the applications given in this paper, it is applied in [CEF] to counting problems in number theory. Representation stability is also used in [C] to give broad generalizations and new proofs of classical homological stability theorems for configuration spaces on oriented manifolds.

We begin with some context and motivation.

Classical homological stability. Let $\{Y_n\}$ be a sequence of groups, or topological spaces, equipped with maps (e.g. inclusions) $\psi_n \colon Y_n \to Y_{n+1}$. The sequence $\{Y_n\}$ is *homologically stable* (over a coefficient ring R) if for each $i \geq 1$ the map

$$(\psi_n)_* \colon H_i(Y_n, R) \to H_i(Y_{n+1}, R)$$

is an isomorphism for n large enough (depending on i). Homological stability is known to hold for certain sequences of arithmetic groups, such as $\{\mathrm{SL}_n \mathbb{Z}\}$ and $\{\mathrm{Sp}_{2n} \mathbb{Z}\}$. It is also known for braid groups, mapping class groups of surfaces with boundary, and for (outer) automorphism groups of free groups, by major results of many people (including Borel, Arnol'd, Harer, Hatcher–Vogtmann and many others; see, e.g. [Coh, Vo] and the references therein). Further, in many of these cases the stable homology groups have been computed.

In contrast, even for $\mathbb{Q}$ coefficients (or for $\mathbb{F}_p$ coefficients in the arithmetic examples), almost nothing is known about the homology of finite index and other natural subgroups of the above-mentioned groups, even in the simplest examples. Indeed, homological stability is known to fail in many cases, and it is not even clear what a closed-form description of the homology might look like. We now consider an example to illustrate this point.

A motivating example. Consider the set X_n of ordered n–tuples of distinct points in the complex plane:

$$X_n := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \middle| z_i \neq z_j \text{ for all } i \neq j\right\}.$$

The set X_n can be considered as a hyperplane complement in $\mathbb{C}^n$. The fundamental group of X_n is the *pure braid group* P_n. It is known that X_n is aspherical, and so $H_i(P_n; \mathbb{Z}) = H_i(X_n; \mathbb{Z})$. The symmetric group S_n acts freely on $\mathbb{C}^n$ by permuting the coordinates, and this action clearly restricts to a free action by homeomorphisms on X_n. The quotient $Y_n := X_n/S_n$ is the space of unordered n–tuples of distinct points in $\mathbb{C}$. The space Y_n is aspherical, and so is a classifying space for its fundamental group B_n, the *braid group*. We have an exact sequence:

$$1 \to P_n \to B_n \to S_n \to 1$$

Arnol'd [Ar] and F. Cohen [Co] proved that the sequence of braid groups $\{B_n\}$ satisfies homological stability with integer coefficients. Over the rationals, they proved for $n \geq 3$ that

$$H_i(B_n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

and so stability holds in a trivial way. In contrast,

$$H_1(P_n; \mathbb{Q}) = \mathbb{Q}^{n(n-1)/2}$$

and so the pure braid groups $\{P_n\}$ do not satisfy homological stability, even for $i = 1$.

Arnol'd also gave a presentation for the cohomology algebra $H^*(P_n; \mathbb{Q})$ (see §4 for the description). But we can try to extract much finer information, using representation theory, as follows. The action of S_n on the space X_n induces an action of S_n on the vector space $H^i(P_n;\mathbb{Q})$, making it an S_n–representation for each $i \geq 0$. Each of these representations is finite-dimensional, and so can be decomposed as a finite direct sum of irreducible S_n–representations. The question of how many of these summands are trivial is already interesting: an easy transfer argument gives that

$$H^i(B_n;\mathbb{Q}) = H^i(P_n;\mathbb{Q})^{S_n};$$

that is, $H^i(B_n;\mathbb{Q})$ is the subspace of S_n–fixed vectors in $H^i(P_n;\mathbb{Q})$. Thus we see that the "trivial piece" of $H^i(P_n;\mathbb{Q})$ already contains the Arnol'd–Cohen computation of $H^i(B_n;\mathbb{Q})$; the other summands evidently contain even deeper information.

Now, the irreducible representations of S_n are completely classified: they are in bijective correspondence with partitions λ of n. Which irreducibles (that is, which partitions) occur in the S_n–representation $H^i(P_n;\mathbb{Q})$? What are their multiplicities? There have been a number of results in this direction (most notably by Lehrer–Solomon [LS]), but an explicit count of the multiplicity of a fixed partition λ is known only for a few λ; an answer for arbitrary λ and arbitrary i seems out of reach.

On the other hand, using the notation $V(a_1,\ldots,a_r)$ to denote the irreducible S_n–representation corresponding to the partition $((n-\sum_{i=1}^r a_i), a_1,\ldots,a_r)$ (see §2.1 below for more details), it is not hard to check that

$$H^1(P_n;\mathbb{Q}) = V(0) \oplus V(1) \oplus V(2) \qquad \text{for } n \geq 4. \tag{1}$$

Note that, with our notation, the right-hand side of (1) has a uniform description, independent of n as long as $n \geq 4$. More interestingly, using work of Lehrer–Solomon and the computer program Magma, we computed the following:

$$\begin{aligned}
H^2(P_4;\mathbb{Q}) &= V(1)^{\oplus 2} \oplus V(1,1) \oplus V(2) \\
H^2(P_5;\mathbb{Q}) &= V(1)^{\oplus 2} \oplus V(1,1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2,1) \\
H^2(P_6;\mathbb{Q}) &= V(1)^{\oplus 2} \oplus V(1,1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2,1)^{\oplus 2} \oplus V(3) \\
H^2(P_n;\mathbb{Q}) &= V(1)^{\oplus 2} \oplus V(1,1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2,1)^{\oplus 2} \oplus V(3) \oplus V(3,1)
\end{aligned} \tag{2}$$

where we carried out the computation in the last line for $n = 7$, 8, and 9. We will see below that the last line of (2) in fact holds for all $n \geq 7$, so the irreducible decomposition of $H^2(P_n;\mathbb{Q})$ *stabilizes*. These low-dimensional ($i = 1$, 2) cases are indicative of a more general pattern. The language needed to describe this pattern is given by the main concept in this paper, which we now describe (in a special case).

Representation stability. Let V_n be a sequence of S_n–representations, equipped with linear maps $\phi_n \colon V_n \to V_{n+1}$, making the following diagram commute for each $g \in S_n$:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ V_n & \xrightarrow[\phi_n]{} & V_{n+1} \end{array}$$

where g acts on V_{n+1} by its image under the standard inclusion $S_n \hookrightarrow S_{n+1}$. We call such a sequence of representations *consistent*.

We want to compare the representations V_n as n varies. However, since V_n and V_{n+1} are representations of different groups, we cannot ask for an isomorphism as representations. But we can ask for injectivity and surjectivity, once they are properly formulated. Moreover, by using the decomposition into irreducibles, we can formulate what it means for V_n and V_{n+1} to be the "same representation".

Definition 1.1 (Representation stability, special case)**.** Let $\{V_n\}$ be a consistent sequence of S_n–representations. We say that the sequence $\{V_n\}$ is *representation stable* if, for sufficiently large n, each of the following conditions holds:

I. **Injectivity:** The maps $\phi_n \colon V_n \to V_{n+1}$ are injective.

II. **Surjectivity:** The span of the S_{n+1}–orbit of $\phi_n(V_n)$ equals all of V_{n+1}.

III. **Multiplicities:** Decompose V_n into irreducible S_n–representations as

$$V_n = \bigoplus_\lambda c_{\lambda,n} V(\lambda)$$

with multiplicities $0 \leq c_{\lambda,n} \leq \infty$. For each λ, the multiplicities $c_{\lambda,n}$ are eventually independent of n.

The idea of representation stability can be extended to other families of groups whose representation theory has a "consistent naming system", for example $\mathrm{GL}_n \mathbb{Q}$, $\mathrm{Sp}_{2n} \mathbb{Q}$ and the hyperoctahedral groups; see §2.3 for the precise definitions.

As an easy example, let $V_n = \mathbb{Q}^n$ denote the standard representation of $\mathrm{GL}_n \mathbb{Q}$. Then the decomposition $V_n \otimes V_n = \mathrm{Sym}^2 V_n \oplus \bigwedge^2 V_n$ into irreducibles shows that the sequence of $\mathrm{GL}_n \mathbb{Q}$–representations $\{V_n \otimes V_n\}$ is representation stable; see Example 2.9.

A natural non-example is the sequence of regular representations $\{\mathbb{Q}S_n\}$ of S_n. These are not representation stable since, for any partition λ, the multiplicity of $V(\lambda)$ in $\mathbb{Q}S_n$ is $\dim(V(\lambda))$, which is not constant, and indeed tends to infinity with n.

In §2.1 and §2.2 we review the representation theory of all the groups we will be considering. In §2.3 we develop the foundations of representation stability, in particular giving a number of useful examples, variations and refinements, such as uniform stability. In particular, we introduce strong stability, used when one wishes to more finely control the G_{n+1}–span of the image of V_n under ϕ_n; this is important for applications. We also develop the idea of "mixed tensor stability", which is meant to capture in certain cases subtle phenomena not detected by representation stability.

With the above language in hand, we can state our first theorem. "Forgetting the $(n+1)^{\text{st}}$ marked point" gives a homomorphism $P_{n+1} \to P_n$ and thus induces a homomorphism $H^i(P_n;\mathbb{Q}) \to H^i(P_{n+1};\mathbb{Q})$. For each fixed $i \geq 1$ the sequence of S_n–representations $\{H^i(P_n;\mathbb{Q})\}$ is consistent in the sense given above. While the exact multiplicities in the decomposition of $H^i(P_n;\mathbb{Q})$ into S_n–irreducible subspaces are far from known, we have discovered the following.

Theorem 4.1, slightly weaker version. *For each fixed $i \geq 0$, the sequence of S_n–representations $\{H^i(P_n;\mathbb{Q})\}$ is representation stable. Indeed the sequence stabilizes once $n \geq 4i$.*

See §4 for the proof. Note that the example in (2) above shows that the "stable range" we give in Theorem 4.1 is close to being sharp.

The obvious explanation for the stability in Theorem 4.1 would be that that each $V(\lambda) \subseteq H^i(P_n;\mathbb{Q})$ includes into $H^i(P_{n+1};\mathbb{Q})$ with S_{n+1}–span equal to $V(\lambda)$, at least for n large enough. But in fact this coherence never happens, even for the trivial representation $V(0)$. Thus the mechanism for stability of multiplicities in $\{H^i(P_n;\mathbb{Q})\}$ must be more subtle, and indeed it is perhaps surprising that this stability occurs at all. See §4.1 for a discussion.

To prove Theorem 4.1, we originally used work of Lehrer–Solomon [LS] to reduce the problem to a statement about stability for certain sequences of induced representations of S_n. We conjectured this stability to D. Hemmer [He], who then proved it (and more). A. Putman has informed us that he has a different approach to Theorem 4.1. In §4 we derive classical homological stability for B_n with twisted coefficients as a corollary of Theorem 4.1. We extend these results to generalized braid groups in §4.2.

Three applications. In joint work [CEF] with Jordan Ellenberg, we use the Grothendieck–Lefschetz trace formula to translate results proved here on the representation-stable cohomology of spaces into counting theorems about points on varieties over finite fields. We then apply this to obtain statistics for polynomials over $\mathbb{F}_q$ and for maximal tori in certain finite groups of Lie type such as $\mathrm{GL}_n(\mathbb{F}_q)$.

For each fixed partition λ, stability for the multiplicity of $V(\lambda)$ in $\{H^i(P_n;\mathbb{Q})\}$ is related to a different counting problem in $\mathbb{F}_q[T]$. For example, Theorem 4.1 for the sign representation $V(1,\ldots,1)$ implies that the discriminant of a random monic squarefree polynomial is equidistributed between residues and non-residues in $\mathbb{F}_q^\times$. Theorem 4.1 for the standard representation $V(1)$ implies that the expected number of linear factors of a random monic squarefree polynomial of degree n is

$$1 - \frac{1}{q} + \frac{1}{q^2} - \frac{1}{q^3} + \cdots \pm \frac{1}{q^{n-2}}.$$

The stability of $\{H^i(P_n;\mathbb{Q})\}$ itself, even without knowing what the stable multiplicities are, already implies that the associated counting problems all have limits as the degree of the polynomials tends to infinity. One can also obtain the Prime Number Theorem, counting the number of irreducible polynomials of degree n, this way. At present this approach reproduces results already known to analytic number theorists, but our methods should generalize to wider classes of examples, such as sections of line bundles on curves other than $\mathbf{P}^1$.

In [CEF] we also give an application of representation stability of the cohomology of flag varieties (see Section 7), obtaining for each $V(\lambda)$ a counting theorem for maximal tori in $\mathrm{GL}_n\,\mathbb{F}_q$ and for Lagrangian tori in $\mathrm{Sp}_{2n}\,\mathbb{F}_q$. For the trivial representation $V(0)$ we obtain Steinberg's theorem that the number of maximal tori in $\mathrm{GL}_n\,\mathbb{F}_q$ is $N = q^{n^2-n}$. The standard representation $V(1)$ gives a formula for the expected number of eigenvectors of a random maximal torus in $\mathrm{GL}_n(\mathbb{F}_q)$ which are defined over $\mathbb{F}_q$. The sign representation gives a theorem of Srinivasan [Sr, Lemma 5]: when splitting a random maximal torus into irreducible factors, the number of factors is more likely to be even than odd, with bias exactly $\frac{1}{\sqrt{N}}$.

Another application of representation stability is given in [C]. Thinking of Theorem 4.1 as a statement about the configuration space of points in the plane, this is generalized to prove representation stability for the cohomology of ordered configuration spaces on an arbitrary orientable manifold. Specializing to the case of stability for the trivial representation already gives new proofs and vast generalizations of classical homological stability theorems of McDuff and Segal for open manifolds. One reason these theorems were not known for general manifolds is that for closed manifolds, there are no maps connecting the unordered configuration spaces for different numbers of points, so it is hard to compare these different spaces, and indeed homological stability often fails integrally. Looking instead at representation stability for the ordered configuration spaces makes it possible to relate different configuration spaces, then push the results down to unordered configuration spaces by taking invariants.

Representation stability in group homology. The example of pure braid groups given above fits into a much more general framework. Suppose Γ is a group with normal subgroup N and quotient $A \coloneqq \Gamma/N$. The conjugation action of Γ on N induces a Γ–action on the group homology (and cohomology) of N, with any coefficients R. This action factors through an A–action on $H_i(N, R)$, making $H_i(N, R)$ into an A–module.

As with pure braid groups, the structure of $H_i(N, R)$ as an A–module encodes fine information. For example, the transfer isomorphism shows that when A is finite and $R = \mathbb{Q}$ the space $H_i(\Gamma; \mathbb{Q})$ appears precisely as the subspace of A–fixed vectors in $H_i(N; \mathbb{Q})$. But there are typically many other summands, and knowing the representation theory of A (over R) gives us a language with which to access these.

The following table summarizes some of the examples fitting in to this framework. Each example will be explained in detail later in this paper: the first in §4, the second and third in §6, and the fourth and fifth in §8.

kernel N	group Γ	acts on	quotient A	$H_1(N, R)$ for big n
P_n	B_n	$\{1, \ldots, n\}$	S_n	$\mathrm{Sym}^2 V/V$
Torelli group $\mathcal{I}_n$	mapping class group Mod_n	$H_1(\Sigma_n, \mathbb{Z})$	$\mathrm{Sp}_{2n}\,\mathbb{Z}$	$\bigwedge^3 V/V$
$\mathrm{IA}(F_n)$	$\mathrm{Aut}(F_n)$	$H_1(F_n, \mathbb{Z})$	$\mathrm{GL}_n\,\mathbb{Z}$	$V^* \otimes \bigwedge^2 V$
congruence subgroup $\Gamma_n(p)$	$\mathrm{SL}_n\,\mathbb{Z}$	$\mathbb{F}_p^n$	$\mathrm{SL}_n\,\mathbb{F}_p$	$\mathfrak{sl}_n\,\mathbb{F}_p$
level p subgroup $\mathrm{Mod}_n(p)$	Mod_n	$H_1(\Sigma_n; \mathbb{F}_p)$	$\mathrm{Sp}_{2n}\,\mathbb{F}_p$	$\bigwedge^3 V/V \oplus\ \mathfrak{sp}_{2n}\,\mathbb{F}_p$

Here $R = \mathbb{Q}$ in the first three examples, $R = \mathbb{F}_p$ in the fourth and fifth, and V stands in each case for the standard representation of A. In the last example p is an odd prime.

In each of the examples given, the groups Γ are known to satisfy classical homological stability. In contrast, the rightmost column shows that none of the groups N satisfies homological stability, even in dimension 1. In fact, except for the first example, almost nothing is known about the A–module $H_i(N, R)$ for $i > 1$, and indeed it is not clear if there is a nice closed form description of these homology groups. However, the appearance of some kind of "stability" can already be seen in the rightmost column, as the names of the irreducible summands of these A–modules are constant for large enough n; this is discussed in detail for each example in the body of the paper.

A crucial observation for us is that each of the groups A in the table above has an inherent stability in the naming of its irreducible representations (over R). For example, an irreducible representation of SL_n is determined by its highest weight vector, and these vectors may be described uniformly without reference to n. For example, for SL_n the irreducible representation $V(L_1 + L_2 + L_3)$ with highest weight $L_1 + L_2 + L_3$ is isomorphic to $\bigwedge^3 V$ regardless of n, where V is the standard representation of SL_n (see Section 2.2 for the representation theory of SL_n). This inherent stability can be used, at least conjecturally, to give a closed form description for $H_i(N, R)$ (for n large enough, depending on i). One idea is that the growth in $\dim_R(H_i(N, R))$ should be fully accounted for by the fact that each element of $H_i(N, R)$ brings along with it an entire A–orbit.

Homology of Lie algebras. In §5 we develop representation stability for Lie algebras and their homology. The main theoretical result here, Theorem 5.3, proves the equivalence between stability for a family of Lie algebras and stability for its homology. Both directions of this implication are applied to give nontrivial results. For example, in Corollary 5.8 we deduce stability for the homology of nilpotent Lie algebras, which is quite complicated, from stability for the homology of free Lie algebras, which is trivial to compute; the proof uses both directions of Theorem 5.3 in an essential way. We also give applications to the adjoint homology of free nilpotent Lie algebras (Corollary 5.9) and the homology of Heisenberg Lie algebras (Examples 5.13 and 5.14).

Although homological stability results for lattices in semisimple Lie groups has been known for some time, we emphasize that there do not seem to have been any stability results for the homology of lattices in nilpotent Lie groups. Since Nomizu proved in the 1950s that the rational homology of a lattice in a nilpotent Lie group N is isomorphic to the Lie algebra homology of the Lie algebra of N, such homological stability results follow from our theorems on nilpotent Lie algebras.

The ubiquity of representation stability. The phenomenon of representation stability occurs in a number of different places in mathematics. The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures. The examples include:

1. Classical representation theory (§3): stability of Schur functors; Littlewood–Richardson rule; Murnaghan's theorem on stability of Kronecker coefficients; other natural constructions. These constructions arise in most other examples, and so their stability underlies the whole theory of representation stability.

2. Cohomology of moduli spaces (§4, §6): pure braid groups and generalized pure braid groups; conjecturally in the Torelli subgroups of mapping class groups $\mathrm{Mod}(S)$ and the analogue for automorphism groups $\mathrm{Aut}(F_n)$ of free groups. We prove representation stability for the homology of pure braid groups in Theorem 4.1 and pure generalized braid groups in Theorem 4.6. In §6 we give a number of conjectures about the stable homology of the Torelli groups and their analogues. Previously there had been few (if any) general suggestions in this direction.

3. Lie algebras (§5): graded components of free Lie algebras; homology of various families of Lie algebras, for example free nilpotent Lie algebras and Heisenberg Lie algebras; Malcev completions of surface groups and (conjecturally) pure braid groups. As discussed in the introduction, the main tool proved is Theorem 5.3. We apply it to prove representation stability for various nilpotent Lie algebras.

4. (Equivariant) cohomology of flag and Schubert varieties (§7). As explained in §7, the space $\mathcal{F}_n$ of complete flags in $\mathbb{C}^n$ admits a nontrivial action of S_n, and the resulting representation on $H^i(\mathcal{F}_n;\mathbb{Q})$ is rather complicated. Similarly, the hyperoctahedral group acts on the space $\mathcal{F}'_n$ of complete flags on Lagrangian subspaces of $\mathbb{C}^{2n}$. For each $i \geq 1$ the natural families $\{H^i(\mathcal{F}_n;\mathbb{Q})\}$ and $\{H^i(\mathcal{F}'_n;\mathbb{Q}))\}$ do not satisfy classical (co)homological stability. However, we prove in each case (see Theorem 7.1 and Theorem 7.3) that these sequences are representation stable.

   Another class of well-studied families of varieties are the *Schubert varieties*. Each permutation w of any finite set determines a family $\{X_w[n]\}$ of Schubert varieties (see §7.2). For each $i \geq 0$ the (T–equivariant, for a certain torus T) cohomology $H^i_T(X_w[n];\mathbb{Q})$ admits a non-obvious action by S_n. While the sequence

$\{H_T^i(X_w[n];\mathbb{Q})\}$ does not satisfy homological stability in the classical sense, we prove in Theorem 7.4 that this sequence is representation stable.

5. Algebraic combinatorics (§7): Lefschetz representations associated to rank-selected posets and cross-polytopes. Here Stanley's counts of multiplicities in terms of various Young tableaux are shown to give representation stability. We also conjecture representation stability for the bigraded pieces of the diagonal coinvariant algebras. This gives an "asymptotic refinement" of the famous $(n+1)^{(n-1)}$ conjecture in algebraic combinatorics (proved by Haiman), as well as conjectures for "higher coinvariant algebras", where very little is known.

6. Homology of congruence subgroups and modular representations. In §8 we study congruence subgroups of certain arithmetic groups and their analogues for $\mathrm{Mod}(S)$ and for $\mathrm{Aut}(F_n)$. Each of these groups Γ admits an action by outer automorphisms by a finite group G of Lie type, such as $G = \mathrm{SL}_n(\mathbb{F}_p)$. This action makes each homology vector space $H_i(\Gamma, \mathbb{F}_p)$ a G–representation. As p divides the order of G, this is a modular representation. Thus, in order to obtain results and conjectures about these important representations, we need to develop a version of our theory using modular representation theory. Here a new phenomenon occurs: *stable periodicity* of a sequence of representations (see §8).

   For each of the sequences of groups Γ above we state a "stable periodicity conjecture" for its homology with $\mathbb{F}_p$ coefficients. The few computations that have been completely worked out are almost all in degree 1, and these use deep mathematics (e.g. the congruence subgroup problem, work of Johnson, etc.). These computations show that our conjectures are satisfied for H_1. See §8 for details.

Historical notes. Various stability phenomena have been known in representation theory at least as far back as the 1930s, when formulas were given for the decomposition of tensor products of irreducible representations of $\mathrm{SL}_n\,\mathbb{Q}$ (by Littlewood–Richardson, see e.g. [FH, Appendix A]) and of the symmetric group S_n (by Murnaghan [Mu]). Some aspects of representation stability can be found in previous work on Lie algebras. Related ideas appear in terms of mixed tensor representations in the work of Hanlon [Han] and R. Brylinski [Bry] on Lie algebra cohomology of non-unital algebras; in Tirao's description in [Ti] of the homology of free nilpotent Lie algebras; and in Hain's description in [Ha] of the associated graded Lie algebra of the fundamental group of a closed surface.

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2 Representation stability

In order to define representation stability and its variants, we will need to be very precise in the labeling of the irreducible representations of the various groups we consider. We begin by reviewing the representation theory of the following families of groups in order to establish uniform notation across the different families.

In this section G_n will always denote one of the following families of groups:

- $G_n = \mathrm{SL}_n \mathbb{Q}$, the special linear group.
- $G_n = \mathrm{GL}_n \mathbb{Q}$, the general linear group.
- $G_n = \mathrm{Sp}_{2n} \mathbb{Q}$, the symplectic group.
- $G_n = S_n$, the symmetric group.
- $G_n = W_n$, the hyperoctahedral group.

By a *representation* of a group G we mean a $\mathbb{Q}$–vector space equipped with a linear action of G. With the exception of Section 8, throughout this paper we work over $\mathbb{Q}$, but the definitions and results hold over any field of characteristic 0, in particular over $\mathbb{C}$. In Section 8 we will extend the definition of representation stability to modular representations of $\mathrm{SL}_n \mathbb{F}_p$ and $\mathrm{Sp}_{2n} \mathbb{F}_p$.

2.1 Symmetric and hyperoctahedral groups

Our basic reference for representation theory is Fulton–Harris [FH]. For hyperoctahedral groups, see Geck–Pfeiffer [GP, §1.4 and §5.5].

Symmetric groups. The irreducible representations of S_n are classified by the partitions λ of n. A *partition* of n is a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0)$ with $\lambda_1 + \cdots + \lambda_\ell = n$; we write $|\lambda| = n$ or $\lambda \vdash n$. These partitions are identified with Young diagrams, where the diagram corresponding to λ has λ_i boxes in the ith row. We identify partitions if their nonzero entries coincide; every partition then can be uniquely written with $\lambda_\ell > 0$, in which case we say that $\ell = \ell(\lambda)$ is the *length* of λ. The irreducible representation corresponding to the partition λ is denoted V_λ. This irreducible representation can be obtained as the image $\mathbb{Q}S_n \cdot c_\lambda$ of a certain idempotent c_λ in the group algebra $\mathbb{Q}S_n$. The fact that every irreducible representation of S_n is defined over $\mathbb{Q}$ implies that any S_n–representation defined over $\mathbb{Q}$ decomposes over $\mathbb{Q}$ into irreducibles. Since every representation of S_n is defined over $\mathbb{Q}$, or alternately since g is conjugate to g^{-1} for all $g \in S_n$, every representation of S_n is self-dual.

For example, the irreducible $V_{(n-1,1)}$ is the standard representation of S_n on $\mathbb{Q}^n/\mathbb{Q}$. The induced representation $\bigwedge^3 V_{(n-1,1)}$ is the irreducible representation $V_{(n-3,1,1,1)}$. To remove the dependence of this notation on n, we make the following definition. If $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash k$ is any partition of a fixed number k, then for any $n \geq k + \lambda_1$ we may define the *padded partition*

$$\lambda[n] = (n - k, \lambda_1, \ldots, \lambda_\ell).$$

The condition $n \geq k + \lambda_1$ is needed so that this sequence is nonincreasing and defines a partition. For $n \geq k + \lambda_1$ we then define $V(\lambda)_n$ to be the irreducible S_n–representation

$$V(\lambda)_n = V_{\lambda[n]}.$$

Every irreducible representation of S_n is of the form $V(\lambda)_n$ for a unique partition λ. When unambiguous, we denote this representation simply by $V(\lambda)$. In this notation, the standard representation is $V(1)$, and the identity $\bigwedge^3 V(1) = V(1,1,1)$ holds whenever both sides are defined.

Hyperoctahedral groups. The hyperoctahedral group W_n is the wreath product $\mathbb{Z}/2\mathbb{Z} \wr S_n$; that is, the semidirect product $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ where the action is by permutations. W_n can also be thought of as the group of signed permutation matrices. General analysis of wreath products shows that the irreducible representations of W_n are classified by *double partitions* (λ^+, λ^-) of n, meaning that $|\lambda^+| + |\lambda^-| = n$. Given any representation V of S_n, we may regard V as a representation of W_n by pullback. The irreducible representation V_λ of S_n yields the irreducible representation $V_{(\lambda,0)}$ of W_n. Let ν be the one-dimensional representation of W_n which is trivial on S_n, while each $\mathbb{Z}/2\mathbb{Z}$ factor acts by -1. Then $V_{(\lambda,0)} \otimes \nu = V_{(0,\lambda)}$. In general, if $\lambda^+ \vdash k$ and $\lambda^- \vdash n - k$, the irreducible $V_{(\lambda^+,\lambda^-)}$ is obtained as the induced representation

$$V_{(\lambda^+,\lambda^-)} = \operatorname{Ind}_{W_k \times W_{n-k}}^{W_n} V_{(\lambda^+,0)} \boxtimes V_{(0,\lambda^-)}$$

where $V_{(\lambda^+,0)} \boxtimes V_{(0,\lambda^-)}$ denotes the vector space $V_{(\lambda^+,0)} \otimes V_{(0,\lambda^-)}$ considered as a representation of $W_k \times W_{n-k}$.

As before, for an arbitrary double partition $\lambda = (\lambda^+, \lambda^-)$ with $|\lambda^+| + |\lambda^-| = k$, for $n \geq k + \lambda_1^+$ we define the padded partition

$$\lambda[n] = ((n-k, \lambda^+), \lambda^-),$$

and define $V(\lambda)_n$ to be the irreducible W_n–representation

$$V(\lambda)_n = V_{\lambda[n]}.$$

Every irreducible representation of W_n is of the form $V(\lambda)_n$ for a unique double partition λ.

2.2 The algebraic groups SL_n, GL_n and Sp_{2n}

In this subsection we recall the representation theory of the algebraic groups SL_n, GL_n and Sp_{2n}.

Special linear groups. We first review the representation theory of $\mathrm{SL}_n \mathbb{Q}$ and $\mathrm{GL}_n \mathbb{Q}$. There is an interplay between two important perspectives here, that of highest weight vectors and that of Schur functors.

Every representation of $\mathrm{SL}_n \mathbb{Q}$ induces a representation of the Lie algebra $\mathfrak{sl}_n\mathbb{Q}$. Fixing a basis gives a triangular decomposition $\mathfrak{sl}_n\mathbb{Q} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, consisting of strictly lower triangular, diagonal, and strictly upper triangular matrices respectively. Given a representation V of $\mathfrak{sl}_n\mathbb{Q}$, a *highest weight vector* is a vector $v \in V$ which is an eigenvector for $\mathfrak{h}$ and is annihilated by $\mathfrak{n}^+$. Every irreducible representation contains a unique highest weight vector and is determined by the corresponding eigenvalue in $\mathfrak{h}^*$, called a *weight*.

Considering the obvious basis for the diagonal matrices, we obtain dual functionals L_i; this yields

$$\mathfrak{h}^* = \mathbb{Q}[L_1, \ldots, L_n]/(L_1 + \cdots + L_n = 0).$$

Every weight lies in the *weight lattice*

$$\Lambda_W = \mathbb{Z}[L_1, \ldots, L_n]/(L_1 + \cdots + L_n).$$

The *fundamental weights* are $\omega_i = L_1 + \cdots + L_i$. A *dominant weight* is a weight that can be written as a nonnegative integral combination $\sum c_i \omega_i$ of the fundamental weights. A highest weight vector always has a dominant weight as its eigenvalue, and every dominant weight is the highest weight of a unique irreducible representation. If $\lambda = \sum c_i \omega_i$ is a dominant weight, we denote by $V(\lambda)_n$ the irreducible representation of $\mathrm{SL}_n \mathbb{Q}$ with highest weight λ. These representations remain distinct and irreducible when restricted to $\mathrm{SL}_n \mathbb{Z}$.

We now give another labeling of the irreducible $\mathrm{SL}_n \mathbb{Q}$–representations. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ be a partition of d. Each such partition determines a *Schur functor* $\mathbb{S}_\lambda$ which attaches to any vector space V the vector space

$$\mathbb{S}_\lambda(V) = V^{\otimes d} \otimes_{\mathbb{Q}S_d} V_\lambda,$$

where $\mathbb{Q}S_d$ acts on $V^{\otimes d}$ by permuting the factors. If $\dim V$ is less than $\ell(\lambda)$, the number of rows of λ, then $\mathbb{S}_\lambda(V)$ is the zero representation. If V is a representation of a group G, the induced action makes $\mathbb{S}_\lambda(V)$ a representation of G as well.

Consider the standard representation of $\mathrm{SL}_n \mathbb{Q}$ on $\mathbb{Q}^n$. For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ with at most n rows, the resulting representation $\mathbb{S}_\lambda(\mathbb{Q}^n)$ is isomorphic to $V(\lambda_1 L_1 + \cdots + \lambda_n L_n)_n$ as $\mathrm{SL}_n \mathbb{Q}$–representations. In particular, $\mathbb{S}_\lambda(\mathbb{Q}^n)$ is irreducible, and all irreducible representations arise this way; see [FH, §6 and §15.3]. For example, let $V = \mathbb{Q}^n$. When $\lambda = d$ is the trivial partition then $\mathbb{S}_\lambda(V) = \mathrm{Sym}^d V$, and when $\lambda = 1 + \cdots + 1$ then $\mathbb{S}_\lambda(V) = \bigwedge^d V$. Note that since $L_1 + \cdots + L_n = 0$, two partitions λ and μ determine the same $\mathrm{SL}_n \mathbb{Q}$–representation if and only if $\lambda_i - \mu_i$ is constant for all $1 \leq i \leq n$. Thus we may always take our partitions to have $\lambda_n = 0$ (see the "important notational convention" remark below).

If $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$ is a partition with k rows, then for any $n > k$ we define

$$V(\lambda)_n \coloneqq \mathbb{S}_\lambda(\mathbb{Q}^n). \tag{3}$$

With this convention, every irreducible representation of $\mathrm{SL}_n \mathbb{Q}$ is of the form $V(\lambda)_n$ for a unique partition λ. As before, we will sometimes refer to $V(\lambda)_n$ as $V(\lambda)$ when the dimension is clear from context. Note that with this terminology $V(3,1)_n$ has the same meaning as $V(3,1,0,0)_n$.

Important notational convention. The right side of (3) makes sense even when $n = k$ or $n < k$. However, we intentionally decline to define $V(\lambda)_n$ when $n \leq k$. The reason is that as noted above, $V(\lambda_1, \ldots, \lambda_n)$ coincides with $V(\lambda_1 - \lambda_n, \ldots, \lambda_n - \lambda_n)$. This coincidence causes confusion with intuitive expectations about multiplicity. For example, we would expect that the multiplicity of the irreducible $\mathrm{SL}_n \mathbb{Q}$–representation $\mathbb{S}_{(2,2,2,2)}(\mathbb{Q}^n)$ in the trivial representation $\mathbb{Q}$ is 0, and this is in fact true for all $n > 4$. However, when $n = 4$ we have $\mathbb{S}_{(2,2,2,2)}(\mathbb{Q}^4) = \mathbb{Q}$, and so the multiplicity in this case is 1. For $n < 4$ the representation $\mathbb{S}_{(2,2,2,2)}(\mathbb{Q}^n)$ is the zero representation, so the multiplicity is not well-defined. Another benefit of this convention is the important fact that every irreducible representation of $\mathrm{SL}_n \mathbb{Q}$ is of the form $V(\lambda)_n$ for a *unique* λ. This notational convention is equivalent to requiring all partitions to have $\lambda_n = 0$, as mentioned above.

General linear groups. Consider the standard representation of $\mathrm{GL}_n \mathbb{Q}$ on $\mathbb{Q}^n$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is a partition with at most n rows, then $V(\lambda)_n = \mathbb{S}_\lambda(\mathbb{Q}^n)$ is an irreducible representation of $\mathrm{GL}_n \mathbb{Q}$. The partition $(1, \ldots, 1)$ with n rows yields the representation $V(1, \ldots, 1)_n = \bigwedge^n \mathbb{Q}^n = D$, the one-dimensional determinant representation of $\mathrm{GL}_n \mathbb{Q}$, and in general for any positive k we have

$$\mathbb{S}_{(\lambda_1+k,\ldots,\lambda_n+k)}(\mathbb{Q}^n) = \mathbb{S}_{(\lambda_1,\ldots,\lambda_n)}(\mathbb{Q}^n) \otimes D^{\otimes k}.$$

A *pseudo-partition* is a sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$, where the integers λ_i are allowed to be negative. The length $\ell(\lambda)$ of a pseudo-partition is the largest i for which $\lambda_i \neq 0$. We extend the definition of $V(\lambda)$ to pseudo-partitions by the above formula. That is, for any pseudo-partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ and any $n \geq k$, we define

$$V(\lambda_1, \ldots, \lambda_k)_n \coloneqq \mathbb{S}_{(\lambda_1-\lambda_k,\ldots,\lambda_k-\lambda_k)}(\mathbb{Q}^n) \otimes D^{\otimes \lambda_k}.$$

Every irreducible representation of $\mathrm{GL}_n \mathbb{Q}$ is of the form $V(\lambda)_n$ for a unique pseudo-partition λ. For example, the dual of $V(\lambda_1, \ldots, \lambda_n)$ is $V(-\lambda_n, \ldots, -\lambda_1)$. As before, the obvious basis for the diagonal matrices yields dual functionals L_i. For a pseudo-partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$, the irreducible representation $V(\lambda)_n$ has heighest weight $\lambda_1 L_1 + \cdots + \lambda_k L_k$. When restricted to $\mathrm{GL}_n \mathbb{Z}$, all of these representations remain irreducible, and $D^{\otimes 2}$ becomes trivial; thus two pseudo-partitions λ and μ determine the same representation of $\mathrm{GL}_n \mathbb{Z}$ if and only if $\lambda_i - \mu_i$ is constant and even for all $1 \leq i \leq n$.

Remark 2.1. Note that for $\mathrm{GL}_n \mathbb{Q}$–representations, $V(3,1)_n$ has the same meaning as $V(3,1,1,1)_n$, while $V(3,1,0)_n$ has the same meaning as $V(3,1,0,0)_n$. The discrepancy between the terminology for representations of $\mathrm{SL}_n \mathbb{Q}$ and $\mathrm{GL}_n \mathbb{Q}$ comes from the fact that for $\mathrm{SL}_n \mathbb{Q}$ we always assume that $\lambda_n = 0$.

Symplectic groups. We now review the representation theory of $\mathrm{Sp}_{2n} \mathbb{Q}$. Every representation of $\mathrm{Sp}_{2n} \mathbb{Q}$ induces a representation of the Lie algebra $\mathfrak{sp}_{2n}\mathbb{Q}$. Again we have a decomposition $\mathfrak{sp}_{2n}\mathbb{Q} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, with $\mathfrak{h}^* = \mathbb{Q}[L_1, \ldots, L_n]$. The fundamental weights are $\omega_i = L_1 + \cdots + L_i$, and so for any dominant weight $\lambda = \sum c_i \omega_i$ there is a unique irreducible representation $V(\lambda)_n$ of $\mathrm{Sp}_{2n} \mathbb{Q}$.

These can be identified explicitly as follows. Let $V = \mathbb{Q}^{2n}$ be the standard representation of $\mathrm{Sp}_{2n} \mathbb{Q}$. For each $1 \leq i < j \leq d$, the symplectic form gives a contraction

$V^{\otimes d} \to V^{\otimes d-2}$ as $\mathrm{Sp}_{2n}\,\mathbb{Q}$–modules. Define $V^{\langle d\rangle} \leq V^{\otimes d}$ to be the intersection of the kernels of these contractions. For any partition $\lambda \vdash d$, the representation $\mathbb{S}_\lambda V$ is realized as the image of $c_\lambda \in \mathbb{Q}S_d$ acting on $V^{\otimes d}$. If k is the number of rows of the partition λ, for any $n \geq k$ we define $V(\lambda)_n$ to be the intersection

$$V(\lambda)_n \coloneqq \mathbb{S}_\lambda V \cap V^{\langle d\rangle}.$$

The notation $\mathbb{S}_{\langle\lambda\rangle} V$ is also used for the intersection $\mathbb{S}_\lambda V \cap V^{\langle d\rangle}$. We remark that this intersection is trivial if n is less than the number of rows of λ. Every irreducible representation of $\mathrm{Sp}_{2n}\,\mathbb{Q}$ is of the form $V(\lambda)_n$ for a unique partition λ. In particular, it follows that each irreducible representation $V(\lambda)_n$ is self-dual. These representations remain distinct and irreducible when restricted to $\mathrm{Sp}_{2n}\,\mathbb{Z}$.

Remark 2.2. There is one issue which can cause confusion when comparing weights for $\mathrm{GL}_{2n}\,\mathbb{Q}$ and $\mathrm{Sp}_{2n}\,\mathbb{Q}$. To clarify, we work out the comparison explicitly in terms of a basis. Let $\{a_1, b_1, \ldots, a_n, b_n\}$ be a symplectic basis for $\mathbb{Q}^{2n}$, meaning that the symplectic form satisfies $\omega(a_i, b_i) = 1$ and $\omega(a_i, b_j) = \omega(a_i, a_j) = \omega(b_i, b_j) = 0$. By abuse of notation, in this remark we also denote by $\{a_1, b_1, \ldots, a_n, b_n\}$ the corresponding basis for $\mathfrak{h}_{\mathfrak{gl}}$, the diagonal matrices in $\mathfrak{gl}_{2n}\mathbb{Q}$, with dual basis $\{a_1^*, b_1^*, \ldots, a_n^*, b_n^*\}$ for $\mathfrak{h}_{\mathfrak{gl}}^*$. These elements, in some order, will be the weights $\{L_1^{\mathfrak{gl}}, \ldots, L_{2n}^{\mathfrak{gl}}\}$, but we defer until later the explicit identification.

If $\mathfrak{h}_{\mathfrak{sp}}$ denotes the diagonal matrices in $\mathfrak{sp}_{2n}\mathbb{Q}$, the dual $\mathfrak{h}_{\mathfrak{sp}}^*$ has basis $\{L_i^{\mathfrak{sp}} = a_i^* - b_i^*\}$. These weights are ordered so that $L_1^{\mathfrak{sp}} > \cdots > L_n^{\mathfrak{sp}}$. The restriction from $\mathfrak{h}_{\mathfrak{gl}}^*$ to $\mathfrak{h}_{\mathfrak{sp}}^*$ maps $a_i^* \mapsto L_i^{\mathfrak{sp}}$ and $b_i^* \mapsto -L_i^{\mathfrak{sp}}$. To correctly compare representations of $\mathrm{GL}_{2n}\,\mathbb{Q}$ with those of $\mathrm{Sp}_{2n}\,\mathbb{Q}$, this restriction should preserve the ordering on weights (for example, so that the notions of "highest weight" agree). This forces us to label the weights of $\mathrm{GL}_{2n}\,\mathbb{Q}$ as

$$L_1^{\mathfrak{gl}} = a_1^*,\ \ldots,\ L_n^{\mathfrak{gl}} = a_n^*,\ \ L_{n+1}^{\mathfrak{gl}} = b_n^*,\ \ldots,\ L_{2n}^{\mathfrak{gl}} = b_1^*.$$

Thus the restriction maps $L_i^{\mathfrak{gl}} \mapsto L_i^{\mathfrak{sp}}$ and $L_{2n-i+1}^{\mathfrak{gl}} \mapsto -L_i^{\mathfrak{sp}}$ for $1 \leq i \leq n$. With respect to the ordered basis $\{a_1, \ldots, a_n, b_n, \ldots, b_1\}$, the subalgebra $\mathfrak{n}^+$ consists of exactly those matrices in $\mathfrak{sp}_{2n}\mathbb{Q}$ that are upper-triangular.

2.3 Definition of representation stability

We are now ready to define the main concept of this paper. Let G_n be one of the families $\mathrm{GL}_n\,\mathbb{Q}$, $\mathrm{SL}_n\,\mathbb{Q}$, $\mathrm{Sp}_{2n}\,\mathbb{Q}$, S_n, or W_n. In this section λ refers to the datum determining the irreducible representations of the corresponding family, namely a pseudo-partition, a partition, or a double partition. For each family we have natural inclusions $G_n \hookrightarrow G_{n+1}$: for S_n and W_n we take the standard inclusions, and for $\mathrm{GL}_n \mathbb{Q}$, $\mathrm{SL}_n \mathbb{Q}$, and $\mathrm{Sp}_{2n} \mathbb{Q}$ we take the upper-left inclusions.

Let $\{V_n\}$ be a sequence of G_n–representations, equipped with linear maps $\phi_n \colon V_n \to V_{n+1}$, making the following diagram commute for each $g \in G_n$:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V_n & \xrightarrow[\phi_n]{} & V_{n+1} \end{array}$$

On the right side we consider g as an element of G_{n+1} by the inclusion $G_n \hookrightarrow G_{n+1}$. This condition is equivalent to saying that ϕ_n, thought of as a map from V_n to the restriction $V_{n+1} \downarrow G_n$, is a map of G_n–representations. We allow the vector spaces V_n to be infinite-dimensional, but we ask that each vector lies in some finite-dimensional representation. This ensures that V_n decomposes as a direct sum of finite-dimensional irreducibles. We call such a sequence of representations *consistent*.

We want to compare the representations V_n as n varies. However, since V_n and V_{n+1} are representations of different groups, we cannot ask for an isomorphism as representations. But we can ask for injectivity and surjectivity, once they are properly formulated. Moreover, using the uniformity of our labeling of irreducible representations, we can formulate what it means for V_n and V_{n+1} to be the "same representation".

Definition 2.3 (Representation stability)**.** Let $\{V_n\}$ be a consistent sequence of G_n–representations. The sequence $\{V_n\}$ is *representation stable* if, for sufficiently large n, each of the following conditions holds.

I. Injectivity: The natural map $\phi_n \colon V_n \to V_{n+1}$ is injective.

II. Surjectivity: The span of the G_{n+1}–orbit of $\phi_n(V_n)$ equals all of V_{n+1}.

III. Multiplicities: Decompose V_n into irreducible representations as

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n$$

with multiplicities $0 \leq c_{\lambda,n} \leq \infty$. For each λ, the multiplicities $c_{\lambda,n}$ are eventually independent of n.

It is not hard to check that, given Condition I for ϕ_n, Condition II for ϕ_n is equivalent to the following when G_n is finite: ϕ_n is a composition of the inclusion $V_n \hookrightarrow \operatorname{Ind}_{G_n}^{G_{n+1}} V_n$ with a surjective G_{n+1}–module homomorphism $\operatorname{Ind}_{G_n}^{G_{n+1}} V_n \to V_{n+1}$.

By requiring Condition III just for the multiplicity of the single irreducible representation $V(\lambda)_n$, we obtain the notion of *λ–representation stable* for a fixed λ. In the presence of Condition IV below, λ–representation stability is exactly equivalent to representation stability for the λ–isotypic components $V_n^{(\lambda)}$.

Remark 2.4. Fix either $G_n = S_n$ or W_n and take for each $n \geq 1$ an exact sequence of groups

$$1 \to A_n \to \Gamma_n \to G_n \to 1.$$

Then an easy transfer argument shows that λ–representation stability of $\{H_i(A_n;\mathbb{Q})\}$ for the trivial representation ($\lambda = 0$) is equivalent to classical homological stability for the sequence $\{H_i(\Gamma_n;\mathbb{Q})\}$.

Remark 2.5. It seems likely that many of the results in this paper can be extended to orthogonal groups and to the corresponding Weyl groups; it would be interesting to know what differences arise, if any.

Uniform stability. In Definition 2.3 we did not require the multiplicities of all the irreducible representations to stabilize simultaneously. We will see that in many cases a stronger form of stability holds, as follows.

Definition 2.6 (Uniform representation stability)**.** A consistent sequence $\{V_n\}$ of G_n–representations is *uniformly representation stable* if Conditions I and II hold for sufficiently large n, and the following condition holds:

III′. Multiplicities (uniform): There is some N, not depending on λ, so that for $n \geq N$ the multiplicities $c_{\lambda,n}$ are independent of n for all λ. In particular, for any λ for which $V(\lambda)_N$ is not defined, $c_{\lambda,n} = 0$ for all $n \geq N$.

For example, if $G_n = \mathrm{GL}_n\,\mathbb{Q}$, the latter condition applies to any partition λ with more than N rows. We will see examples below both of uniform and nonuniform representation stability.

Multiplicity stability. It sometimes happens that for a sequence $\{V_n\}$ of G_n–representations there are no natural maps $V_n \to V_{n+1}$. For example, this is the situation for the Torelli groups of closed surfaces (see Section 6 below). In this case we can still ask whether the decomposition of V_n into irreducibles stabilizes in terms of multiplicities.

Definition 2.7 ((Uniform) multiplicity stability)**.** A sequence of G_n–representations V_n is called *multiplicity stable* (respectively *uniformly multiplicitly stable*) if Condition III (respectively Condition III′) holds.

Reversed maps. The definitions above capture the behavior of a sequence of representations, one including into the next. In a number of contexts (see, e.g., § 7 below) we are given sequences of representations with the maps going the other way, from $V_{n+1} \to V_n$. In this case we need to alter the definition of representation stability, in particular injectivity and surjectivity.

Definition 2.8 (Representation stability with maps reversed)**.** A consistent sequence of G_n–representations $\{V_n\}$ with maps $\phi_n \colon V_n \leftarrow V_{n+1}$ is *representation stable* if for sufficiently large n, Condition III holds, and the following conditions hold:

I′. **Surjectivity:** The map $\phi_n \colon V_n \leftarrow V_{n+1}$ is surjective.

II′. **Injectivity:** There exists a subspace V_{n+1} which maps isomorphically under ϕ_n to V_n, and whose G_{n+1}–orbit spans V_{n+1}.

We remark that Definition 2.8 is *not* equivalent to representation stability for the dual sequence $\phi_n^* : V_n^* \to V_{n+1}^*$. For an explanation, and a way to handle dual sequences, see the discussion of "mixed tensor stability" in §2.4.

Examples of representation stability. We will see many examples of representation stability below; indeed much of this paper is an exploration of such examples. For now, we mention the following simple examples.

Example 2.9. Let $V_n = \mathbb{Q}^n$ be the standard representation of $\mathrm{GL}_n \mathbb{Q}$. Then $\{V_n \otimes V_n\}$ is uniformly representation stable. This follows easily from the decomposition

$$V_n \otimes V_n = \mathrm{Sym}^2 V_n \oplus \textstyle\bigwedge^2 V_n$$

of $V_n \otimes V_n$ into irreducibles. We will see in Section 3 that $\{V_n \otimes V_n\}$ will be uniformly representation stable for any sequence $\{V_n\}$ of uniformly stable representations.

In the other direction, we have the following.

Non-example 2.10. Let $V_n = \mathbb{Q}^n$ be the standard representation of $\mathrm{SL}_n \mathbb{Q}$, and let $W_n = \bigwedge^* V_n$. Then $\{W_n\}$ is a stable sequence of $\mathrm{SL}_n \mathbb{Q}$–representations, but it is not a uniformly stable sequence: the multiplicity of the irreducible representation $V(1, \ldots, 1)_n$, with k occurrences of 1, does not stabilize until $n > k$.

Non-example 2.11. Let G_n be either S_n or W_n. Then the sequence of regular representations $\{\mathbb{Q}G_n\}$ is not representation stable, or even λ–representation stable for any partition or double partition λ. This follows from the standard fact that the multiplicity of $V(\lambda)_n$ in the regular representation equals $\dim(V(\lambda)_n)$, which is not constant, and indeed tends to infinity with n.

2.4 Strong and mixed tensor stability

In this subsection we define two variations of representation stability. Both variations will be used later in the paper in the analysis of certain examples. The reader might want to skip this subsection until encountering those examples and move to §3.

Strong stability. Conditions I and II together give a kind of "isomorphism" between representations of different groups, but they give no information about the subrepresentations of the V_n. Condition III better captures the internal structure of the representations V_n, but ignores the maps between the representations. For example, Condition III alone does not rule out the possibility that the maps $\phi_n\colon V_n \to V_{n+1}$ are all zero. The following condition combines these approaches to give careful control over the behavior of a subrepresentation under inclusion. We require that for every irreducible $V(\lambda)_n \subset V_n$, the G_{n+1}–span of the image $\phi_n(V(\lambda)_n)$ is isomorphic to $V(\lambda)_{n+1}$.

Definition 2.12 (Strong representation stability)**.** A consistent sequence $\{V_n\}$ of G_n–representations is *strong representation stable* if for sufficiently large n, not depending on λ, Conditions I, II, and III$'$ hold (that is, $\{V_n\}$ is uniformly representation stable), and the following condition holds:

- **IV. Type-preserving:** For any subrepresentation $W \subset V_n$ so that $W \approx V(\lambda)_n$, the span of the G_{n+1}–orbit of $\phi_n(W)$ is isomorphic to $V(\lambda)_{n+1}$.

It is possible to embed the $\mathrm{GL}_n\,\mathbb{Q}$–module $\bigwedge^i\mathbb{Q}^n = V(\omega_i)_n$ into the $\mathrm{GL}_{n+1}\,\mathbb{Q}$–module $\bigwedge^{i+1}\mathbb{Q}^{n+1} = V(\omega_{i+1})_{n+1}$ by $v \mapsto v \wedge x_{n+1}$. This embedding respects the group actions, but the $\mathrm{GL}_{n+1}\,\mathbb{Q}$–span of the image is all of $\bigwedge^{i+1}\mathbb{Q}^{n+1}$; similar embeddings occur for other pairs of irreducible representations. Condition IV rules out this type of phenomenon. One example of a uniformly stable sequence of S_n–representations that is not strongly stable is given by the cohomology of pure braid groups; see §4.1.

Remark 2.13. For applications, we will need the stronger statement that any subspace isomorphic to $V(\lambda)_n^{\oplus k}$ has G_{n+1}–span isomorphic to $V(\lambda)_{n+1}^{\oplus k}$, where the multiplicity k may be greater than 1. Fortunately, this stronger statement follows from Condition IV above. First, the maps $V_n \to V_{n+1}$ are injective (apply Condition IV to any W contained in the kernel). Furthermore, for a fixed λ, Condition IV implies that the inclusions $V_n \hookrightarrow V_{n+1}$ restrict to inclusions of λ–isotypic components $V_n^{(\lambda)} \hookrightarrow V_{n+1}^{(\lambda)}$.

It is thus clear that the G_{n+1}–span of $V(\lambda)_n^{\oplus k}$ is $V(\lambda)_{n+1}^{\oplus \ell}$ with $\ell \leq k$. The potential problem is that two independent subrepresentations $W, W' \approx V(\lambda)_n \subset V_n$ could both map into the same $V(\lambda)_{n+1} \subset V_{n+1}$. This is ruled out by the following property, shared by each of our families of groups: the restriction $V(\lambda)_{n+1} \downarrow G_n$ contains the irreducible G_n–representation $V(\lambda)_n$ with multiplicity 1. Thus the multiplicity of $V(\lambda)_n$ in $V(\lambda)_{n+1}^{\oplus \ell} \downarrow G_n$ is ℓ. But as G_n–representations, we have an inclusion

$$V(\lambda)_n^{\oplus k} \hookrightarrow \left(V(\lambda)_{n+1}^{\oplus \ell} \downarrow G_n\right),$$

which implies $k \leq \ell$, verifying the stronger statement as desired.

For $G_n = \mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$, the property mentioned above can be seen from the formula (8) given in the proof of Theorem 3.1(6) below. For $G_n = \mathrm{Sp}_{2n} \mathbb{Q}$, it follows from (9) below. For $G_n = S_n$, this is the classical *branching rule* [FH, Equation 4.42]:

$$V(\lambda)_{n+1} \downarrow S_n = V(\lambda)_n \oplus \bigoplus_{\mu} V(\mu)_n$$

where μ ranges over those partitions obtained by removing one box from λ. For $G_n = W_n$, the branching rule has the form [GP, Lemma 6.1.3]:

$$V(\lambda^+, \lambda^-)_{n+1} \downarrow W_n = V(\lambda^+, \lambda^-)_n \oplus \bigoplus_{\mu^+} V(\mu^+, \lambda^-)_n \oplus \bigoplus_{\mu^-} V(\lambda^+, \mu^-)_n$$

where μ^+ is obtained from λ^+, and μ^- is obtained from λ^-, by removing one box.

It follows that assuming surjectivity, Condition IV also implies Condition III′. Conversely, as long as the V_n are finite-dimensional, or even have finite multiplicities $0 \leq c_{\lambda,n} < \infty$, Conditions III′ and IV together imply Condition II.

An equivalent formulation of Condition IV. When G_n is $\mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$, Condition IV can be stated in a more familiar basis-dependent form. Let P_{n+1} be the n–dimensional subgroup of $\mathrm{SL}_{n+1} \mathbb{Q}$ preserving and acting trivially on $\mathbb{Q}^n < \mathbb{Q}^{n+1}$; that is, agreeing with the identity outside the rightmost column. Then assuming uniform multiplicity stability, Condition IV can be stated as follows.

Proposition 2.14. *For $G_n = \mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$, let $\{V_n\}$ be a uniformly multiplicity stable sequence of G_n–representations. Assume that the maps $\phi_n \colon V_n \hookrightarrow V_{n+1}$ are injective. The sequence $\{V_n\}$ is type-preserving (satisfies Condition IV) for sufficiently large n if and only if the following condition is satisfied for sufficiently large n.*

IV′. P_{n+1} acts trivially on the image $\phi(V_n)$ of V_n in V_{n+1}.

Condition IV′ is in practice much easier to check than Condition IV. As we will see in Theorem 3.1, Condition IV′ is also preserved by many natural constructions. It is equivalent to the statement that ϕ_n takes highest weight vectors in V_n to highest weight vectors in V_{n+1}.

Proof. Within this proof, let $\mathfrak{p}_{n+1}$ be the Lie algebra of P_{n+1}. Explicitly, $\mathfrak{p}_{n+1}$ is the span of the elementary matrices $E_{i,n+1}$ with $1 \le i \le n$. The subgroup P_{n+1} was chosen exactly so that $\mathfrak{n}^+_{n+1}$ is spanned by $\mathfrak{n}^+_n$ together with $\mathfrak{p}_{n+1}$.

IV′ $\implies$ IV. In fact, we need only assume that P_{n+1} acts trivially on the image of each highest weight vector. Consider a highest weight vector $v \in V_n$, so v is an eigenvector for $\mathfrak{h}_n$ with weight $\lambda \in \mathfrak{h}^*_n$, and v is annihilated by $\mathfrak{n}^+_n$. By possibly rechoosing v, we may assume that $\phi(v)$ is an eigenvector for $\mathfrak{h}_{n+1}$ with weight $\lambda' \in \mathfrak{h}^*_{n+1}$. The consistency of the map $V_n \to V_{n+1}$ implies that under the restriction map $\mathfrak{h}^*_{n+1} \to \mathfrak{h}^*_n$, the weight λ' restricts to λ. The condition that P_{n+1} acts trivially on $\phi(V_n)$ implies that $\mathfrak{p}_{n+1}$ annihilates $\phi(V_n)$. It follows that $\mathfrak{n}^+_{n+1} = \mathfrak{n}^+_n \oplus \mathfrak{p}_{n+1}$ annihilates $\phi(v)$, so $\phi(v)$ is a highest weight vector for G_{n+1}. By assumption $\{V_n\}$ is uniformly multiplicity stable. This implies that once n is sufficiently large, the only weight λ' occurring in V_{n+1} which restricts to $\lambda \in \mathfrak{h}^*_n$ is the weight satisfying $V(\lambda)_{n+1} = V(\lambda')_{n+1}$. Thus we see that $\phi(v)$ spans the subrepresentation $V(\lambda)_{n+1}$, as desired. Since this holds for all highest weight vectors v, and each irreducible subrepresentation is the span of a highest weight vector, Condition IV follows.

IV $\implies$ IV′. Conversely, if $\phi \colon V_n \hookrightarrow V_{n+1}$ is type-preserving, let $v \in V_n$ be a highest weight vector for G_n spanning $V(\lambda)_n$, and consider its image in V_{n+1}. Certainly $\phi(v)$ remains a highest weight vector for G_n with weight λ. By Condition IV, its G_{n+1}–span is isomorphic to $V(\lambda)_{n+1} = V(\lambda')_{n+1}$. Let $w \in V(\lambda)_{n+1}$ be the G_{n+1}–highest weight vector with heighest weight λ'. Then w is evidently a highest weight vector for G_n with weight λ as well. But as noted in Remark 2.13, the restriction $V(\lambda)_{n+1} \downarrow G_n$ contains $V(\lambda)_n$ with multiplicity 1, so $V(\lambda)_{n+1}$ contains a unique G_n–highest weight vector with weight λ. Thus $\phi(v)$ must coincide with w, and in particular $\phi(v)$ is a highest weight vector for G_{n+1}.

This implies that P_{n+1} acts trivially on $\phi(v)$ for each highest weight vector v. It remains to show that P_{n+1} acts trivially on the entire image of V_n, that is on the G_n–span of the highest weight vectors $\phi(v)$. This is a general fact of representation theory. Since P_{n+1} is contained in $\mathrm{SL}_{n+1}\mathbb{Q}$, we may assume that $G_n = \mathrm{SL}_n\mathbb{Q}$. For the rest of the argument we identify $\mathfrak{h}_{n+1}^* = \mathbb{Z}[L_1, \ldots, L_{n+1}]/(L_1 + \cdots + L_{n+1})$ with $\mathbb{Z}[L_1, \ldots, L_n]$ by setting $L_{n+1} = 0$. Restrict to the inclusion of a single irreducible $V(\lambda)_n \subset V(\lambda)_{n+1}$, with highest weight vector v of weight $\lambda = \lambda_1 L_1 + \cdots + \lambda_n L_n$. Let $k \coloneqq \sum \lambda_i$ be the sum of the coefficients.

The irreducible representation $V(\lambda)_{n+1}$ is the span of v under $\mathfrak{n}_{n+1}^-$, which is spanned by the elementary matrices $\{E_{j,i} | 1 \leq i < j \leq n+1\}$. If $j \leq n$ the matrix $E_{j,i}$ has weight $L_j - L_i$, while $E_{n+1,i}$ has weight $L_{n+1} - L_i = -L_i$. Adding the former does not change the sum of the coefficients, while adding the latter decreases the sum, so every weight $\mu = \mu_1 L_1 + \cdots + \mu_n L_n$ occurring in $V(\lambda)_{n+1}$ has $\sum \mu_i \leq k$. The subspace $V(\lambda)_n$ is the span of v under $\mathfrak{n}_n^-$, which is spanned by the $\{E_{j,i} | 1 \leq i < j \leq n\}$ with roots $\{L_j - L_i | 1 \leq i < j \leq n\}$. Thus the weights μ occurring in $V(\lambda)_n$ all have $\sum \mu_i = k$. Applying any matrix $E_{i,n+1} \in \mathfrak{p}_{n+1}$ with weight $L_i - L_{n+1} = L_i$ to such a vector would yield a vector with weight

$$\mu_1 L_1 + \cdots + (\mu_i + 1) L_i + \cdots + \mu_n L_n.$$

The sum of the coefficients of such a weight is $k+1$, and we have already said that no such weight occurs in $V(\lambda)_{n+1}$. It follows that $\mathfrak{p}_{n+1}$ must annihilate every element of $V(\lambda)_n$, and thus P_{n+1} acts trivially on $V(\lambda)_n \subset V(\lambda)_{n+1}$, as desired. We conclude that P_{n+1} acts trivially on $\phi(V_n) \subset V_{n+1}$. □

Remark 2.15. There is no such nice formulation of Condition IV for representations of $\mathrm{Sp}_{2n}\mathbb{Q}$. Indeed we will see in Theorem 3.1 that if $\{V_n\}$ is a strongly stable sequence of $\mathrm{SL}_n\mathbb{Q}$–representations, then for any Schur functor $\mathbb{S}_\lambda$, the sequence $\{\mathbb{S}_\lambda(V_n)\}$ is strongly stable as well. The proof hinges upon the equivalence of Conditions IV and IV′.

The corresponding fact for $\mathrm{Sp}_{2n}\mathbb{Q}$ is false. For example, $\{V_n = \mathbb{Q}^{2n}\}$ is certainly strongly stable. However, $\{\bigwedge^2 V_n = \bigwedge^2 \mathbb{Q}^{2n}\}$ is not strongly stable. The unique trivial subrepresentation of $\bigwedge^2 \mathbb{Q}^{2n}$ is spanned by $a_1 \wedge b_1 + \cdots + a_n \wedge b_n$. However, this vector is not taken to a trivial subrepresentation of $\bigwedge^2 \mathbb{Q}^{2n+2}$. In fact, the $\mathrm{Sp}_{2n+2}\mathbb{Q}$–span of $a_1 \wedge b_1 + \cdots + a_n \wedge b_n$ is all of $\bigwedge^2 \mathbb{Q}^{2n+2}$. This failure is related to the fact, described in Remark 2.2, that the upper-left inclusion $\mathrm{Sp}_{2n}\mathbb{Q} \subset \mathrm{Sp}_{2n+2}\mathbb{Q}$ does not respect the ordering of the roots. Of course there does exist some map $V(0)_n \oplus V(\lambda_2)_n \to V(0)_{n+1} \oplus V(\lambda_2)_{n+1}$ which is type-preserving, but viewed as a map $\bigwedge^2 \mathbb{Q}^{2n} \to \bigwedge^2 \mathbb{Q}^{2n+2}$ this map appears wholly unnatural.

Mixed tensor representations. There are certain natural families of representations with an inherent "stability", which indeed satisfy the definition of representation stability given above, but for trivial reasons that do not capture the real nature of their stability. For example, the dual of the standard representation of $\mathrm{GL}_n \mathbb{Q}$ has highest weight $-L_n$. In terms of pseudo-partitions, the dual representation $V(1,0,\ldots,0)_n^*$ is the representation $V(0,\ldots,0,-1)_n$, which is given by a different pseudo-partition for each n. So in the sequence of representations $\{V_n = (\mathbb{Q}^n)^*\}$, for each λ the irreducible $V(\lambda)_n$ appears in V_n for at most one n, from which it follows that the sequence $\{V_n\}$ does fit the definition of representation stable given above. However, the "stable representation" is trivial, since each representation $V(\lambda)$ eventually has multiplicity 0.

To accurately capture the stability of this sequence, as well as other natural sequences such as $\{V_n^* \otimes \bigwedge^2 V_n\}$ and the adjoint representations $\{\mathfrak{sl}_n\mathbb{Q}\}$, we will use mixed tensor representations to define a stronger condition than representation stability. Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_\ell)$, for $n \geq k + \ell$ the *mixed tensor representation* $V(\lambda; \mu)_n$ is the irreducible representation of $\mathrm{GL}_n \mathbb{Q}$ with highest weight

$$\lambda_1 L_1 + \cdots + \lambda_k L_k - \mu_\ell L_{n-\ell+1} - \cdots - \mu_1 L_n.$$

Equivalently, $V(\lambda;\mu)_n$ is the irreducible representation corresponding to the pseudo-partition $(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0, -\mu_\ell, \ldots, -\mu_1)$. Note that when restricted to $\mathrm{SL}_n \mathbb{Q}$, this representation corresponds to the partition

$$(\mu_1 + \lambda_1, \ldots, \mu_1 + \lambda_k, \mu_1, \ldots, \mu_1, \mu_1 - \mu_\ell, \ldots, \mu_1 - \mu_2, 0).$$

Definition 2.16 (Mixed tensor stable)**.** A consistent sequence of $\mathrm{GL}_n \mathbb{Q}$–representations or $\mathrm{SL}_n \mathbb{Q}$–representations $\{V_n\}$ is called *mixed representation stable* if Conditions I and II are satisified for large enough n, and if in addition the following condition is satisfied:

MTIII. For all partitions λ and μ, the multiplicity of the mixed tensor representation $V(\lambda;\mu)_n$ in V_n is eventually constant.

Note that mixed tensor stability implies representation stability. As an example of mixed tensor stability, consider the adjoint representation of $\mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$ on $\mathfrak{sl}_n\mathbb{Q}$. This corresponds to the partition $(2,1,\ldots,1,0)$, or to the pseudo-partition $(1,0,\ldots,0,-1)$.

Thus $\mathfrak{sl}_n\mathbb{Q}$ is the mixed tensor representation $V(1;1)_n$. Similarly, the dual $(\mathbb{Q}^n)^*$ of the standard representation is $V(0;1)_n$. In general, the dual of $V(\lambda;\mu)_n$ is $V(\mu;\lambda)_n$, so in particular if a sequence $\{V_n\}$ is representation stable, the sequence of duals $\{V_n^*\}$ is mixed tensor stable. We remark that a sequence which is mixed representation stable is essentially never type-preserving.

Mixed representation stability is used in Sections 6 and 8. This notion was applied by Hanlon [Han] to Lie algebra cohomology over non-unital algebras, and futher applied by R. Brylinski [Bry].

3 Stability in classical representation theory

In this section we discuss examples of representation stability in classical representation theory. We remark that the definition of representation stability itself already relies upon an inherent stability in the classification of irreducible representations of the groups G_n, in the sense that the system of names of representations of the varying groups G_n can be organized in a coherent way.

3.1 Combining and modifying stable sequences

The ubiquity of representation stability would be unlikely were it not that many of the natural constructions in classical representation theory preserve representation stability. We now formalize this. Many of the results follow from well-known classical theorems.

Theorem 3.1. *Suppose that $G_n = \mathrm{SL}_n\,\mathbb{Q}$ and that $\{V_n\}$ and $\{U_n\}$ are multiplicity stable sequences of finite-dimensional G_n–representations. Fix partitions λ and μ. Then the following sequences of G_n–representations are multiplicity stable.*

1. ***Tensor products:*** $\{V_n \otimes U_n\}$.
2. ***Schur functors:*** $\{\mathbb{S}_\lambda(V_n)\}$.
3. ***Schur functors of direct sums:*** $\{\mathbb{S}_\lambda(V_n \oplus U_n)\}$.
4. ***Schur functors of tensor products:*** $\{\mathbb{S}_\lambda(V_n \otimes U_n\})$.
5. ***Compositions of Schur functors:*** $\{\mathbb{S}_\lambda(\mathbb{S}_\mu(V_n))\}$.
   For example, $\{\mathrm{Sym}^r(\bigwedge^s(V_n))\}$ *for each fixed* $r, s \geq 0$.

If G_n is $\mathrm{SL}_n \mathbb{Q}$, $\mathrm{GL}_n \mathbb{Q}$ or $\mathrm{Sp}_{2n} \mathbb{Q}$ and $\{V_n\}$ and $\{U_n\}$ are uniformly multiplicity stable sequences of finite-dimensional G_n–representations, then all the preceding examples are uniformly multiplicity stable, as are the following two examples.

6. ***Shifted sequences:*** *The restrictions $\{V_n \downarrow G_{n-k}\}$ for any fixed $k \geq 0$.*

7. ***Restrictions:*** *The restrictions $\{V_n \downarrow \mathrm{SL}_n \mathbb{Q}\}$ and $\{V_{2n} \downarrow \mathrm{Sp}_{2n} \mathbb{Q}\}$.*

If G_n is $\mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$ and $\{V_n\}$ and $\{U_n\}$ are strongly stable, then the resulting sequences in Parts 1–5 are strongly stable.

We also have a version of Theorem 3.1 for S_n–representations.

Theorem 3.2. *Let $\{V_n\}$ and $\{W_n\}$ be consistent sequences of S_n–representations that are uniformly multiplicity stable. Then the following sequences of S_n–representations are uniformly multiplicity stable.*

1. ***Tensor products:*** $\{V_n \otimes W_n\}$

2. ***Shifted sequences:*** *The restrictions $\{V_n \downarrow S_{n-k}\}$ for any fixed $k \geq 0$.*

Before proving Theorem 3.1 and Theorem 3.2, we give a number of examples in order to illustrate the necessity of various hypotheses in the theorems.

Non-example 3.3. In the final part of the statement of Theorem 3.1, we need to assume strong stability even to conclude standard representation stability of the various combinations of representations. This strong assumption is used via the "type-preserving" condition, and without this assumption stability may not hold. Perhaps surprisingly, the issue is not the stability of the multiplicities, but surjectivity. Here is a simple example which illustrates the problem. This example is not injective, but it can easily be made so; we do this below. Let

$$V_n = V(0)_n \oplus V(1)_n = \mathbb{Q} \oplus \mathbb{Q}^n,$$

with maps $V_n \to V_{n+1}$ defined by

$$\mathbb{Q} \oplus \mathbb{Q}^n \ni (a, v) \mapsto (a, ax_{n+1}) \in \mathbb{Q} \oplus \mathbb{Q}^{n+1}$$

where x_{n+1} is the basis vector $x_{n+1} = (0, \ldots, 0, 1)$. The tensor product $V_n \otimes V_n$ decomposes into irreducibles as $V(0) \oplus V(1)^{\oplus 2} \oplus V(2) \oplus V(1,1)$, where the last two factors come from the decomposition $\mathbb{Q}^n \otimes \mathbb{Q}^n = \mathrm{Sym}^2 \mathbb{Q}^n \oplus \bigwedge^2 \mathbb{Q}^n$. It is easy to check that the

$\mathrm{SL}_{n+1}\,\mathbb{Q}$–span of the image of $V_n \otimes V_n$ in $V_{n+1} \otimes V_{n+1}$ is exactly $V(0) \oplus V(1)^{\oplus 2} \oplus V(2)$; the $\bigwedge^2 \mathbb{Q}^n$ factor is inaccessible.

For an example which is actually representation stable, let

$$V_n = V(0)_n \oplus V(1)_n \oplus V(2)_n = \mathbb{Q} \oplus \mathbb{Q}^n \oplus \mathrm{Sym}^2\,\mathbb{Q}^n$$

with $V_n \hookrightarrow V_{n+1}$ defined by $(a, v, w) \mapsto (a, ax_{n+1}, w + v \cdot x_{n+1})$. The sequence $\{V_n\}$ is consistent and uniformly representation stable. The tensor product $V_n \otimes V_n$ contains $V(1,1) = \bigwedge^2 \mathbb{Q}^n$ with multiplicity 1, but the $\mathrm{SL}_{n+1}\,\mathbb{Q}$ image of $V_n \otimes V_n$ does not contain this factor. Thus the sequence $\{V_n \otimes V_n\}$ is not surjective in the sense of Definition 2.3.

Non-example 3.4. Even when the sequence $\{V_n\}$ is type-preserving (and so strongly stable), restrictions often fail to be surjective. Take $V_n = V(1)_n = \mathbb{Q}^n$, which certainly is strongly stable. The restriction $W_n = V_{n+1} \downarrow \mathrm{GL}_n\,\mathbb{Q}$ splits as $V(1)_n \oplus V(0)_n = \mathbb{Q}^n \oplus \mathbb{Q}$, which is multiplicity stable. But the image of W_n in $W_{n+1} = \mathbb{Q}^{n+2}$ is invariant under $\mathrm{GL}_{n+1}\,\mathbb{Q}$, and so the sequence $\{W_n\}$ is not stable due to the failure of surjectivity.

Proof of Theorem 3.1. In each case, injectivity is either trivial or follows from the functoriality of the Schur functor $\mathbb{S}_\lambda$. The proofs of multiplicity stability generally separate into two parts. First, we check stability when each V_n is a single irreducible; the proof in this case often corresponds to a classical fact of representation theory. Second, we promote this to the general case when $\{V_n\}$ is an arbitrary representation stable sequence. This sometimes requires bootstrapping off the first step of other parts.

To reduce confusion, we have labeled the two steps of the proofs separately as (for example) Parts 1a and 1b. For simplicity, we refer to “stability” and “uniform stability” in the course of the proofs, but we reiterate that we are not claiming surjectivity, so these should properly be references to “multiplicity stability”. We defer the discussion of strong stability until after the claims have been verified in the stable and uniformly stable cases.

1. In general, the problem of decomposing the tensor product of two irreducible representations is called the *Clebsch–Gordan problem.* The quintessential example of stability is the Littlewood–Richardson rule, which answers the Clebsch–Gordan problem for SL_n and shows that the multiplicities in the decomposition are independent of n. Given two partitions λ and μ, and a partition $\nu \vdash |\lambda| + |\mu|$, the *Littlewood–Richardson coefficient* $C^\nu_{\lambda\mu}$ is the number of ways that ν can be obtained as a strict μ–expansion of λ (see [FH, Appendix A] for full definitions). The tensor product then decomposes as

[FH, Equation 6.7]

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus C^\nu_{\lambda\mu}\, \mathbb{S}_\nu(V). \tag{4}$$

1a. We first verify the claim in the case of a single irreducible. We show that in each case, the tensor $V(\lambda)_n \otimes V(\mu)_n$ decomposes as $\bigoplus N^\nu_{\lambda\mu} V(\nu)_n$ for some constant $N^\nu_{\lambda\mu}$ independent of n. For $\mathrm{SL}_n \mathbb{Q}$, by the Littlewood–Richardson rule $V(\lambda)_n \otimes V(\mu)_n$ decomposes as $\bigoplus C^\nu_{\lambda\mu} V(\nu)_n$, so we may take $N^\nu_{\lambda\mu} = C^\nu_{\lambda\mu}$. For $\mathrm{GL}_n \mathbb{Q}$, recall that D denotes the determinant representation, and note that for fixed ℓ, the representation $D^\ell \otimes V_n$ is stable if and only if V_n is stable. Every irreducible $V(\lambda)_n$ can be written as $\mathbb{S}_{\overline{\lambda}} \mathbb{Q}^n \otimes D^\ell$ for a unique partition $\overline{\lambda}$ and integer ℓ (namely $\ell = \lambda_n$ and $\overline{\lambda}_i = \lambda_i - \lambda_n$). Then

$$V(\lambda)_n \otimes V(\mu)_n = \mathbb{S}_{\overline{\lambda}}(\mathbb{Q}^n) \otimes D^\ell \otimes \mathbb{S}_{\overline{\mu}}(\mathbb{Q}^n) \otimes D^m = D^{\ell+m} \otimes \bigoplus C^{\overline{\nu}}_{\overline{\lambda}\overline{\mu}}\, \mathbb{S}_{\overline{\nu}}(\mathbb{Q}^n).$$

The decomposition of the right side into irreducibles $V(\nu)_n = D^{\ell+m} \otimes \mathbb{S}_{\overline{\nu}}(\mathbb{Q}^n)$ is independent of n. Thus we may take $N^\nu_{\lambda\mu} = C^{\overline{\nu}}_{\overline{\lambda}\overline{\mu}}$ for those ν with $\nu_n = \lambda_n + \mu_n$, and $N^\nu_{\lambda\mu} = 0$ otherwise.

For $\mathrm{Sp}_{2n} \mathbb{Q}$, the corresponding formula is [FH, Equation 25.27]:

$$\mathbb{S}_{\langle\lambda\rangle}(\mathbb{Q}^{2n}) \otimes \mathbb{S}_{\langle\mu\rangle}(\mathbb{Q}^{2n}) = \bigoplus \sum_{\zeta,\sigma,\tau} C^\lambda_{\zeta\sigma} C^\mu_{\zeta\tau} C^\nu_{\sigma\tau}\, \mathbb{S}_{\langle\nu\rangle}(\mathbb{Q}^{2n}) \tag{5}$$

where the sum is over all partitions ζ, σ, τ. Thus $N^\nu_{\lambda\mu} = \sum_{\zeta,\sigma,\tau} C^\lambda_{\zeta\sigma} C^\mu_{\zeta\tau} C^\nu_{\sigma\tau}$.

1b. Now consider arbitrary consistent sequences $\{V_n\}$ and $\{U_n\}$. If these sequences are uniformly representation stable, their decompositions $V_n = \bigoplus c_{\lambda,n} V(\lambda)_n$ and $U_n = \bigoplus d_{\mu,n} V(\mu)_n$ are eventually independent of n. Thus the decomposition of the tensor product as

$$V_n \otimes U_n = \left(\bigoplus c_{\lambda,n} V(\lambda)_n\right) \otimes \left(\bigoplus d_{\mu,n} V(\mu)_n\right) = \bigoplus_\nu \sum_{\lambda,\mu} c_{\lambda,n} d_{\mu,n} N^\nu_{\lambda\mu} V(\nu)_n$$

is eventually independent of n.

For $\mathrm{SL}_n \mathbb{Q}$ the assumption of uniform stability is not necessary. In this case $N^\nu_{\lambda\mu}$ is the Littlewood–Richardson coefficient, which is nonzero only if $|\nu| = |\lambda| + |\mu|$. Thus for fixed ν, only finitely many pairs (λ, μ) can contribute to the $\sum_{\lambda,\mu} c_{\lambda,n} d_{\mu,n} N^\nu_{\lambda\mu} V(\nu)_n$ term above. Thus we may take n large enough that these finitely many coefficients $c_{\lambda,n}$ and $d_{\mu,n}$ are all independent of n, and the multiplicity $\sum_{\lambda,\mu} c_{\lambda,n} d_{\mu,n} N^\nu_{\lambda\mu}$ of $V(\nu)_n$ is eventually independent of n as desired.

2a. The classical *plethysm* problem is to decompose the composition of two Schur functors:

$$\mathbb{S}_\lambda(\mathbb{S}_\mu V) = \bigoplus M^\nu_{\lambda\mu}\, \mathbb{S}_\nu V \tag{6}$$

To compute the coefficients $M^\nu_{\lambda\mu}$ is difficult, but it is known that such coefficients exist, and are nonzero only when $|\nu| = |\lambda| \cdot |\mu|$ [FH, Exercise 6.17a]. It immediately follows that the sequence $\{\mathbb{S}_\lambda(V(\mu)_n\}$ of $\mathrm{SL}_n\,\mathbb{Q}$–representations $\mathbb{S}_\lambda(V(\mu)_n) = \bigoplus M^\nu_{\lambda\mu} V(\nu)_n$ is representation stable. For $\mathrm{GL}_n\,\mathbb{Q}$, write $V(\mu)_n = \mathbb{S}_{\overline{\mu}}\,\mathbb{Q}^n \otimes D^\ell$ for some partition λ and integer ℓ. Note that in general, if ρ acts on V diagonally by multiplication by R, then the action of ρ on $\mathbb{S}_\lambda V$ will be multiplication by $R^{|\lambda|}$. Since the center of $\mathrm{GL}_n\,\mathbb{Q}$ acts diagonally, it follows that

$$\mathbb{S}_\lambda(V(\mu)_n) = \mathbb{S}_\lambda(\mathbb{S}_{\overline{\mu}}(\mathbb{Q}^n) \otimes D^\ell) = \mathbb{S}_\lambda(\mathbb{S}_{\overline{\mu}}(\mathbb{Q}^n)) \otimes D^{\ell|\lambda|} = \bigoplus M^{\overline{\nu}}_{\lambda\overline{\mu}}\, \mathbb{S}_{\overline{\nu}}(\mathbb{Q}^n) \otimes D^{\ell|\lambda|}$$

and thus $\{\mathbb{S}_\lambda(V(\mu)_n)\}$ is representation stable. For the symplectic group the stability of the plethysm

$$\mathbb{S}_\lambda(\mathbb{S}_{\langle\mu\rangle}(\mathbb{Q}^{2n})) = \bigoplus L^\nu_{\lambda\mu}\, \mathbb{S}_{\langle\nu\rangle}(\mathbb{Q}^{2n})$$

was only proved recently by Kabanov [Kab, Theorem 7]. If μ has $\ell = \ell(\mu)$ rows, the coefficients $L^\nu_{\lambda\mu}$ are independent of n once $n \geq \ell|\lambda|$ and are nonzero only for those ν with at most $\ell|\lambda|$ rows.

2b and 3. We now verify Parts 2 and 3 in parallel by induction on total multiplicity. We do this first under the assumption of uniform stability, so that total multiplicity is well-defined. We then explain how to extend this to all representation stable sequences in the case of $\mathrm{SL}_n\,\mathbb{Q}$. In general, when a Schur functor is applied to a direct sum we have the decomposition [FH, Exercise 6.11]

$$\mathbb{S}_\lambda(V \oplus U) = \bigoplus C^\lambda_{\mu\nu}(\mathbb{S}_\mu V \otimes \mathbb{S}_\nu U). \tag{7}$$

We have already verified Part 2 when V_n has total multiplicity 1, and Part 3 reduces to Part 2 when $V_n \oplus U_n$ has total multiplicity 1. We now prove Part 3 when $V_n \oplus U_n$ has total multiplicity k by strong induction. Assume that $\{V_n\}$ and $\{U_n\}$ are uniformly representation stable sequences, and that neither is eventually zero. So we may assume that Part 2 of the theorem holds for $\{V_n\}$ and for $\{U_n\}$ by induction. By Part 2 we have that $\{\mathbb{S}_\mu V_n\}$ and $\{\mathbb{S}_\nu U_n\}$ are each uniformly stable. By Part 1, the tensor product $\{\mathbb{S}_\mu V_n \otimes \mathbb{S}_\nu U_n\}$ is uniformly stable. Thus the sum

$$\mathbb{S}_\lambda(V_n \oplus U_n) = \bigoplus C^\lambda_{\mu\nu}(\mathbb{S}_\mu V_n \otimes \mathbb{S}_\nu U_n)$$

is uniformly stable, verifying Part 3. To verify Part 2 when V_n has total multiplicity k, write $V_n = U_n \oplus W_n$ with each factor uniformly stable and apply Part 3. Although the splitting $V_n = U_n \oplus W_n$ might not respect the maps $V_n \to V_{n+1}$, we are only concerned with multiplicities at this point so this is not a problem. When we revisit this issue later, it will be under the assumption of strong stability, in which case $\{V_n\}$ does split as a sum of consistent, strongly stable sequences $\{U_n\}$ and $\{W_n\}$.

We now consider the case when $G_n = \mathrm{SL}_n \mathbb{Q}$ and the sequences are not necessarily uniformly stable. For a fixed finite-dimensional $\mathrm{SL}_n \mathbb{Q}$–representation $V = \bigoplus c_\eta V(\eta)$, consider decomposing $\mathbb{S}_\lambda(V) = \mathbb{S}_\lambda(\bigoplus c_\eta V(\eta))$ by repeatedly applying the formula (7) for $\mathbb{S}_\lambda(V \oplus W)$. We obtain a decomposition of the form

$$\mathbb{S}_\lambda(V) = \bigoplus X_\bullet \, \mathbb{S}_{\mu_\bullet} V(\eta_\bullet) \otimes \cdots \otimes \mathbb{S}_{\mu_\bullet} V(\eta_\bullet),$$

where the $V(\eta_\bullet)$ range over the irreducible summands of V. Consider the individual terms $\mathbb{S}_\mu V(\eta)$. As we noted above, the decomposition $\mathbb{S}_\mu V(\eta) = \bigoplus M^\zeta_{\mu\eta} V(\zeta)$ only contains those $V(\zeta)$ with $|\zeta| = |\mu| \cdot |\eta|$. Furthermore, recall that the coefficients $C^\lambda_{\mu\nu}$ are only nonzero if $|\mu| + |\nu| = |\lambda|$ (this is where we use that $G_n = \mathrm{SL}_n \mathbb{Q}$). It follows that the irreducibles $V(\nu)$ appearing in a tensor $V(\zeta_1) \otimes \cdots V(\zeta_k)$ all satisfy $|\nu| = |\zeta_1| + \cdots + |\zeta_k|$. Combining this, we obtain the key point of the argument: when considering the multiplicity of $V(\nu)$ in $\mathbb{S}_\lambda(\bigoplus c_\eta V(\eta))$, *we need only consider those $V(\eta)$ with $|\eta| \le |\nu|$.*

Using this observation, we reduce this case to the uniformly stable case as follows. For fixed ν, replace $V_n = \bigoplus c_{\eta,n} V(\eta)$ with

$$V_n^{\le \nu} = \bigoplus_{|\eta| \le |\nu|} c_{\eta,n} V(\eta).$$

If the sequence $\{V_n\}$ is representation stable, then since only finitely many η satisfy $|\eta| \le |\nu|$, the sequence $\{V_n^{\le \nu}\}$ is uniformly stable. Thus applying Part 2, we conclude that $\{\mathbb{S}_\lambda(V_n^{\le \nu})\}$ is uniformly stable; in particular, the multiplicity of $V(\nu)$ is eventually constant. By the observation, this is the same as the multiplicity of $V(\nu)$ in $\mathbb{S}_\lambda(V_n)$. Thus $\{\mathbb{S}_\lambda(V_n)\}$ is multiplicity stable, as desired.

4. In general, when a Schur functor is applied to a tensor product, we have the decomposition [FH, Exercise 6.11b]

$$\mathbb{S}_\lambda(V \otimes W) = \bigoplus D^\lambda_{\mu\nu} (\mathbb{S}_\mu V \otimes \mathbb{S}_\nu W),$$

where the sum is over partitions with $|\mu| = |\nu| = |\lambda|$ and the coefficients are defined as follows. Let $d = |\lambda|$; given a partition $\eta \vdash d$, let C_η be the conjugacy class in S_d

whose cycle decomposition is encoded by η. Let χ_λ be the character of the irreducible S_d–representation $V(\lambda)$. Then

$$D^\lambda_{\mu\nu} = \sum_{\eta\vdash d} \frac{\chi_\lambda(C_\eta)\chi_\mu(C_\eta)\chi_\nu(C_\eta)}{|Z_{S_d}(C_\eta)|}$$

where $Z_{S_d}(C_\eta)$ is the centralizer in S_d of a representative of C_η. Now assume that $\{V_n\}$ and $\{U_n\}$ are uniformly stable, and consider $\mathbb{S}_\lambda(V_n \otimes U_n)$. By Part 2, $\{\mathbb{S}_\mu V_n\}$ and $\{\mathbb{S}_\nu U_n\}$ are uniformly stable for each μ and ν. By Part 1 we have that $\{\mathbb{S}_\mu V_n \otimes \mathbb{S}_\nu U_n\}$ is uniformly stable. Thus the sum

$$\mathbb{S}_\lambda(V_n \oplus U_n) = \bigoplus D^\lambda_{\mu\nu}(\mathbb{S}_\mu U_n \otimes \mathbb{S}_\nu W_n)$$

is uniformly stable. The case when $G_n = \mathrm{SL}_n \mathbb{Q}$ and uniform stability is not assumed proceeds exactly as in Part 2, since the coefficents $D^\lambda_{\mu\nu}$ are only nonzero if $|\mu| = |\nu| = |\lambda|$.

5. Stability for the composition of Schur functors $\mathbb{S}_\lambda(\mathbb{S}_\mu(V_n))$ can be deduced from the plethysm decomposition in (6), or just by applying Part 2 twice, first to $\{\mathbb{S}_\mu(V_n)\}$ and then to $\{\mathbb{S}_\lambda(\mathbb{S}_\mu(V_n))\}$.

6. For the restriction of $V(\lambda)_n = \mathbb{S}_\lambda(\mathbb{Q}^n)$ from $\mathrm{GL}_n \mathbb{Q}$ to $\mathrm{GL}_{n-k} \mathbb{Q}$, the restriction decomposes as [FH, Exercise 6.12]

$$\mathbb{S}_\lambda(\mathbb{Q}^n) \downarrow \mathrm{GL}_{n-k} \mathbb{Q} = \bigoplus_\nu \big(\sum_\mu C^\lambda_{\mu\nu} \dim \mathbb{S}_\mu(\mathbb{Q}^k)\big)\, \mathbb{S}_\nu(\mathbb{Q}^{n-k}). \tag{8}$$

Note that $\dim \mathbb{S}_\mu(\mathbb{Q}^k)$ does not depend on n. The claim for a single irreducible representation of $\mathrm{SL}_n \mathbb{Q}$ immediately follows:

$$V(\lambda)_n \downarrow \mathrm{SL}_{n-k} \mathbb{Q} = \bigoplus_\nu \big(\sum_\mu C^\lambda_{\mu\nu} \dim \mathbb{S}_\mu(\mathbb{Q}^k)\big) V(\nu)_{n-k}$$

For $\mathrm{GL}_{n-k} \mathbb{Q}$ it follows after noting that the determinant representation restricts to the determinant representation: $D \downarrow \mathrm{GL}_{n-k} \mathbb{Q} = D$, so if $V(\lambda)_n = \mathbb{S}_{\overline{\lambda}}(\mathbb{Q}^n) \otimes D^\ell$ we get

$$\mathbb{S}_{\overline{\lambda}}(\mathbb{Q}^n) \otimes D^\ell \downarrow \mathrm{GL}_{n-k} \mathbb{Q} = \bigoplus_{\overline{\nu}} \big(\sum_\mu C^{\overline{\lambda}}_{\mu\overline{\nu}} \dim \mathbb{S}_\mu(\mathbb{Q}^k)\big)\, \mathbb{S}_{\overline{\nu}}(\mathbb{Q}^{n-k}) \otimes D^\ell.$$

The claim for a uniformly stable sequence $\{V_n\}$ follows by taking n large enough that the decomposition $V_n = \bigoplus c_{\lambda,n} V(\lambda)_n$ is independent of n. Note that uniform stability is necessary here even for $\mathrm{SL}_n \mathbb{Q}$. Indeed, from (8) we see that for every partition λ

with $\ell(\lambda) \leq k$, the restriction $\mathbb{S}_\lambda(\mathbb{Q}^n) \downarrow \mathrm{SL}_{n-k}\mathbb{Q}$ contains the trivial representation with multiplicity $\dim \mathbb{S}_\lambda(\mathbb{Q}^k)$. Thus the multiplicity of $V(0)$ in $V_n \downarrow \mathrm{SL}_{n-k}\mathbb{Q}$ is at least the total multiplicity of subrepresentations $V(\lambda)_n$ of V_n with $\ell(\lambda) \leq k$, which need not be eventually constant if we do not assume uniform stability.

For $\mathrm{Sp}_{2n}\mathbb{Q}$, we consider the restriction to $\mathrm{Sp}_{2n-2}\mathbb{Q}$. For a single irreducible representation $V(\lambda)_n$ of $\mathrm{Sp}_{2n}\mathbb{Q}$, the restriction decomposes as [FH, Equation 25.36]

$$V(\lambda)_n \downarrow \mathrm{Sp}_{2n-2}\mathbb{Q} = \bigoplus_\nu N_\lambda^\nu V(\nu)_{n-1}, \tag{9}$$

where the sum is over partitions with $\nu_n = 0$. The coefficient N_λ^ν is the number of sequences $p_1, \ldots, p_n$ satisfying:

$$\begin{aligned} \lambda_1 \geq p_1 \geq \lambda_2 \geq p_2 \geq \cdots \geq \lambda_n \geq p_n \\ p_1 \geq \nu_1 \geq p_2 \geq \cdots \geq \nu_{n-1} \geq p_n \geq \nu_n = 0 \end{aligned}$$

Note that if $\lambda_k = 0$, then $p_i = 0$ for $i \geq k$, and thus any ν contributing to this sum has $\nu_i = 0$ for $i > k$. It follows that for fixed λ, the collection of ν that contribute to this sum is independent of n once $n \geq \ell(\lambda)$, so the collection of $p_1, \ldots, p_n$ above and the multiplicities N_λ^ν are also eventually independent of n. Thus $\{V(\lambda)_n \downarrow \mathrm{Sp}_{2n-2}\mathbb{Q}\}$ is stable, and as above it follows that $\{V_n \downarrow \mathrm{Sp}_{2n-2}\mathbb{Q}\}$ is uniformly stable if $\{V_n\}$ is uniformly stable. Uniform stability for the restriction to $\mathrm{Sp}_{2n-2k}\mathbb{Q}$ now follows by induction.

7. Every irreducible $\mathrm{GL}_n\mathbb{Q}$–representation $V(\lambda)_n$ remains irreducible when restricted to $\mathrm{SL}_n\mathbb{Q}$; the resulting representation is $V(\overline{\lambda})_n$, where $\overline{\lambda}$ is the partition defined by $\overline{\lambda}_i = \lambda_i - \lambda_n$. If $V_n = \bigoplus_\lambda c_{\lambda,n} V(\lambda)_n$, the restriction $V_n \downarrow \mathrm{SL}_n\mathbb{Q}$ is $\bigoplus_\mu \sum_\lambda c_{\lambda,n} V(\mu)$ where the sum is over those λ with $\overline{\lambda} = \mu$. For fixed μ, the collection of such λ is independent of n. Thus if $\{V_n\}$ is uniformly stable and $c_{\lambda,n}$ are eventually independent of n, the same is true of the multiplicities $\sum_\lambda c_{\lambda,n}$ of $V(\mu)_n$.

It thus suffices to consider the restriction from $\mathrm{SL}_{2n}\mathbb{Q}$ to $\mathrm{Sp}_{2n}\mathbb{Q}$. Littlewood proved that if λ is a partition with at most n rows, the restriction of the irreducible $V(\lambda)_{2n} = \mathbb{S}_\lambda(\mathbb{Q}^{2n})$ decomposes as [FH, Equation 25.39]

$$\mathbb{S}_\lambda(\mathbb{Q}^{2n}) \downarrow \mathrm{Sp}_{2n}\mathbb{Q} = \bigoplus_\mu \sum_\eta C_{\eta\mu}^\lambda \mathbb{S}_{\langle\mu\rangle}(\mathbb{Q}^{2n}), \tag{10}$$

where the sum is over all partitions $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \cdots)$ where each number appears an even number of times. Note that this formula is independent of n once $n \geq \ell(\lambda)$ (so that the formula applies). Thus the sequence $\{V(\lambda)_{2n} \downarrow \mathrm{Sp}_{2n}\mathbb{Q}\}$ is representation stable. If $\{V_n = \bigoplus c_{\lambda,n} V(\lambda)_n\}$ is uniformly stable, let N be the largest number of rows of any partition λ for which the eventual multiplicity of $V(\lambda)_n$ is positive. Assume that the decomposition of V_n stabilizes once $n \geq N$. Then for $n \geq N$, we may apply Littlewood's rule to conclude that

$$V_n \downarrow \mathrm{Sp}_{2n} \mathbb{Q} = \bigoplus_{\mu} \sum_{\lambda,\eta} c_{\lambda,n} C^{\lambda}_{\eta\mu} V(\mu)_n$$

is independent of n. We conclude that the sequence $\{V_n \downarrow \mathrm{Sp}_{2n} \mathbb{Q}\}$ is uniformly representation stable.

Strong stability. We now consider the case when $G_n = \mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$ and $\{V_n\}$ and $\{U_n\}$ are strongly stable, meaning they are not only uniformly stable but also type-preserving. Recall that for $G_n = \mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$, this implies Condition IV′: that $P_{n+1} < G_{n+1}$ acts trivially on $V_n \subset V_{n+1}$. This property is preserved by direct sum and by tensor product: if P_{n+1} acts trivially on $V_n \subset V_{n+1}$ and on $U_n \subset U_{n+1}$, it acts trivially on $V_n \oplus U_n \subset V_{n+1} \oplus U_{n+1}$ and $V_n \otimes U_n \subset V_{n+1} \otimes U_{n+1}$. The functoriality of $\mathbb{S}_\lambda$ implies that if P_{n+1} acts trivially on $V_n \subset V_{n+1}$, it acts trivially on $\mathbb{S}_\lambda(V_n) \subset \mathbb{S}_\lambda(V_{n+1})$. The constructions in Parts 1–5 are obtained by composing these operations, so we conclude that Condition IV′ holds for each of the resulting sequences.

We have already proved above that the resulting sequences in Parts 1–5 are uniformly multiplicity stable (Condition III′). Thus we may apply Proposition 2.14 to conclude that these sequences are type preserving (Condition IV). Finally, by Remark 2.13 Conditions III′ and IV together imply surjectivity (Condition II). This concludes the proof of strong stability in Parts 1–5, and thus completes the proof of Theorem 3.1. □

Proof of Theorem 3.2.

1. For irreducible representations $V(\lambda)_n$ and $V(\mu)_n$ of S_n, Murnaghan proved that the decomposition of the tensor product $V(\lambda)_n \otimes V(\mu)_n$ into irreducibles $V(\nu)_n$ is eventually independent of n (see Section 1 of [Mu]), and Briand–Orellana–Rosas have recently proved that the decomposition of $V(\lambda)_n \otimes V(\mu)_n$ stabilizes once $n \geq |\lambda| + |\mu| + \lambda_1 + \mu_1$ [BOR, Theorem 1.2]. If $\{V_n = \bigoplus c_{\lambda,n} V(\lambda)_n\}$ and $\{W_n = \bigoplus d_{\mu,n} V(\mu)_n\}$ are uniformly multiplicity stable, taking n large enough that the decomposition of $V(\lambda)_n \otimes V(\mu)_n$ stabilizes for all λ and ν occurring in V_n and W_n, it follows by distributivity that $\{V_n \otimes W_n\}$ is uniformly multiplicity stable.

2. For $k = 1$, we repeat from Remark 2.13 the branching rule for restrictions from S_n to S_{n-1}:

$$V(\lambda)_n \downarrow S_{n-1} = V(\lambda)_{n-1} \oplus \bigoplus_{\mu} V(\mu)_{n-1}$$

where μ ranges over those partitions obtained by removing one box from λ. It is immediate that uniform multiplicity stability is preserved. For restrictions from S_n to S_{n-k} with $k > 1$ a similar formula can be given explicitly [FH, Exercise 4.44], but to conclude stability we can just inductively apply the result for $k = 1$. $\square$

3.2 Reversing the Clebsch–Gordan problem

We conclude this section by discussing the possibility of reversing the conclusion of Theorem 3.1(1). This idea will play an important role in Section 5.

Theorem 3.5. *Let $G_n = \mathrm{SL}_n \mathbb{Q}$, $\mathrm{GL}_n \mathbb{Q}$, or $\mathrm{Sp}_{2n} \mathbb{Q}$. If $\{W_n\}$ and $\{V_n \otimes W_n\}$ are nonzero and multiplicity stable as G_n–representations, then $\{V_n\}$ is multiplicity stable. This remains true if "multiplicity stable" is replaced by "uniformly multiplicity stable".*

Proof. We will prove the theorem in the following form: given the irreducible decompositions of W and of $V \otimes W$, the irreducible decomposition of V can be determined, and without reference to n. This will be formalized in the course of the proof. In the remark following the proof, we sketch a constructive way to determine the decomposition of V. But the theorem as stated follows from more general properties, as we now explain.

First, we will use that the representation ring is a domain for any such group G_n. Recall that the *representation ring* R_n consists of formal differences $V - U$ of representations of G_n, with addition given by direct sum and multiplication given by tensor product. Complete reducibility implies that as a group, R_n is the free abelian group on the irreducible representations. The ring structure is more complicated, but we can in fact describe R_n explicitly. Indeed, let Λ_n be the weight lattice in $\mathfrak{h}_n^*$. Any representation V determines a "character" in the group ring $\mathbb{Z}[\Lambda_n]$, where the coefficient of the weight $L \in \Lambda_n$ is the dimension of the eigenspace $V^{(L)}$:

$$V \mapsto \sum_{L \in \Lambda_n} \dim V^{(L)} \cdot L$$

The highest weight decomposition as described in Section 2.2 implies that a representation is determined by its character; that is, the induced ring homomorphism $R_n \hookrightarrow \mathbb{Z}[\Lambda_n]$ is injective. This would suffice for our purposes, but we can say more: any such character is invariant under the Weyl group W_n, and in fact R_n is exactly the subring $\mathbb{Z}[\Lambda_n]^{W_n}$ of invariants ([FH], Theorem 23.24, combined with Exercise 23.36(d) for $\mathrm{GL}_n \mathbb{Q}$).

Since R_n is a domain, V_n is the unique solution in R_n to the equation $x \cdot [W_n] = [V_n \otimes W_n]$. It remains to see that given the decompositions of W_n and $V_n \otimes W_n$, the solution to this equation does not depend on n. To do this, we need to relate the representation rings R_n and R_{n+1}. There is a natural homomorphism $R_{n+1} \to R_n$ given by restriction from G_{n+1} to G_n, but this is *not* the map we want, since restriction does not take irreducibles to irreducibles. Instead, assume first that $G_n = \mathrm{SL}_n \mathbb{Q}$; by identifying $V(\lambda)_n$ with λ, we get an identification of R_n with the free abelian group $\mathbb{Z}[\{\lambda | \ell(\lambda) < n\}]$ on partitions with fewer than n rows. The map we want is simply the projection

$$R_{n+1} = \mathbb{Z}[\{\lambda | \ell(\lambda) < n+1\}] \overset{\pi}{\twoheadrightarrow} \mathbb{Z}[\{\lambda | \ell(\lambda) < n\}] = R_n$$

which sends $\lambda \mapsto 0$ if $\ell(\lambda) = n$ and $\lambda \mapsto \lambda$ otherwise. With respect to this basis of partitions, multiplication in the ring is given by the Littlewood–Richardson coefficients; in a sense, Theorem 3.1(1) is based on the fact that this map is a ring homomorphism. This projection has a right inverse $i \colon R_n \to R_{n+1}$ defined by the inclusion $\{\lambda | \ell(\lambda) < n\} \subset \{\lambda | \ell(\lambda) < n+1\}$; this is not a ring homomorphism, however. Note that uniform multiplicity stability is equivalent to $i([V_n]) = [V_{n+1}]$ for large enough n.

Assume that the sequences in question are uniformly stable, and that n is large enough that the decompositions of W_n and $V_n \otimes W_n$ have stabilized, meaning

$$i([W_n]) = [W_{n+1}] \quad \text{and} \quad i([V_n \otimes W_n]) = [V_{n+1} \otimes W_{n+1}].$$

Then we have $\pi([W_{n+1}]) = [W_n]$ and $\pi([V_{n+1} \otimes W_{n+1}]) = [V_n \otimes W_n]$. Thus $[V_{n+1}]$ projects to a solution of $x \cdot [W_n] = [V_n \otimes W_n]$, so by uniqueness we have $\pi([V_{n+1}]) = [V_n]$.

We want to prove that $i([V_n]) = [V_{n+1}]$. Suppose not; that is, assume the difference $[V_{n+1}] - i([V_n])$ is not zero. Since $\pi([V_{n+1}]) = [V_n]$, this difference consists of all those irreducibles $V(\lambda)_{n+1}$ contained in V_{n+1} having $\ell(\lambda) = n$. It is easy to check from the definition of the Littlewood–Richardson coefficients that if a representation V contains such a $V(\lambda)_{n+1}$, then for any nonzero representation W the tensor $V \otimes W$ also contains some $V(\mu)_{n+1}$ with $\ell(\mu) = n$. Applying this to $V_{n+1} \otimes W_{n+1}$ gives that $[V_{n+1} \otimes W_{n+1}] \neq i([V_n \otimes W_n])$, contradicting the uniform stability of $V_n \otimes W_n$. We conclude that $[V_{n+1}] = i([V_n])$ and so $\{V_n\}$ is uniformly stable, as desired.

For $G_n = \mathrm{Sp}_{2n} \mathbb{Q}$, the argument proceeds identically, except that R_n is identified with the free abelian group $\mathbb{Z}[\{\lambda | \ell(\lambda) \leq n\}]$ on partitions with at most n rows. We can deduce from (5) the desired property that if $\ell(\lambda) = n+1$, $V(\lambda)_{n+1} \otimes W$ contains some $V(\mu)_{n+1}$

with $\ell(\mu) = n + 1$. For $G_n = \mathrm{GL}_n \mathbb{Q}$, R_n is the free abelian group $\mathbb{Z}[\{\lambda | \ell(\lambda) \leq n\}]$ on *pseudo*-partitions with at most n rows, and we also have to modify the maps between R_n and R_{n+1}. In this case the inclusion $i\colon R_n \to R_{n+1}$ takes $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ to $i(\lambda) = (\lambda_1 \geq \cdots \geq \lambda_n \geq \lambda_n)$; the projection $\pi\colon R_{n+1} \to R_n$ sends the pseudo-partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq \lambda_{n+1})$ to 0 if $\lambda_n \neq \lambda_{n+1}$, and to $\pi(\lambda) = (\lambda_1 \geq \cdots \geq \lambda_n)$ if $\lambda_n = \lambda_{n+1}$. The argument above then goes through, with the role of the partitions with $\ell(\lambda) = n$ played by the pseudo-partitions having $\lambda_n \neq \lambda_{n+1}$.

We sketch a compactness argument to extend this to the case when the sequences are multiplicity stable but not uniformly so. Consider for example the ideal of R_n spanned by partitions λ with $|\lambda| > k$. The corresponding quotients have basis the partitions λ with $|\lambda| \leq k$. Since this set is finite, the corresponding subset of the multiplicities converges uniformly, and we can apply the argument above. Letting $k \to \infty$, we conclude that $\{V_n\}$ is multiplicity stable. □

Remark 3.6. A related theorem also holds: if $\{V_n \otimes V_n\}$ is multiplicity stable, then $\{V_n\}$ is multiplicity stable, and similarly for uniform stability. Both this claim and Theorem 3.5 can be proved constructively, by an algorithm which we now sketch. The Littlewood–Richardson coefficients have the following property with respect to the lexicographic order on partitions: given λ and μ, the largest partition occurring in $V(\lambda) \otimes V(\mu)$ is $\lambda + \mu$, with multiplicity 1. Thus the largest partition occurring in $V_n \otimes V_n$ will be $\lambda + \lambda$, where λ is the largest partition occurring in V_n. The next largest must be $\lambda + \mu$, where μ is the next largest partition in V_n. Continue, at each stage finding the largest irreducible in $V_n \otimes V_n$ not yet accounted for by those irreducibles already found. The algorithm pivots on partitions of the form $\lambda + \mu$, so since $\ell(\lambda) \leq \ell(\lambda + \mu)$ and $\ell(\mu) \leq \ell(\lambda + \mu)$, the steps in the algorithm will not depend on n (if the sequences are not uniformly stable, we also need a compactness argument as above).

4 Cohomology of pure braid and related groups

Let P_n denote the pure braid group, as discussed in the introduction. As explained there, the action of S_n on the configuration space X_n makes each cohomology group $H^i(X_n; \mathbb{Q}) = H^i(P_n; \mathbb{Q})$ into an S_n–representation. Explicit formulas for the multiplicity of an irreducible $V(\lambda)$ in $H^i(P_n; \mathbb{Q})$ are not known. However, we do have the following.

Theorem 4.1. *For each fixed $i \geq 0$, the sequence of S_n–representations $\{H^i(P_n; \mathbb{Q})\}$ is uniformly representation stable, and in fact stabilizes once $n \geq 4i$.*

For example, for $n \geq 4$,

$$H^1(P_n;\mathbb{Q}) = V(0) \oplus V(1) \oplus V(2)$$

and thanks to computations by Hemmer, for $n \geq 7$ we have:

$$H^2(P_n;\mathbb{Q}) = V(1)^{\oplus 2} \oplus V(1,1)^{\oplus 2} \oplus V(2)^{\oplus 2} \oplus V(2,1)^{\oplus 2} \oplus V(3) \oplus V(3,1). \tag{11}$$

As mentioned in the introduction, it is tempting to guess that the reason for the stability in (11) is that each factor $V(\lambda) \subset H^2(P_n;\mathbb{Q})$ has S_{n+1}–span inside $H^2(P_{n+1};\mathbb{Q})$ isomorphic to $V(\lambda)$. In the terminology of §2.4, we would hope that the natural homomorphism $H^2(P_n;\mathbb{Q}) \to H^2(P_{n+1};\mathbb{Q})$ is type-preserving and so the sequence $\{H^2(P_n;\mathbb{Q})\}$ is strongly stable. However, this is false for $\{H^2(P_n;\mathbb{Q})\}$ and indeed is false for $\{H^i(P_n;\mathbb{Q})\}$ for every $i \geq 1$.

We can see this failure explicitly for $H^1(P_n;\mathbb{Q})$ as follows. We will see below that $H^1(P_n;\mathbb{Q})$ has basis $\{w_{ij} | 1 \leq i < j \leq n\}$; after identifying $w_{ji} = w_{ij}$, the group S_n acts on this basis by permuting the indices. Thus the unique trivial subrepresentation $V(0) \subset H^1(P_n;\mathbb{Q})$ is spanned by the vector

$$v = \sum_{1 \leq i < j \leq n} w_{ij}.$$

This vector is, up to a scalar, the sum $\sum_{\sigma \in S_n} \sigma \cdot w_{12}$, and thus is certainly S_n–invariant. But after including it into $H^1(P_{n+1};\mathbb{Q})$ this vector is not invariant under S_{n+1} (for example, it does not involve any basis elements of the form $w_{i,n+1}$). In fact, it is not too hard to check that the S_{n+1}–span of this vector is $V(0) \oplus V(1) \subset H^1(P_{n+1};\mathbb{Q})$.

In trying to prove Theorem 4.1, we were able to use work of Orlik–Solomon [OS] and Lehrer–Solomon [LS] to reduce the problem to a stability statement for certain induced representations of symmetric groups. We conjectured the following theorem to D. Hemmer in certain special cases. The theorem was then proved by Hemmer in much greater generality than we had hoped. This result itself provides another example of representation stability.

We begin by presenting Hemmer's result, which we will use in the proof of Theorem 4.1. Fix a subgroup H of the symmetric group S_k, and fix any representation V of H. For $n \geq k$ we may extend the action of H on V to the subgroup $H \times S_{n-k} < S_n$ by letting S_{n-k} act trivially on V; this representation of $H \times S_{n-k}$ is denoted $V \boxtimes \mathbb{Q}$. Finally, we may consider the induced representation $\mathrm{Ind}_{H \times S_{n-k}}^{S_n}(V \boxtimes \mathbb{Q})$, which is a representation of S_n.

Theorem 4.2 (Hemmer [He])**.** *Fix $k \geq 1$, a subgroup $H < S_k$, and a representation V of H. Then the sequence of S_n–representations $\{\mathrm{Ind}_{H\times S_{n-k}}^{S_n}(V \boxtimes \mathbb{Q})\}$ is uniformly representation stable. The decomposition of this sequence stabilizes once $n \geq 2k$.*

Injectivity and surjectivity are immediate from the definition of induced representation; indeed $\mathrm{Ind}_{H\times S_{n-k}}^{S_n}(V \boxtimes \mathbb{Q})$ sits inside $\mathrm{Ind}_{H\times S_{n-k+1}}^{S_{n+1}}(V \boxtimes \mathbb{Q})$ as the S_n–span of $V \boxtimes \mathbb{Q}$. For the proof of uniform multiplicity stability, see Hemmer [He].

4.1 Stability of the cohomology of pure braid groups

With the above tool in hand, we can now prove representation stability for $\{H^i(P_n;\mathbb{Q})\}$ for each fixed $i \geq 0$.

Proof of Theorem 4.1. We continue with the notation given in the introduction. The projections of configuration spaces $X_{n+1} \to X_n$ given by forgetting the last coordinate give surjections $\psi_n\colon P_{n+1} \to P_n$. These surjections induce maps

$$\psi_n^*\colon H^*(P_n;\mathbb{Q}) \to H^*(P_{n+1};\mathbb{Q}).$$

We will prove representation stability with respect to these maps. For each pair $j \neq k$, let $w_{jk} \in H^1(P_n;\mathbb{Q})$ be the cohomology class represented by the differential form $\frac{1}{2\pi i}\frac{dz_j - dz_k}{z_j - z_k}$ on $X_n \subset \mathbb{C}^n$. Note that $w_{jk} = w_{kj}$. The vector space $H^1(P_n;\mathbb{Q})$ is spanned by the vectors w_{jk}, and the map $H^1(P_n;\mathbb{Q}) \to H^1(P_{n+1};\mathbb{Q})$ sends $w_{jk} \in H^1(P_n;\mathbb{Q})$ to $w_{jk} \in H^1(P_{n+1};\mathbb{Q})$. Furthermore, Arnol'd proved that $H^*(P_n;\mathbb{Q})$ is generated as a $\mathbb{Q}$–algebra by $H^1(P_n;\mathbb{Q})$, subject only to the relations

$$R_{jkl}\colon \quad w_{jk} \wedge w_{kl} + w_{kl} \wedge w_{lj} + w_{lj} \wedge w_{jk} = 0.$$

This implies that $H^i(P_n;\mathbb{Q})$ has basis

$$\big\{w_{j_1k_1} \wedge \cdots \wedge w_{j_ik_i} \big| k_1 < \cdots < k_i, \text{ and } j_m < k_m \text{ for all } m\big\}.$$

Injectivity of each ψ_n^* is then immediate. To prove surjectivity of ψ_n^* (in the sense of the definition of representation stability), consider an arbitrary basis element for $H^i(P_{n+1};\mathbb{Q})$. Note that for $n \geq 2i$, no basis element can involve all the numbers from 1 to $n+1$ as indices. It follows that by applying some element of S_{n+1}, we may assume that our basis element can be written without $n+1$ as an index. But such an element is in the subalgebra of $H^*(P_{n+1};\mathbb{Q})$ spanned by the image of $H^1(P_n)$, and thus is contained in the image of $H^i(P_n;\mathbb{Q})$, as desired.

We now prove uniform stability of multiplicities; we will defer the computation of the stable range until afterwards. The work of Orlik–Solomon on the cohomology of hyperplane complements implies that $H^*(P_n;\mathbb{Q})$ splits into pieces "supported on the top cohomology of Young subgroups", as follows. For details of what follows, see Lehrer–Solomon [LS]. Any subset of $\{1,\ldots,n\}$ of cardinality k determines a projection $P_n \to P_k$ by forgetting the other $n-k$ coordinates (strands). Given a partition $\mathcal{S}$ of $\{1,\ldots,n\}$ into subsets, the product over all these projections gives a projection of P_n onto the group $P_{\mathcal{S}}$ defined as the product of the pure braid groups of sizes corresponding to elements of the partition. For concreteness we illustrate this explicitly for the partition of $\{1,\ldots,n\}$ into $\{1,\ldots,k\}$ and $\{k+1,\ldots,n\}$, which determines a projection $P_n \to P_{\mathcal{S}} = P_k \times P_{n-k}$. There is always a splitting $P_{\mathcal{S}} \to P_n$, given in this case for example by realizing P_k and P_{n-k} disjointly. Note that the partition may contain subsets of size 1. For example, the partition of $\{1,\ldots,n\}$ into $\{1,\ldots,k\},\{k+1\},\ldots,\{n\}$ determines the group $P_k \times P_1 \times \cdots P_1 \approx P_k$.

We refer to these groups $P_{\mathcal{S}}$ as *Young subgroups* of P_n, by analogy with Young subgroups of symmetric groups, such as $S_k \times S_{n-k} < S_n$. This is a slight abuse of notation, since the embedding of $P_{\mathcal{S}}$ as a subgroup is not unique; the important thing is the projection $P_n \to P_{\mathcal{S}}$. The projection onto such a Young subgroup gives an inclusion $H^*(P_{\mathcal{S}};\mathbb{Q}) \to H^*(P_n;\mathbb{Q})$. We now consider the image in $H^*(P_n;\mathbb{Q})$ of the top cohomology of $P_{\mathcal{S}}$. For example, the cohomological dimension of $P_k \times P_{n-k}$ is $(k-1)+(n-k-1) = n-2$, and we consider the image of the top cohomology $H^{n-2}(P_k \times P_{n-k};\mathbb{Q})$ inside $H^{n-2}(P_n;\mathbb{Q})$. For each partition $\mathcal{S}$ of $\{1,\ldots,n\}$ into i subsets, the corresponding Young subgroup $P_{\mathcal{S}}$ is the product of i pure braid groups, so the image of its top cohomology determines a subspace $H^{\mathcal{S}}(P_n)$ of $H^{n-i}(P_n;\mathbb{Q})$. Orlik–Solomon [OS, Proposition 2.10] implies that $H^*(P_n;\mathbb{Q})$ splits as an S_n–module as a direct sum

$$H^*(P_n;\mathbb{Q}) = \bigoplus_{\mathcal{S}} H^{\mathcal{S}}(P_n)$$

over all partitions $\mathcal{S}$ of $\{1,\ldots,n\}$, and that S_n permutes the summands according to its action on $\{1,\ldots,n\}$.

Every partition $\mathcal{S}$ of $\{1,\ldots,n\}$ determines a partition $\overline{\mathcal{S}}$ of n, listing the sizes of the subsets in $\mathcal{S}$. The term $H^{\mathcal{S}}(P_n)$ contributes to $H^i(P_n;\mathbb{Q})$ exactly if $|\mathcal{S}| = \ell(\overline{\mathcal{S}}) = n-i$. The action of S_n on $\{1,\ldots,n\}$ induces an action on partitions $\mathcal{S}$ of $\{1,\ldots,n\}$, and the summands $H^{\mathcal{S}}(P_n)$ are permuted according to this action. In particular, for a fixed $\mu \vdash n$, the direct sum $\bigoplus_{\overline{\mathcal{S}}=\mu} H^{\mathcal{S}}(P_n)$ is a subrepresentation of $H^i(P_n;\mathbb{Q})$. We will need explicit orbit representations, so for any partition $\mu \vdash n$, let $\mathcal{S}_\mu$ be the partition of

$\{1, \ldots, n\}$ given by

$$\{1, \ldots, \mu_1\}, \{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}, \ldots, \{\mu_1 + \cdots + \mu_{n-1} + 1, \ldots, n\}.$$

This gives for each μ an orbit representative $\mathcal{S}_\mu$ with $\overline{\mathcal{S}_\mu} = \mu$. For a fixed μ, the subrepresentation $\bigoplus_{\overline{\mathcal{S}}=\mu} H^{\mathcal{S}}(P_n)$ is generated by one summand $H^{\mathcal{S}_\mu}(P_n)$ and is the direct sum of its translates. Thus by the definition of induced representation we have

$$\bigoplus_{\overline{\mathcal{S}}=\mu} H^{\mathcal{S}}(P_n) = \operatorname{Ind}_{\operatorname{Stab}(\mathcal{S}_\mu)}^{S_n} H^{\mathcal{S}_\mu}(P_n). \tag{12}$$

We would like to apply Theorem 4.2 to the terms (12). Consider the projection onto the Young subgroup $P_n \to P_k \times P_{n-k}$. Pulling back by the projection $P_{n+1} \to P_n$, this pulls back to the projection $P_{n+1} \to P_k \times P_{n-k} \times P_1$. In general, the Young subgroup $P_{\mathcal{S}} < P_n$ pulls back to $P_{\mathcal{S}\langle n+1\rangle} < P_{n+1}$, where $\mathcal{S}\langle n+1\rangle$ is the partition $\mathcal{S} \cup \{n+1\}$. Note that if $\overline{\mathcal{S}} = \mu = (\mu_1, \ldots, \mu_{n-i})$, then $\overline{\mathcal{S}\langle n+1\rangle} = \mu\langle n+1\rangle \coloneqq (\mu_1, \ldots, \mu_{n-i}, 1)$. For larger $m \geq n$, we define $\mathcal{S}\langle m\rangle \coloneqq \mathcal{S} \cup \{n+1\} \cup \cdots \cup \{m\}$ and $\mu\langle m\rangle \coloneqq (\mu_1, \ldots, \mu_{n-i}, 1, \ldots, 1)$ similarly.

Since $\mathcal{S}\langle n+1\rangle$ is a partition of $\{1, \ldots, n+1\}$ into $(n+1) - i$ sets, $H^{\mathcal{S}\langle n+1\rangle}(P_{n+1})$ is contained in $H^i(P_{n+1}; \mathbb{Q})$, and in fact the natural map $H^*(P_n; \mathbb{Q}) \to H^*(P_{n+1}; \mathbb{Q})$ restricts to an isomorphism $H^{\mathcal{S}}(P_n) \to H^{\mathcal{S}\langle n+1\rangle}(P_{n+1})$.

Certainly not every partition of $\{1, \ldots, n+1\}$ contains the singleton set $\{n+1\}$. But fixing i, every partition of $n+1$ with $(n+1) - i$ entries must have some entry equal to 1 once $n \geq 2i$. This means that any such partition is equal to $\mu\langle n+1\rangle$ for some $\mu \vdash n$. Note that we chose the definition of $\mathcal{S}_\mu$ so that $\mathcal{S}_{\mu\langle n+1\rangle} = \mathcal{S}_\mu\langle n+1\rangle$. Thus writing the decomposition

$$H^i(P_n; \mathbb{Q}) = \bigoplus_{\substack{\mu \vdash n \\ \ell(\mu) = n-i}} \bigoplus_{\overline{\mathcal{S}}=\mu} H^{\mathcal{S}}(P_n) = \bigoplus_{\substack{\mu \vdash n \\ \ell(\mu) = n-i}} \operatorname{Ind}_{\operatorname{Stab}(\mathcal{S}_\mu)}^{S_n} H^{\mathcal{S}_\mu}(P_n)$$

we have for $n \geq 2i$ a decomposition of $H^i(P_{n+1}; \mathbb{Q})$ over the same partitions μ:

$$\begin{aligned} H^i(P_{n+1}; \mathbb{Q}) &= \bigoplus_{\substack{\nu \vdash n+1 \\ \ell(\mu) = n+1-i}} \operatorname{Ind}_{\operatorname{Stab}(\mathcal{S}_\nu)}^{S_{n+1}} H^{\mathcal{S}_\nu}(P_{n+1}) \\ &= \bigoplus_{\substack{\mu \vdash n \\ \ell(\mu) = n-i}} \operatorname{Ind}_{\operatorname{Stab}(\mathcal{S}_\mu\langle n+1\rangle)}^{S_{n+1}} H^{\mathcal{S}_\mu\langle n+1\rangle}(P_{n+1}) \end{aligned}$$

We already mentioned above that $H^{\mathcal{S}_\mu\langle n+1\rangle}(P_{n+1}) \approx H^{\mathcal{S}_\mu}(P_n)$. The set of partitions $\mu \vdash n$ with $\ell(\mu) = n - i$ rows is finite and independent of n (subtracting 1 from each entry yields a partition of i), so it suffices to prove for each μ that the sequence $\{\mathrm{Ind}_{\mathrm{Stab}(\mathcal{S}_\mu\langle m\rangle)}^{S_m} H^{\mathcal{S}_\mu}(P_m)\}$ is uniformly representation stable as $m \to \infty$.

Note that the stabilizer $\mathrm{Stab}(\mathcal{S})$ of a partition $\mathcal{S}$ of $\{1, \ldots, n\}$ need not preserve the individual subsets making up $\mathcal{S}$, only the overall decomposition into subsets. Thus if $\mathcal{S}$ has m_j subsets of size j, the stabilizer $\mathrm{Stab}(\mathcal{S})$ will be a product of wreath products $S_j \wr S_{m_j} = (S_j)^{m_j} \rtimes S_{m_j}$, where the $(S_j)^{m_j}$ factor acts on the subsets of size j, and the S_{m_j} factor permutes them. In particular, the $S_1 \wr S_{m_1} = S_{m_1}$ factor acts by permuting the singleton sets in $\mathcal{S}_\mu$. This corresponds to rearranging the $P_1 \times \cdots P_1$ factors in the Young subgroup $P_{\mathcal{S}}$. From this we see that the S_{m_1} factor of $\mathrm{Stab}(\mathcal{S})$ acts trivially on $H^{\mathcal{S}}(P_n)$.

If we write $\mathrm{Stab}(\mathcal{S}_\mu) = H \times S_{m_1}$, we have $\mathrm{Stab}(\mathcal{S}_\mu\langle n+1\rangle) = H \times S_{m_1+1}$, and so on. Take $k = n - m_1$ and let $\nu \vdash k$ be the partition obtained from μ by deleting those entries equal to 1. The subgroup H is exactly $H_\nu \coloneqq \mathrm{Stab}(\mathcal{S}_\nu) < S_k$, and identifying $H^{\mathcal{S}_\mu}(P_n)$ with $H^{\mathcal{S}_\nu}(P_k)$, the sequence in question can be written as $\{\mathrm{Ind}_{H_\nu \times S_{n-k}}^{S_n} H^{\mathcal{S}_\nu}(P_k) \boxtimes \mathbb{Q}\}$. Thus Theorem 4.2 applies and gives that this sequence is uniformly multiplicity stable, as desired.

To compute the stable range, it suffices to bound the number $k = |\nu|$ which appears in the last paragraph of the proof. It is not hard to check that for a fixed i, the maximum k occurs for the partition $\mu = (2, 2, \ldots, 2, 1, \ldots, 1)$, corresponding to Young subgroups isomorphic to $P_2 \times \cdots \times P_2$. For such μ we have $\nu = (2, \ldots, 2)$ with $\ell(\nu) = i$, and thus the maximal k is $k = 2i$. Since the stability range is Theorem 4.2 is $n \geq 2k$, we conclude that $\{H^i(P_n; \mathbb{Q})\}$ stabilizes once $n \geq 4i$, as claimed. □

Remark 4.3. By a careful analysis of the individual pieces $H^{\mathcal{S}}(P_n)$, Lehrer–Solomon [LS] decompose $H^i(P_n; \mathbb{Q})$ into a direct sum of representations induced from 1–dimensional representations of certain centralizers in S_n. Though we did not need this description to prove that $\{H^i(P_n; \mathbb{Q})\}$ is representation stable, it is indispensable when actually computing multiplicities of irreducibles. We revisit these multiplicities in [CEF], where we explicitly compute the multiplicities of certain irreducible representations in $H^i(P_n; \mathbb{Q})$ and give arithmetic consequences of their stable values.

Arnol'd [Ar] (see also F. Cohen [Co]) established homological stability for the integral homology groups $H_i(B_n; \mathbb{Z})$. He also showed that $H_i(B_n; \mathbb{Z})$ is finite for $i \geq 2$, so that $H_i(B_n; \mathbb{Q})$ is trivial in this range. As a corollary of Theorem 4.1, we obtain homological stability for B_n with twisted coefficients. Any representation of S_n can be regarded as a representation of B_n by composing with the standard projection $B_n \to S_n$.

Corollary 4.4. *For any partition λ the sequence $\{H_*(B_n; V(\lambda)_n)\}$ of twisted homology groups satisfies classical homological stability: for each fixed $i \geq 0$, once $n \geq 4i$ there is an isomorphism*

$$H_i(B_n; V(\lambda)_n) \approx H_i(B_{n+1}; V(\lambda)_{n+1}). \tag{13}$$

Proof. There are no natural maps between the homology groups in (13), but we show that their dimension is eventually constant. Since P_n has finite index in B_n and our coefficients are vector spaces over $\mathbb{Q}$, the transfer map gives an isomorphism

$$H_i(B_n; V(\lambda)_n) \approx H_i(P_n; V(\lambda)_n)^{S_n}$$

with the S_n–invariants in $H_i(P_n; V(\lambda)_n)$. Since the action of B_n on $V(\lambda)_n$ factors through S_n, the representation $V(\lambda)_n$ is trivial when restricted to P_n. Thus

$$H_i(P_n; V(\lambda)_n)^{S_n} \approx \big(H_i(P_n; \mathbb{Q}) \otimes V(\lambda)_n\big)^{S_n}.$$

Recall from Section 2 that every representation of S_n is self-dual. Schur's lemma thus gives that $V(\mu) \otimes V(\nu)$ contains the trivial representation if and only if $\mu = \nu$, in which case the trivial representation appears with multiplicity 1. It follows that the dimension of $\big(H_i(P_n; \mathbb{Q}) \otimes V(\lambda)_n\big)^{S_n}$ is exactly the multiplicity of $V(\lambda)_n$ in $H_i(P_n; \mathbb{Q})$, which is the same as the multiplicity of $V(\lambda)_n$ in $H^i(P_n; \mathbb{Q})$. By Theorem 4.1, this multiplicity is constant once $n \geq 4i$, as desired. ☐

Remark 4.5. The space X_n is the configuration space of n ordered points in $\mathbb{C}$, and Theorem 4.1 states that its cohomology groups $\{H^i(X_n; \mathbb{Q})\}$ are representation stable. Similarly, the space Y_n is the configuration space of n unordered points in $\mathbb{C}$, and Corollary 4.4 gives classical homological stability for the sequence $\{H_i(Y_n; \mathbb{Q})\}$. These results are extended to configuration spaces of arbitrary orientable manifolds in [C].

4.2 Generalized braid groups

The *generalized pure braid group* of type B_n is the fundamental group $WP_n := \pi_1(X'_n)$ of the configuration space

$$X'_n := \big\{\mathbf{z} \in \mathbb{C}^{2n} \big| z_i \neq z_j, z_i \neq -z_j, z_i \neq 0\big\}.$$

This configuration space is aspherical, so $H^*(WP_n; \mathbb{Q}) = H^*(X'_n; \mathbb{Q})$. The hyperoctahedral group W_n acts on X'_n by permuting and negating the coordinates; this induces an action of W_n on $H^*(WP_n; \mathbb{Q})$. The quotient $Y'_n := X'_n/W_n$ is the space of unordered

n–tuples of distinct sets $\{z, -z\}$ with $z \neq 0$. Identifying each set $\{z, -z\}$ with the point z^2, the space Y'_n is identified with the space of unordered n–tuples of distinct nonzero points. Thus the *generalized braid group* $WB_n := \pi_1(Y'_n)$ can be identified with the subgroup $B_{1,n} < B_{n+1}$ which is the preimage of the stabilizer $\mathrm{Stab}(1) < S_{n+1}$. See the survey by Vershinin [Ve] for an overview of generalized braid groups.

Theorem 4.6. *For each fixed $i \geq 0$, the sequence $\{H^i(WP_n; \mathbb{Q})\}$ of W_n–representations is uniformly representation stable.*

Proof. The results of Orlik–Solomon are a bit more involved in this case, so we cover the necessary definitions in more detail. See Douglass [Do] for an excellent exposition of these results in the case of $H^*(X'_n; \mathbb{Q})$. Let $\mathcal{H}$ be the set of hyperplanes defined by the equations $z_i = z_j$, $z_i = -z_j$, and $z_i = 0$. To each $H \in \mathcal{H}$ defined by the linear equation $L = 0$ we associate the element $w_H \in H^1(X'_n; \mathbb{Q})$ represented by $\frac{1}{2\pi i} \frac{dL}{L}$. The action of W_n on the elements w_H is the same as the action of W_n on $\mathcal{H}$.

Brieskorn [Bri] proved that $H^*(WP_n; \mathbb{Q}) = H^*(X'_n; \mathbb{Q})$ is generated by the w_H, subject to certain relations [OS, Theorem 5.2]. Injectivity in the definition of representation stability follows from the naturality of these relations; this is essentially the observation that any relation supported on the image of $H^*(X'_n; \mathbb{Q})$ in $H^*(X'_{n+1}; \mathbb{Q})$ already holds in $H^*(X'_n; \mathbb{Q})$. For surjectivity, simply note that any monomial of length i involves at most $2i$ coordinates, and thus up to the W_{n+1}–action is contained in $H^i(X'_n; \mathbb{Q})$ once $n \geq 2i$.

Let $\mathcal{J}$ be the set of intersections of hyperplanes in $\mathcal{H}$. The *support* of a monomial $w_{H_1} \cdots w_{H_k}$ is the intersection $H_1 \cap \cdots \cap H_k \in \mathcal{J}$. Orlik–Solomon [OS, Proposition 2.10] prove that $H^*(X'_n; \mathbb{Q})$ splits as a direct sum over $J \in \mathcal{J}$

$$H^*(X'_n; \mathbb{Q}) = \bigoplus_{J \in \mathcal{J}} \left\langle w_{H_1} \cdots w_{H_k} \middle| H_1 \cap \cdots \cap H_k = J \right\rangle$$

of the subspace spanned by monomials with support J. The factors are permuted according to the action of W_n on $\mathcal{J}$. Let H^J be the summand spanned by all monomials $w_{H_1} \cdots w_{H_k}$ with $H_1 \cap \cdots \cap H_k = J$, so the splitting above is just

$$H^*(X'_n; \mathbb{Q}) = \bigoplus_{J \in \mathcal{J}} H^J.$$

To write this as a sum of induced representations, we need to understand the orbits of the W_n–action on $\mathcal{J}$.

Consider the elements of $\mathcal{J}$, that is the subspaces which occur as intersections of the defining hyperplanes $\mathcal{H}$. A representative example is the subspace defined by the equations

$$z_1 = -z_2 = -z_3, \quad z_4 = z_5 = -z_6, \quad z_7 = z_8 = 0.$$

In general, any element $J \in \mathcal{J}$ splits into disjoint sets of indices in this way. One subset corresponds to the ℓ coordinates which are equal to 0, for some $0 \leq \ell \leq n$. On each other subset, the indices are split into two parts as in $z_1 = -z_2 = -z_3$. Since W_n not only can permute coordinates but can also negate them, this internal division is not preserved by W_n. The orbits of W_n acting on $\mathcal{J}$ correspond to pairs (λ, ℓ) where λ is a partition not involving 1 and $|\lambda| + \ell \leq n$. We let $m = m(J)$ denote $|\lambda| + \ell$. For example, the subspace above corresponds to $((3,3),2)$, with representative

$$J = J_{((3,3),2)} \colon \qquad z_1 = z_2 = z_3, \quad z_4 = z_5 = z_6, \quad z_7 = z_8 = 0.$$

For this subspace J, the stabilizer $\mathrm{Stab}_{W_n}(J)$ is $(S_3 \wr W_2) \times W_2 \times W_{n-8}$. All we will need is that in general, the stabilizer $\mathrm{Stab}_{W_n}(J)$ splits as a product $G_J \times W_{n-m(J)}$, where $n - m(J)$ is the number of unrestricted coordinates.

Let H^J be the subspace spanned by all monomials $w_{H_1} \cdots w_{H_k}$ with $H_1 \cap \cdots \cap H_k = J$. The splitting

$$H^*(X'_n; \mathbb{Q}) = \bigoplus_{J \in \mathcal{J}} H^J$$

can be rewritten as a sum over W_n–orbit representatives of induced representations

$$H^*(X'_n; \mathbb{Q}) = \bigoplus_{J = J_{(\lambda,\ell)}} \mathrm{Ind}_{\mathrm{Stab}_{W_n}(J)}^{W_n} H^J.$$

Unfortunately, unlike the case of $H^{\mathcal{S}}(P_n)$ in the previous section, the subspace H^J is not homogeneous: if k denotes the number of entries in λ then H^J has components in $H^i(X'_n; \mathbb{Q})$ for all $m(J) - k \leq i \leq m(J)$. For such i, define the *homogeneous component* $H^{J|i} < H^i(X'_n; \mathbb{Q})$ by

$$H^{J|i} \coloneqq \big\langle w_{H_1} \cdots w_{H_i} \big| H_1 \cap \cdots \cap H_i = J \big\rangle.$$

We obtain the decomposition

$$H^i(X'_n; \mathbb{Q}) = \bigoplus_{\substack{J = J_{(\lambda,\ell)} \\ m(J) - k \leq i \leq m(J)}} \mathrm{Ind}_{\mathrm{Stab}_{W_n}(J)}^{W_n} H^{J|i}.$$

Note that for fixed i, the set of pairs (λ, ℓ) with $|\lambda| + \ell - k \leq i \leq |\lambda| + \ell$ is finite and eventually independent of n; that is, the collection of orbit representatives $J = J_{(\lambda,\ell)}$

for which H^J contributes to $H^i(X_n';\mathbb{Q})$ does not depend on n. Thus it would suffice to prove that for each such J and i, the sequence of representations $\operatorname{Ind}_{\operatorname{Stab}_{W_n}(J)}^{W_n} H^{J|i}$ is uniformly representation stable.

We now mimic Hemmer's proof of Theorem 4.2 to finish the proof. Fix $J = J_{(\lambda,\ell)}$ and i such that $m-k \leq i \leq m$, and take $n > m$. Recall that $\operatorname{Stab}_{W_n}(J)$ splits as $G_J \times W_{n-m}$, where $n-m$ is the number of unrestricted coordinates. Since $H^{J|i}$ is spanned by monomials for which the intersection $H_1 \cap \cdots \cap H_i$ is equal to J, no unrestricted coordinate appears in any element of $H^{J|i}$. It follows that the W_{n-m} factor above acts trivially on $H^{J|i}$. Thus we may consider $H^{J|i}$ as the representation $H^{J|i} \boxtimes \mathbb{Q}$ of $G_J \times W_{n-m}$. Factor the desired induction as

$$\begin{aligned}\operatorname{Ind}_{\operatorname{Stab}_{W_n}(J)}^{W_n} H^{J|i} &= \operatorname{Ind}_{W_m\times W_{n-m}}^{W_n} \operatorname{Ind}_{G_J\times W_{n-m}}^{W_m\times W_{n-m}} H^{J|i} \boxtimes \mathbb{Q} \\ &= \operatorname{Ind}_{W_m\times W_{n-m}}^{W_n} \Big(\big(\operatorname{Ind}_{G_J}^{W_m} H^{J|i} \big) \boxtimes \mathbb{Q} \Big)\end{aligned}$$

Let $V_{J|i}$ be the representation $\operatorname{Ind}_{G_J}^{W_m} H^{J|i}$ of W_m; note that this does not depend on n. Consider the decomposition of $V_{J|i}$ into irreducible representations $V_{(\lambda^+,\lambda^-)}$ of W_m. Since only finitely many irreducibles (λ^+,λ^-) occur in $V_{J|i}$, it suffices to prove uniform stability for each factor $\operatorname{Ind}_{W_m\times W_{n-m}}^{W_n} \big(V_{(\lambda^+,\lambda^-)} \boxtimes \mathbb{Q}\big)$. The Littlewood–Richardson rule generalizes to hyperoctahedral groups as [GP, Lemma 6.1.3], giving:

$$\operatorname{Ind}_{W_m\times W_{n-m}}^{W_n} \big(V_{(\lambda^+,\lambda^-)} \boxtimes V_{(\mu^+,\mu^-)}\big) = \bigoplus_{\nu^+,\nu^-} C^{\nu^+}_{\lambda^+,\mu^+} C^{\nu^-}_{\lambda^-,\mu^-} V_{(\nu^+,\nu^-)}.$$

Applying this to the trivial representation $\mathbb{Q} = V_{((n-m),(0))}$ yields

$$\operatorname{Ind}_{W_m\times W_{n-m}}^{W_n} \big(V_{(\lambda^+,\lambda^-)} \boxtimes \mathbb{Q}\big) = \bigoplus_{\nu} C^{\nu}_{\lambda^+,(n-m)} V_{(\nu,\lambda^-)} = \bigoplus_{\nu} V_{(\nu,\lambda^-)}$$

where the last sum is over those partitions μ obtained from λ^+ by adding $n-m$ boxes, no two in the same column. For fixed λ^+ and large enough n, say $n-m > |\lambda^+|$, any such ν must have multiple boxes added to the first row. This yields a bijection between the partitions ν of $j \coloneqq n - |\lambda^-|$ appearing in this decomposition and their stabilizations $\nu[j+1]$ appearing in the decomposition

$$\operatorname{Ind}_{W_m\times W_{n+1-m}}^{W_{n+1}} \big(V_{(\lambda^+,\lambda^-)} \boxtimes \mathbb{Q}\big) = \bigoplus_{\nu} V_{(\nu[j+1],\lambda^-)},$$

implying that this sequence of induced representations is uniformly multiplicity stable. This completes the proof that $H^i(WP_n;\mathbb{Q}) = H^i(X_n';\mathbb{Q})$ is uniformly representation stable. □

This gives the following corollary, by the same argument as Corollary 4.4. We remark that an explicit stability range for Theorem 4.6 and Corollary 4.7 can be extracted from the proof of Theorem 4.6.

Corollary 4.7. *For any double partition $\lambda = (\lambda^+, \lambda^-)$ the sequence $\{H_*(B_{1,n}; V(\lambda)_n) = H_*(WB_n; V(\lambda)_n)\}$ of twisted homology groups satisfies classical homological stability: for each fixed $i \geq 0$ and sufficiently large n (depending on i), there is an isomorphism*

$$H_i(B_{1,n}; V(\lambda)_n) \approx H_i(B_{1,n+1}; V(\lambda)_{n+1}).$$

An explicit stability range for Theorem 4.6 and Corollary 4.7 can be extracted from the proof.

4.3 Groups of string motions

A 3–dimensional analogue of the pure braid group is $P\Sigma_n$, the group of pure string motions. Let X''_n be the space of embeddings of n disjoint unlinked loops into 3–space, and define $P\Sigma_n \coloneqq \pi_1(X''_n)$. The hyperoctahedral group W_n acts on X''_n by permuting the labels and reversing the orientations, inducing an W_n–action on $H^*(P\Sigma_n; \mathbb{Q})$. We remark that $P\Sigma_n$ can also be identified with McCool's *pure symmetric automorphism group*, consisting of those automorphisms of the free group F_n sending each generator to a conjugate of itself. The quotient $Y''_n \coloneqq X''_n / W_n$ is the space of n unordered unoriented unlinked loops, and its fundamental group $B\Sigma_n \coloneqq \pi_1(Y''_n)$ is the group of *string motions*.

The cohomology ring of $P\Sigma_n$ has been computed by Jensen–McCammond–Meier [JMM], who prove that $H^*(P\Sigma_n; \mathbb{Q})$ is generated by classes $\alpha_{ij} \in H^1(P\Sigma_n; \mathbb{Q})$ for all $i \neq j$, $1 \leq i, j \leq n$, subject to the relations

$$\alpha_{ij} \wedge \alpha_{ji} = 0 \qquad \text{and} \qquad \alpha_{ij} \wedge \alpha_{jk} + \alpha_{kj} \wedge \alpha_{ik} + \alpha_{ik} \wedge \alpha_{ij}.$$

The action of W_n is as follows: S_n acts by permuting the indices, while negating the jth coordinate negates generators of the form α_{ij} and fixes all other generators. There is a natural embedding $P_n \hookrightarrow P\Sigma_n$, and the induced surjection $H^*(P\Sigma_n; \mathbb{Q}) \to H^*(P_n; \mathbb{Q})$ maps $\alpha_{ij} \mapsto w_{ij}$. Based on the results of the previous sections, it is natural to make the following conjecture.

Conjecture 4.8. *For each fixed $i \geq 0$, the sequence of W_n–representations $\{H^i(P\Sigma_n; \mathbb{Q})\}$ is uniformly representation stable.*

Note that some element of W_n negates α_{ij} and preserves α_{ji}, but both are mapped to $w_{ij} = w_{ji} \in H^1(P_n; \mathbb{Q})$. Thus the action of W_n on $H^*(P\Sigma_n; \mathbb{Q})$ does not descend to the action of S_n on $H^*(P_n;\mathbb{Q})$, though of course the restricted action of S_n on $H^*(P\Sigma_n;\mathbb{Q})$ does.

5 Lie algebras and their homology

In this section we show how the phenomenon of representation stability occurs in the theory of Lie algebras. Our main result, Theorem 5.3 below, relates representation stability for a sequence of Lie algebras to representation stability for their homology groups. We then give a number of applications, some of which were already known by other methods.

5.1 Graded Lie algebras and Lie algebra homology

Lie algebra homology. Given a Lie algebra $\mathcal{L}$ over $\mathbb{Q}$, its *Lie algebra homology* $H_*(\mathcal{L};\mathbb{Q})$ is computed by the chain complex

$$\cdots \longrightarrow \bigwedge^3\mathcal{L} \xrightarrow{\partial_3} \bigwedge^2\mathcal{L} \xrightarrow{\partial_2} \mathcal{L} \xrightarrow{\partial_1} \mathbb{Q}, \tag{14}$$

where the differential is given by

$$\partial_i(x_1\wedge\cdots\wedge x_i) = \sum_{j<k}(-1)^{j+k+1}[x_j,x_k]\wedge x_1\wedge\cdots\wedge\widehat{x}_j\wedge\cdots\wedge\widehat{x}_k\wedge\cdots\wedge x_i.$$

Note that if $\mathrm{GL}(\mathcal{L})$ denotes the group of Lie algebra automorphisms of $\mathcal{L}$, the induced action of $\mathrm{GL}(\mathcal{L})$ on $\bigwedge^i\mathcal{L}_n$ commutes with ∂. Thus an action of any group G on $\mathcal{L}$ by automorphisms induces an action of G on $H_i(\mathcal{L};\mathbb{Q})$ for each i.

Homology with coefficients. If M is an $\mathcal{L}$–module, the *homology* $H_*(\mathcal{L};M)$ *with coefficients in* M is the homology of the complex

$$\cdots\to\bigwedge^3\mathcal{L}\otimes M\to\bigwedge^2\mathcal{L}\otimes M\to\mathcal{L}\otimes M\to M\to 0,$$

where the differential is the sum of the previous differential on $\bigwedge^*\mathcal{L}$, extended by the identity to $\bigwedge^*\mathcal{L}\otimes M$, plus $\partial'_i\colon \bigwedge^i\mathcal{L}\otimes M\to\bigwedge^{i-1}\mathcal{L}\otimes M$ defined by

$$\partial'_i(x_1\wedge\cdots\wedge x_i\otimes m) = \sum(-1)^{j+1}x_1\wedge\cdots\wedge\widehat{x}_j\wedge\cdots\wedge x_i\otimes x_j\cdot m.$$

A common example is the *adjoint homology* $H_*(\mathcal{L};\mathcal{L})$. If G acts on $\mathcal{L}$ by automorphisms and acts $\mathcal{L}$–equivariantly on M, meaning that $(g\cdot x)\cdot(g\cdot m) = g\cdot(x\cdot m)$, then ∂' commutes with the action of G, inducing an action of G on $H_i(\mathcal{L};M)$ for each i.

Graded Lie algebras. A Lie algebra $\mathcal{L}$ is called a *graded Lie algebra* if it decomposes into homogeneous components $\mathcal{L} = \bigoplus_{j\geq 1} \mathcal{L}^j$ so that $[\mathcal{L}^j, \mathcal{L}^k] \subset \mathcal{L}^{j+k}$. This induces a grading $\bigwedge^i \mathcal{L} = \bigoplus_j (\bigwedge^i \mathcal{L})^j$ under which, for example, the subspace $\bigwedge^3 \mathcal{L}^2 \subset \bigwedge^3 \mathcal{L}$ has degree 6. From the definition above we see that the differential ∂ preserves this grading. Thus it descends to a grading $H_i(\mathcal{L};\mathbb{Q}) = \bigoplus_j H_i(\mathcal{L};\mathbb{Q})^j$ of the Lie algebra homology. If $M = \bigoplus_{j\geq 0} M^j$ is a graded $\mathcal{L}$–module, meaning that $\mathcal{L}^j \cdot M^k \subset M^{j+k}$, then we similarly obtain a grading $H_i(\mathcal{L};M) = \bigoplus H_i(\mathcal{L};M)^j$.

Definition 5.1 (Consistent sequence of Lie algebras)**.** Let G_n be $\mathrm{SL}_n \mathbb{Q}$, $\mathrm{GL}_n \mathbb{Q}$, or $\mathrm{Sp}_{2n} \mathbb{Q}$. Consider a sequence of Lie algebras $\{\mathcal{L}_n\}$ with injections $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+1}$ and with each $\mathcal{L}_n$ equipped with an action of G_n by Lie algebra automorphisms. We call the sequence $\{\mathcal{L}_n\}$ *consistent* if each of the following holds:

1. $\mathcal{L}_n$ is consistent when considered as a sequence of G_n–representations.
2. Each $\mathcal{L}_n$ is graded, and both the maps $\mathcal{L}_n \to \mathcal{L}_{n+1}$ and the action of G_n preserve the grading.
3. The graded components $\mathcal{L}_n^j$ are finite-dimensional.

It will also be useful to allow our coefficient modules to vary with n.

Definition 5.2 (Admissible coefficients)**.** A sequence $\{M_n\}$ of nonzero graded $\mathcal{L}_n$–modules with maps $M_n \to M_{n+1}$ and equivariant G_n–actions is *admissible* if the following conditions hold: for each $j \geq 0$ the sequence $\{M_n^j\}$ is strongly stable; each M_n^j is finite-dimensional; and M_n^j is eventually nonzero for at least one $j \geq 0$.

Our main result in this section is the following. It proves among other things that strong representation stability for a sequence of Lie algebras is actually equivalent to strong stability for its homology. Each direction of this equivalence has applications.

Theorem 5.3 (Stability of Lie algebras and their homology)**.** *Let $G_n = \mathrm{SL}_n \mathbb{Q}$ or $\mathrm{GL}_n \mathbb{Q}$, and let $\{\mathcal{L}_n\}$ be a consistent sequence of graded Lie algebras with G_n–actions which is type-preserving (satisfies Condition IV). The following are equivalent:*

1. *For each fixed $j \geq 0$ the sequence $\{\mathcal{L}_n^j\}$ is strongly stable.*
2. *For each fixed $i, j \geq 0$ the sequence $\{H_i(\mathcal{L}_n;\mathbb{Q})^j\}$ is strongly stable.*
3. *For each fixed $i, j \geq 0$ the sequence $\{H_i(\mathcal{L}_n;\mathcal{L}_n)^j\}$ of graded adjoint homology groups is strongly stable.*

4. *For one admissible sequence of coefficients $\{M_n\}$, the sequence $\{H_i(\mathcal{L}_n; M_n)^j\}$ is strongly stable for each fixed $i, j \geq 0$.*

5. *For every admissible sequence of coefficients $\{M_n\}$, the sequence $\{H_i(\mathcal{L}_n; M_n)^j\}$ is strongly stable for each fixed $i, j \geq 0$.*

Proof. We will prove the equivalence for $G_n = \mathrm{SL}_n \mathbb{Q}$ and $\mathrm{GL}_n \mathbb{Q}$ simultaneously. Note that by taking coefficients $M = \mathbb{Q}$ concentrated in grading 0 with trivial $\mathcal{L}$–action, (2) is a special case of (4), so (2) $\implies$ (4). We will begin by proving that (4) $\implies$ (1). We will then modify this argument slightly to prove that (3) $\implies$ (1). Note that under the assumption of (1), $\{\mathcal{L}_n\}$ is an admissible sequence of coefficients, so that under this assumption (3) follows from (5). Thus once we have proved that (1) $\implies$ (5), it immediately follows that (1) $\implies$ (3). Since (5) also trivially implies (2) and (4), this will complete the proof of equivalence.

We first describe the complex computing graded homology. Let $\mathcal{L} = \bigoplus_{j\geq 1} \mathcal{L}^j$ be a graded Lie algebra, and $M = \bigoplus_{j\geq 0} M^j$ a graded $\mathcal{L}$–module. Since the differential preserves the grading, we can decompose the complex (14) computing $H_*(\mathcal{L}; M)$ into its graded pieces. The slice of this complex in grading k, which computes $H_*(\mathcal{L}; M)^k$, has the form:

$$\begin{aligned} 0 \longrightarrow \textstyle\bigwedge^k \mathcal{L}^1 \otimes M^0 \longrightarrow \left(\textstyle\bigwedge^{k-2}\mathcal{L}^1 \otimes \mathcal{L}^2 \otimes M^0\right) \oplus \left(\textstyle\bigwedge^{k-1}\mathcal{L}^1 \otimes M^1\right) \longrightarrow \cdots \\ \cdots \longrightarrow \bigoplus_{1\leq j,j'<k} \mathcal{L}^j \otimes \mathcal{L}^{j'} \otimes M^{k-j-j'} \longrightarrow \bigoplus_{1\leq j\leq k} \mathcal{L}^j \otimes M^{k-j} \longrightarrow M^k \to 0 \end{aligned} \tag{15}$$

Here the $\mathcal{L}^j \otimes \mathcal{L}^{j'} \otimes M^{k-j-j'}$ term is actually $\bigwedge^2 \mathcal{L}^j \otimes M^{k-2j}$ when $j = j'$. Note the following key property: the graded piece $\mathcal{L}^k$ only appears in the second-to-last term, in the term $\mathcal{L}^k \otimes M^0$; all other terms involve $\mathcal{L}^j$ only for smaller j (for $j < k$).

(4) $\implies$ (1). Assume that $\{\mathcal{L}_n\}$ is a consistent G_n–sequence of graded Lie algebras, that $\{M_n\}$ is an admissible G_n–sequence of graded $\mathcal{L}_n$–modules, and that $\{H_i(\mathcal{L}_n; M_n)^j\}$ is strongly stable for each fixed $i, j \geq 0$. Assume for now that M_n^0 is eventually nonzero. We prove that $\{\mathcal{L}_n^j\}$ is strongly stable by induction. Since we have assumed that $\mathcal{L}_n \hookrightarrow \mathcal{L}_{n+1}$ is type-preserving, it suffices to prove that $\{\mathcal{L}_n^j\}$ is uniformly multiplicity stable. In the next two sections of the proof only, we will abbreviate "uniformly multiplicity stable" to "stable". Furthermore, since stability is always taken over sequences with respect to n, we suppress the subscript n for readability. To sum up, within the next two sections "$\{H_1(\mathcal{L}; M)^1\}$ is stable" means "the sequence of G_n–representations $\{H_1(\mathcal{L}_n; M_n)^1\}$ is uniformly multiplicity stable".

We first prove that $\{\mathcal{L}^1\}$ is stable. The following sequence computes $H_*(\mathcal{L}; M)^1$:

$$0 \to \mathcal{L}^1 \otimes M^0 \xrightarrow{\partial} M^1 \to 0$$

By assumption $\{M^1\}$ is stable, as are $\{\ker \partial = H_1(\mathcal{L}; M)^1\}$ and $\{M^1 / \operatorname{im} \partial = H_0(\mathcal{L}; M)^1\}$. We see that $\{\operatorname{im} \partial\}$ is also stable, and thus $\{\mathcal{L}^1 \otimes M^0 = \ker \partial \oplus \operatorname{im} \partial\}$ is stable as well. We now appeal to Theorem 3.5, which states that if $\{M^0\}$ and $\{\mathcal{L}^1 \otimes M^0\}$ are stable, then $\{\mathcal{L}^1\}$ is stable as well.

The argument in the inductive step is similar. Assume that $\{\mathcal{L}^j\}$ is stable for each $j < k$. Consider the sequence (15) computing $H_*(\mathcal{L}; M)^k$. Since $\{M^j\}$ is stable for each fixed $j \geq 0$, by repeatedly applying Theorems 3.1(1) and 3.1(2) we conclude that each term of (15) is stable except possibly the term $\{\mathcal{L}^k \otimes M^0\}$. We now proceed along this complex from left to right, comparing the complex itself with its homology. Start with ∂_k, whose domain is $\bigwedge^k \mathcal{L}^1 \otimes M^0$. Since $\{\bigwedge^k \mathcal{L}^1 \otimes M^0\}$ and $\{\ker \partial_k = H_k(\mathcal{L}; M)^k\}$ are stable, so is $\{\operatorname{im} \partial_k\}$. Since $\{\operatorname{im} \partial_k\}$ and $\{\ker \partial_{k-1} / \operatorname{im} \partial_k = H_{k-1}(\mathcal{L}; M)^k\}$ are stable, so is $\{\ker \partial_{k-1}\}$. Since $\{\ker \partial_{k-1}\}$ and the domain of ∂_{k-1} are stable, so is $\{\operatorname{im} \partial_{k-1}\}$. Continuing along the complex, we have by induction that $\{\operatorname{im} \partial_2\}$ is stable, as is $\{\ker \partial_1 / \operatorname{im} \partial_2 = H_1(\mathcal{L}; M)^k\}$, so $\{\ker \partial_1\}$ is stable. Now moving to the right side, $\{M^k\}$ and $\{M^k / \operatorname{im} \partial_1 = H_0(\mathcal{L}; M)^k\}$ are stable, so $\{\operatorname{im} \partial_1\}$ is stable. Combining these claims, we see that $\{\ker \partial_1 \oplus \operatorname{im} \partial_1 = \bigoplus_{1 \leq j \leq k} \mathcal{L}^j \otimes M^{k-j}\}$ is stable. Since all but one term in this sum is stable, the remaining term $\{\mathcal{L}^k \otimes M^0\}$ is stable as well. Applying Theorem 3.5, we conclude that $\{\mathcal{L}^k\}$ is stable.

In the previous two paragraphs we assumed that M^0_n was eventually nonzero, but a similar argument applies in general. For example, consider the case when M^0_n is zero for all n, but M^1_n is eventually nonzero. Then every term containing M^0 vanishes in the complex (15) computing $H_*(\mathcal{L}; M)^k$. Among the remaining terms, $\mathcal{L}^k$ no longer appears, and $\mathcal{L}^{k-1}$ appears only in the term $\mathcal{L}^{k-1} \otimes M^1$. Thus assuming that $\{L^j\}$ is stable for $j < k-1$, an argument like the one above shows that $\{\mathcal{L}^{k-1}\}$ is stable. This completes the proof that (4) $\implies$ (1).

(3) $\implies$ (1). This proof is exactly like the proof that (4) $\implies$ (1), except that we do not know at the beginning that $\mathcal{L}_n$ is an admissible sequence of coefficients. To prove this, first note that the complex computing $H_1(\mathcal{L}; \mathcal{L})^1$ is just $0 \to \mathcal{L}^1 \to 0$, so $\mathcal{L}^1$ must be stable. Since $\mathcal{L}$ has positive grading, the complex computing $H_k(\mathcal{L}; \mathcal{L})^k$ has the form

$$0 \longrightarrow \bigwedge^{k-1} \mathcal{L}^1 \otimes \mathcal{L}^1 \to \cdots \to \bigoplus_{0<j<k} \mathcal{L}^j \otimes \mathcal{L}^{k-j} \longrightarrow \mathcal{L}^k \longrightarrow 0$$

In particular, $\mathcal{L}^k$ appears only in the last term. By induction, every term except possibly the last is stable, and the homology in each dimension is stable, so as above we can conclude that $\mathcal{L}^k$ is stable, as desired. Note that we do not need a separate argument for the case when $\mathcal{L}^1$ is trivial.

(1) $\implies$ (5). Assume that $\{\mathcal{L}_n^j\}$ is strongly stable for each $j \geq 0$. Let $N_n^{i,j}$ be the piece of $\bigwedge^i \mathcal{L}_n \otimes M_n$ in grading j, so that the complex (15) computing $H_*(\mathcal{L}_n; M_n)^k$ has the form

$$0 \to N_n^{k,k} \to N_n^{k-1,k} \to \cdots \to N_n^{2,k} \to N_n^{1,k} \to N_n^{0,k} \to 0.$$

We have already encountered these subspaces; for example, $N_n^{k,k} = \bigwedge^k \mathcal{L}_n^1 \otimes M_n^0$ and $N_n^{1,k} = \bigoplus_{1 \leq j \leq k} \mathcal{L}_n^j \otimes M_n^{k-j}$. We have already assumed that $\{\mathcal{L}_n\}$ and $\{M_n\}$ are strongly stable. If both are finite-dimensional, then by Theorems 3.1(1) and 3.1(2), the sequence $\{\bigwedge^i \mathcal{L}_n \otimes M_n\}$ is strongly stable for each fixed $i \geq 0$. Even if $\mathcal{L}_n$ is not finite-dimensional, for fixed i, j the term $N_n^{i,j}$ only involves finitely many graded pieces $\mathcal{L}_n^\bullet$ and $M_n^\bullet$, as in the example $N_n^{1,k} = \bigoplus_{1 \leq j \leq k} \mathcal{L}_n^j \otimes M_n^{k-j}$ above. Since each graded piece is assumed finite-dimensional, we may repeatedly apply Theorem 3.1 to conclude that $\{N_n^{i,j}\}$ is strongly stable, and in particular satisfies Condition IV.

Let ∂_i^n be the differential $\bigwedge^i \mathcal{L}_n \otimes M_n \to \bigwedge^{i-1} \mathcal{L}_n \otimes M_n$, and let $(\partial_i^n)^j \colon N_n^{i,j} \to N_n^{i-1,j}$ be the restriction to $N_n^{i,j}$. The commutativity of

$$\begin{array}{ccc} \bigwedge^i \mathcal{L}_n \otimes M_n & \xrightarrow{\partial_i^n} & \bigwedge^{i-1} \mathcal{L}_n \otimes M_n \\ \downarrow & & \downarrow \\ \bigwedge^i \mathcal{L}_{n+1} \otimes M_{n+1} & \xrightarrow[\partial_i^{n+1}]{} & \bigwedge^{i-1} \mathcal{L}_{n+1} \otimes M_{n+1} \end{array} \tag{16}$$

implies that under the vertical inclusions, $\ker \partial_i^n$ maps to $\ker \partial_i^{n+1}$, and that $\operatorname{im} \partial_i^n$ maps to $\operatorname{im} \partial_i^{n+1}$. Restricting to grading j, we similarly conclude that $\ker(\partial_i^n)^j$ maps to $\ker(\partial_i^{n+1})^j$ and that $\operatorname{im}(\partial_i^n)^j$ maps to $\operatorname{im}(\partial_i^{n+1})^j$ under the inclusions $N_n^{i,j} \hookrightarrow N_{n+1}^{i,j}$.

Recall that Condition IV for $\{N_n^{i,j}\}$ says that for any subspace isomorphic to $V(\lambda)_n^k$ in $N_n^{i,j}$, its G_{n+1}–span in $N_{n+1}^{i,j}$ is isomorphic to $V(\lambda)_{n+1}^k$. Applying this to $\ker(\partial_i^n)^j$ and $\operatorname{im}(\partial_i^n)^j$, the observation above implies that for fixed i, j and λ, the multiplicity of $V(\lambda)_n$ in $\ker(\partial_i^n)^j$ and in $\operatorname{im}(\partial_i^n)^j$ is nondecreasing in n. The sum of these representations is $N_n^{i,j}$, whose decomposition is eventually constant by uniform multiplicity stability. Once the decomposition of $N_n^{i,j}$ has stabilized, an increase in $\ker(\partial_i^n)^j$ would necessitate a corresponding decrease in $\operatorname{im}(\partial_i^n)^j$, contradicting the observation for $\operatorname{im}(\partial_i^n)^j$, and vice versa. We conclude that $\{\ker(\partial_i^n)^j\}$ and $\{\operatorname{im}(\partial_i^n)^j\}$ are uniformly multiplicity stable for each i and j, stabilizing once $N_n^{i,j}$ does. Thus for each i and j the quotient $\{H_i(\mathcal{L}_n; M_n)^j = \ker(\partial_{i-1}^n)^j / \operatorname{im}(\partial_i^n)^j\}$ is uniformly multiplicity stable, as desired.

Since $\{N_n^{i,j}\}$ is uniformly multiplicity stable, for fixed $i, j \geq 0$ and sufficiently large n we have the following property: only finitely many partitions λ occur in $N_n^{i,j}$ (meaning the multiplicity of $V(\lambda)_n$ in $N_n^{i,j}$ is nonzero). This property passes to the subquotient $H_i(\mathcal{L}_n; M_n)^j$. But a sequence $\{H_i(\mathcal{L}_n; M_n)^j\}$ which is multiplicity stable yet involves only finitely many irreducibles is necessarily uniformly multiplicity stable, and so we can promote Condition III to Condition III′.

By assumption $\mathcal{L}_n$ and M_n are strongly stable. By Proposition 2.14, this implies that P_{n+1} acts trivially on the image of $\mathcal{L}_n$ in $\mathcal{L}_{n+1}$ and of M_n in M_{n+1}. As noted above, this implies that P_{n+1} acts trivially on the image of $\bigwedge^i \mathcal{L}_n \otimes M_n$ in $\bigwedge^i \mathcal{L}_{n+1} \otimes M_{n+1}$ for each i. But this condition passes to subquotients, so P_{n+1} acts trivially on the image of $H_i(\mathcal{L}_n; M_n)$ in $H_i(\mathcal{L}_{n+1}; M_{n+1})$, verifying Condition IV for the sequences $\{H_i(\mathcal{L}_n; M_n)\}$ and $\{H_i(\mathcal{L}_n; M_n)^j\}$. Since the $N_n^{i,j}$ are finite-dimensional, the same is true of their subquotients $H_i(\mathcal{L}_n; M_n)^j$. By Remark 2.13, for a finite-dimensional sequence Conditions III′ and IV together imply Conditions I and II. This concludes the proof of strong stability of $\{H_i(\mathcal{L}_n; M_n)^j\}$. □

Symplectic Lie algebras. In the proof of (1) $\implies$ (5) of Theorem 5.3 we used the assumption that $\{\mathcal{L}_n\}$ is type-preserving. For representations of $\operatorname{Sp}_{2n} \mathbb{Q}$ we do not have the appropriate analogue of Proposition 2.14, so the argument does not work in this case. But examining the proof above, we did not use that $\{\mathcal{L}_n\}$ is type-preserving in the implications (4) $\implies$ (1) or (3) $\implies$ (1). We needed only Theorem 3.5 and Theorem 3.1 for uniform multiplicity stable sequences, and these theorems apply to $\operatorname{Sp}_{2n} \mathbb{Q}$–representations as well. Thus we deduce the following from the proof of Theorem 5.3.

Theorem 5.4. *Let $\{\mathcal{L}_n\}$ be a consistent $\operatorname{Sp}_{2n} \mathbb{Q}$–sequence of graded Lie algebras. If the sequence $\{H_i(\mathcal{L}_n; \mathbb{Q})^j\}$ is uniformly multiplicity stable for each fixed $i, j \geq 0$, then the sequence $\{\mathcal{L}_n^j\}$ is uniformly multiplicity stable for each fixed $j \geq 0$.*

5.2 Simple representation stability

Certain classical families of representations satisfy a stronger form of stability, which is in some sense as close to actual stability as a sequence of $\operatorname{GL}_n \mathbb{Q}$–representations can be. Consider a partition λ with $\ell = \ell(\lambda)$ rows. As noted above, $\mathbb{S}_\lambda(\mathbb{Q}^n)$ is trivial for $n < \ell$, and for such n there is no irreducible representation which could be called $V(\lambda)_n$. A sequence is called simply representation stable if this is the only obstruction to having constant multiplicities.

Definition 5.5 (Simple representation stability)**.** A consistent sequence $\{V_n\}$ of $\mathrm{GL}_n\,\mathbb{Q}$–representations is called *simply representation stable* if for all $n \geq 1$ it satisfies Conditions I and II, and if in addition it satisfies the following:

SIII. For each partition λ with $\ell = \ell(\lambda)$ nonzero rows, the multiplicity of the irreducible representation $\mathbb{S}_\lambda(\mathbb{Q}^n)$ in V_n is constant for all $n \geq \ell$. For any pseudo-partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ which is not a partition (meaning $\lambda_\ell < 0$), the multiplicity of $V(\lambda)_n$ in V_n is 0.

SIV. For any subrepresentation $W \subset V_n$ so that $W \approx \mathbb{S}_\lambda(\mathbb{Q}^n)$, the span of the $\mathrm{GL}_{n+1}\,\mathbb{Q}$–orbit of $\phi_n(W)$ is isomorphic to $\mathbb{S}_\lambda(\mathbb{Q}^{n+1})$.

If we interpret $V(\lambda)_n$ as being trivial when n is less than the number of rows of λ, then simple representation stability says there is a decomposition

$$V_n = \bigoplus c_\lambda\, \mathbb{S}_\lambda(\mathbb{Q}^n) = \bigoplus c_\lambda V(\lambda, 0)_n$$

over partitions λ which is totally independent of n and preserved by the maps $V_n \hookrightarrow V_{n+1}$. Then Theorem 5.3 has the following strengthening.

Theorem 5.6. *Let $\{\mathcal{L}_n\}$ be a consistent $\mathrm{GL}_n\,\mathbb{Q}$–sequence of graded Lie algebras which is type-preserving, and $\{M_n\}$ an admissible sequence of coefficients which is simply stable. Then Theorem 5.3 remains true if "strongly stable" is replaced everywhere by "simply stable".*

Proof. We sketch the proof. The characterization of Proposition 2.14 still holds: given Conditions I, II, and SIII, Condition SIV is equivalent to Condition IV$'$. Examining the proof of Theorem 3.1, and in particular that the formulas (4) and (6) are independent of n, we conclude that if $\{V_n\}$ and $\{U_n\}$ are simply stable, the same is true of $\{V_n \otimes U_n\}$ and $\{\mathbb{S}_\lambda(V_n)\}$. Similarly, from the proof of Theorem 3.5, we conclude that if $\{U_n\}$ and $\{V_n \otimes U_n\}$ satisfy Condition SIII, the same is true of $\{V_n\}$. This has the folllowing implications for the proof of Theorem 5.3.

In the proofs of (4) $\implies$ (1) and of (3) $\implies$ (1) we can replace "stable" with "simply stable" everywhere, and the argument remains valid. For (1) $\implies$ (5), if the sequence $\{\mathcal{L}_n\}$ is simply stable, the same is true of $\{\bigwedge^i \mathcal{L}_n \otimes M_n\}$ and of $\{N_n^{i,j}\}$. As before, the multiplicity of $\mathbb{S}_\lambda(\mathbb{Q}^n)$ in $\ker(\partial_i^n)^j$ and $\mathrm{im}(\partial_i^n)^j$ is nondecreasing. Their sum is the multiplicity of $\mathbb{S}_\lambda(\mathbb{Q}^n)$ in $N_n^{i,j}$, which by simple stability of $\{N_n^{i,j}\}$ is finite and constant, so the same is true for $\ker \partial_i^n$ and $\operatorname{im} \partial_i^n$. Since this holds for each i, we conclude that the multiplicity of $\mathbb{S}_\lambda(\mathbb{Q}^n)$ in $H_i(\mathcal{L}_n; M_n)^j$ is constant, as desired. Condition SIV follows as before, and we conclude that $\{H_i(\mathcal{L}_n; M_n)^j\}$ is simply stable. □

5.3 Applications and examples

In this subsection we give a number of applications of Theorem 5.3, Theorem 5.4, and Theorem 5.6.

Free Lie algebras. Let V_n be a $\mathbb{Q}$–vector space with basis $x_1, \ldots, x_n$. Let $\mathcal{L}(V_n) = \mathcal{L}(x_1, \ldots, x_n)$ be the free Lie algebra on V_n. The action of $\mathrm{GL}(V_n) \approx \mathrm{GL}_n \mathbb{Q}$ on V_n induces an action of $\mathrm{GL}_n \mathbb{Q}$ on $\mathcal{L}(V_n)$. The Lie algebra $\mathcal{L}(V_n)$ has a natural grading

$$\mathcal{L}(V_n) = \bigoplus_{i \geq 1} \mathcal{L}_i(V_n)$$

which is preserved by the action of $\mathrm{GL}_n \mathbb{Q}$. The obvious inclusion of $V_n \hookrightarrow V_{n+1}$ induces natural maps $\mathcal{L}(V_n) \hookrightarrow \mathcal{L}(V_{n+1})$ and $\mathcal{L}_i(V_n) \hookrightarrow \mathcal{L}_i(V_{n+1})$. These inclusions are respected by the inclusion of $\mathrm{GL}_n \mathbb{Q} \hookrightarrow \mathrm{GL}_{n+1} \mathbb{Q}$.

The free Lie algebra $\mathcal{L}(V_n)$ has the following homology groups for all $n \geq 1$:

$$H_i(\mathcal{L}(V_n); \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ V_n & i = 1 \\ 0 & i \geq 2 \end{cases}$$

This follows from (18) below, and can also be checked directly. As $\mathrm{GL}_n \mathbb{Q}$–representations, these are $\mathbb{Q} = \mathbb{S}_{(0)}(\mathbb{Q}^n)$, with grading 0, and $V_n = \mathbb{S}_{(1)}(\mathbb{Q}^n)$, with grading 1. Thus $\{H_*(\mathcal{L}(V_n); \mathbb{Q})\}$ is simply stable, so Theorem 5.6 gives the following corollary.

Corollary 5.7. *For each fixed $m \geq 1$, the sequence of $\mathrm{GL}_n \mathbb{Q}$–representations $\{\mathcal{L}_m(V_n)\}$ of degree m components of the free Lie algebras $\mathcal{L}(V_n)$ is simply representation stable.*

In fact, the multiplicities of $V(\lambda)_n$ in $\mathcal{L}_m(V_n)$ are known, at least in some sense. For the irreducible representation $V(\lambda)_n$ to appear in $\mathcal{L}_m(V_n)$ it is necessary that λ be a partition of m. For such λ, Bakhturin [Ba, Proposition 3, §3.4.5] gives the following formula for the multiplicity. Let χ_λ denote the character of the irreducible representation of S_m associated to λ, and let τ be the m–cycle $(1\,2\,\ldots\,m)$. Then the multiplicity of $V(\lambda)_n$ in $\mathcal{L}_m(V_n)$ is

$$c_\lambda := \frac{1}{m} \sum_{d|m} \mu(d) \chi_\lambda(\tau^{m/d}).$$

Despite this formula, due to the dependence on the values of irreducible characters χ_λ of the symmetric group, explicitly calculating these multiplicities remains an active area of research.

Free nilpotent Lie algebras. Let $\mathcal{N}_k(n)$ be the level k truncation of the free Lie algebra of rank n, meaning:

$$\mathcal{N}_k(n) = \mathcal{L}(V_n)/\mathcal{L}_{k+1}(V_n) = \bigoplus_{i \leq k} \mathcal{L}_i(V_n).$$

This is the free k–step nilpotent Lie algebra on V_n. Since $\mathcal{N}_k(n)$ is a truncation of $\mathcal{L}(V_n)$, Corollary 5.7 tells us that for each fixed i the sequence of i^{th} graded pieces $\{\mathcal{N}_k(n)^i = \mathcal{L}_i(V_n)\}$ is simply stable. Thus as a corollary of Theorem 5.6, we obtain the following theorem of Tirao [Ti, Theorem 2.9].

Corollary 5.8 (Tirao)**.** *Fix any $k \geq 1$. Then for each fixed $i \geq 0$ the sequence of $\mathrm{GL}_n(\mathbb{Q})$–representations $\{H_i(\mathcal{N}_k(n);\mathbb{Q})\}$ is simply stable.*

The novelty of our deduction of Corollary 5.8 is that we start with the homology of the free Lie algebra, which is quite easy to compute, then we move to the free Lie algebra itself using one direction of Theorem 5.3, then to its nilpotent truncation, then to the homology of that truncation using the reverse implication in Theorem 5.3.

We note that, while the $\mathcal{N}_k(n)$ are not themselves complicated, their homology is quite complicated. For $k = 2$, it follows from work of Kostant (see [CT1, Theorem 3.1]) that the multiplicity of $V(\lambda)_n$ in $H_i(\mathcal{N}_2(n);\mathbb{Q})$ is 0 unless λ is self-conjugate (i.e., its Young diagram is symmetric under reflection across the diagonal) and has exactly i boxes above the diagonal, in which case the multiplicity is 1. No formula is known in general, but for some small values of i and k, Tirao [Ti] computes the decomposition of this homology explicitly. For example, he proves that (in our terminology):

$$\begin{aligned}
H_3(\mathcal{N}_3(2);\mathbb{Q}) &= V(4,2)\\
H_3(\mathcal{N}_3(3);\mathbb{Q}) &= V(4,2) \oplus V(2,2,2) \oplus V(3,1,1) \oplus V(3,3,1) \oplus V(4,2,1) \oplus V(5,1,1)\\
H_3(\mathcal{N}_3(4);\mathbb{Q}) &= V(4,2) \oplus V(2,2,2) \oplus V(3,1,1) \oplus V(3,3,1) \oplus V(4,2,1) \oplus V(5,1,1)\\
&\quad \oplus V(3,1,1,1) \oplus V(3,2,1,1)\\
H_3(\mathcal{N}_3(n);\mathbb{Q}) &= V(4,2) \oplus V(2,2,2) \oplus V(3,1,1) \oplus V(3,3,1) \oplus V(4,2,1) \oplus V(5,1,1)\\
&\quad \oplus V(3,1,1,1) \oplus V(3,2,1,1) \oplus V(2,2,1,1,1) \qquad \text{for } n \geq 5
\end{aligned}$$

Note that, as guaranteed by simple stability, each representation $V(\lambda)_n$ first appears in $H_3(\mathcal{N}_3(n))$ when n is the number of rows of λ, and persists with the same multiplicity thereafter.

Also from Theorem 5.6, we obtain the following corollary on the homology of $\mathcal{N}_k(n)$ with coefficients in the adjoint representation.

Corollary 5.9. *Fix $k \geq 1$. Then for each fixed $i \geq 0$ the sequence of* $\mathrm{GL}_n(\mathbb{Q})$*–representations $\{H_i(\mathcal{N}_k(n); \mathcal{N}_k(n))\}$ is simply stable.*

The adjoint homology of $\mathcal{N}_2(n)$ was studied in Cagliero–Tirao [CT1], and in this case Corollary 5.9 can be deduced from [CT1, Theorem 4.4] combined with the description of $H_i(\mathcal{N}_2(n); \mathbb{Q})$ above. For $k \geq 3$, to the best of our knowledge this result was not previously known.

Continuous cohomology and pseudo-nilpotent groups. The *continuous cohomology* of a group Γ is the direct limit

$$H^*_{\text{cts}}(\Gamma; \mathbb{Q}) = \varinjlim H^*(\Gamma/K; \mathbb{Q})$$

of the cohomology of all its finitely generated nilpotent quotients Γ/K. The basic properties of continuous cohomology are established in Hain [Ha2]. There is an obvious comparison map $H^*_{\text{cts}}(\Gamma; \mathbb{Q}) \to H^*(\Gamma; \mathbb{Q})$, which is always an isomorphism on H^0 and H^1, and is always injective on H^2. A finitely generated group Γ is called *pseudo-nilpotent* if this map is an isomorphism in every degree.

Nomizu's theorem implies that for finitely generated groups, $H^*_{\text{cts}}(\Gamma; \mathbb{Q})$ coincides with the continuous cohomology $H^*_{\text{cts}}(\mathfrak{g}; \mathbb{Q})$ of the Malcev Lie algebra $\mathfrak{g}$ of Γ. The Malcev Lie algebra is a certain pronilpotent $\mathbb{Q}$–Lie algebra associated to Γ. We will only need the following property. Recall that the *lower central series*

$$\Gamma = \Gamma_1 > \Gamma_2 > \cdots$$

of a group Γ is defined inductively by $\Gamma_1 \coloneqq \Gamma$ and $\Gamma_{n+1} \coloneqq [\Gamma, \Gamma_n]$. The *associated graded (rational) Lie algebra* $\mathrm{gr}(\Gamma)$ is the $\mathbb{Q}$–Lie algebra defined by

$$\mathrm{gr}(\Gamma) \coloneqq \bigoplus_{n=1}^{\infty} (\Gamma_n/\Gamma_{n+1}) \otimes \mathbb{Q}$$

where the Lie bracket is induced by the group commutator. The Malcev Lie algebra $\mathfrak{g}$ of Γ has the property that the graded Lie algebra associated to its lower central series is isomorphic to $\mathrm{gr}(\Gamma)$. For the groups we consider, we have an isomorphism $H^*_{\text{cts}}(\mathfrak{g}; \mathbb{Q}) \approx H^*(\mathrm{gr}(\Gamma); \mathbb{Q})$. Note that the automorphism group $\mathrm{Aut}(\Gamma)$ acts on Γ, preserves each Γ_i, and so acts on $\mathrm{gr}(\Gamma)$. It is well known that this action factors through the representation

$$\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma/[\Gamma, \Gamma]). \tag{17}$$

For the free group F_n it is well known that $\mathrm{gr}(F_n) = \mathcal{L}(H_1(F_n;\mathbb{Q})) = \mathcal{L}(V_n)$. Since free groups are pseudo-nilpotent (see [Ha2, Corollary 5.10]), we conclude that

$$H^*(\mathrm{gr}(F_n);\mathbb{Q}) = H^*(F_n;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^n. \tag{18}$$

In this case the representation (17) gives the natural action of $\mathrm{GL}_n \mathbb{Z}$ on $\mathcal{L}(V_n)$, which extends to a representation of $\mathrm{GL}_n \mathbb{Q}$. In Corollary 5.7, we noted that the isomorphism $H^*(\mathrm{gr}(F_n);\mathbb{Q}) = H^*(F_n;\mathbb{Q})$ implies that $\{H^i(\mathrm{gr}(F_n);\mathbb{Q})\}$ is simply stable for each i, and so we concluded that the sequence of graded components $\{\mathrm{gr}(F_n)^j\}$ is simply stable for each $j \geq 0$ as well.

We would like to mimic this argument for surface groups. Let $\pi_g = \pi_1(S_g)$ be the fundamental group of the closed, connected, orientable surface of genus $g \geq 2$. Labute [La] proved that $\mathrm{gr}(\pi_g)$ is the quotient of the free Lie algebra $\mathcal{L}(H_1(S_g;\mathbb{Q}))$ by the ideal generated by the symplectic form:

$$\mathrm{gr}(\pi_g) \approx \mathcal{L}(H_1(S_g;\mathbb{Q}))/([a_1, b_1] + \cdots + [a_g, b_g])$$

Further, in this case the representation (17) is known to factor through the integral symplectic group $\mathrm{Sp}_{2g} \mathbb{Z}$. Hain proved [Ha2, Proposition 5.11] that π_g is pseudo-nilpotent, and that the continuous cohomology of its Malcev Lie algebra coincides with $H^*(\mathrm{gr}(\pi_g);\mathbb{Q})$. Thus $H^*(\mathrm{gr}(\pi_g);\mathbb{Q}) \approx H^*(S_g;\mathbb{Q})$, which as an $\mathrm{Sp}_{2g} \mathbb{Q}$–representation decomposes as follows for all $g \geq 1$:

$$H^i(S_g;\mathbb{Q}) = \begin{cases} V(0) & i = 0, 2 \\ V(1) & i = 1 \\ 0 & i \geq 3 \end{cases}$$

We do not have maps $\pi_g \to \pi_{g+1}$, but we do have surjections $\pi_{g+1} \to \pi_g$ inducing surjections $\mathrm{gr}(\pi_{g+1}) \to \mathrm{gr}(\pi_g)$, which induce maps $H^*(\mathrm{gr}(\pi_g)) \to H^*(\mathrm{gr}(\pi_{g+1}))$. For each i, this makes the cohomology groups $\{H^i(\mathrm{gr}(\pi_g);\mathbb{Q}) = H^i(S_g;\mathbb{Q})\}$ into a uniformly stable sequence of $\mathrm{Sp}_{2g} \mathbb{Q}$–representations. Theorem 5.4 works just as well for cohomology, so we obtain as a corollary the following result of Hain [Ha, Corollary 8.5].

Corollary 5.10 (Hain)**.** *For each fixed $i \geq 0$, the sequence of $\mathrm{Sp}_{2g} \mathbb{Q}$–representations given by the graded components $\{\mathrm{gr}(\pi_g)^i\}$ are uniformly representation stable for all j.*

Homology of symplectic Lie algebras. Many sequences of Lie algebras $\mathcal{L}_n$ are naturally $\mathrm{Sp}_{2n}\,\mathbb{Q}$–representations and in fact are uniformly stable, such as the Heisenberg Lie algebras considered below. We would like to conclude stability for the homology groups of this sequence of Lie algebras $\{H_i(\mathcal{L}_n;\mathbb{Q})\}$ as we did in Corollary 5.8 above. But without a good notion of strong stability for $\mathrm{Sp}_{2n}\,\mathbb{Q}$–representations, the proof of the necessary implication in Theorem 5.3, namely (1) $\implies$ (5), does not work. However, in specific cases the argument can be successfully modified.

We will give a concrete example of such a modification, but first we extract from the proof of (1) $\implies$ (5) in Theorem 5.3 exactly where strong stability was used. Ignoring the grading for the moment, the homology $H_*(\mathcal{L}_n;\mathbb{Q})$ is computed by the rows of the complex

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \bigwedge^3\mathcal{L}_n & \xrightarrow{\partial_3} & \bigwedge^2\mathcal{L}_n & \xrightarrow{\partial_2} & \mathcal{L}_n & \xrightarrow{\partial_1} & \mathbb{Q} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \bigwedge^3\mathcal{L}_{n+1} & \xrightarrow{\partial_3} & \bigwedge^2\mathcal{L}_{n+1} & \xrightarrow{\partial_2} & \mathcal{L}_{n+1} & \xrightarrow{\partial_1} & \mathbb{Q} \end{array} \tag{19}$$

We know stability holds for each term $\{\bigwedge^i\mathcal{L}_n\}$, but the differentials between them introduce a possible source of instability. To draw conclusions about stability for $H_i(\mathcal{L}_n;\mathbb{Q}) = \ker\partial_i/\operatorname{im}\partial_{i+1}$, we need control over how the differentials ∂_i interact with the vertical maps $\bigwedge^i\mathcal{L}_n \hookrightarrow \bigwedge^i\mathcal{L}_{n+1}$. For example, we do not know that $\{\ker\partial_i\}$ is stable.

In Theorem 5.3, the type-preserving assumption guaranteed that the vertical maps preserved isotypic components. Then since the vertical maps are injective, the commutativity of (19) implied that even if $\{\ker\partial_i\}$ and $\{\operatorname{im}\partial_i\}$ did not stabilize immediately, their decompositions were nondecreasing in n. Thus once the terms, which here would be $\{\bigwedge^i\mathcal{L}_n = \ker\partial_i \oplus \operatorname{im}\partial_i\}$, stabilized, the summands $\{\ker\partial_i\}$ and $\{\operatorname{im}\partial_i\}$ were forced to stabilize as well. However, for $\mathrm{Sp}_{2n}\,\mathbb{Q}$–representations the vertical maps for $\bigwedge^i\mathcal{L}_n$ are almost never type-preserving, as we will see in detail below. Thus a new idea is needed.

Let $H_n \coloneqq V(\lambda_1)_n = \mathbb{Q}^{2n}$ be the standard representation of $\mathrm{Sp}_{2n}\,\mathbb{Q}$. For $i \leq n$, we have the decomposition into irreducibles

$$\bigwedge^i H_n = V(\lambda_i)_n \oplus V(\lambda_{i-2})_n \oplus \cdots V(\lambda_\varepsilon)_n$$

where $\varepsilon = 0$ or 1 if i is even or odd respectively. The inclusion $\bigwedge^i H_n \hookrightarrow \bigwedge^i H_{n+1}$ does *not* respect this decomposition. In fact, we have the following:

Lemma 5.11. *For $n > i$ and any irreducible representation $V(\lambda_k)_n \subset \bigwedge^i H_n$ with $i > k$, the $\mathrm{Sp}_{2n+2}\,\mathbb{Q}$–span of $V(\lambda_k)_n$, considered as a subspace of $\bigwedge^i H_{n+1}$, is isomorphic to*

$$V(\lambda_{k+2})_{n+1} \oplus V(\lambda_k)_{n+1}.$$

Proof. For an overview of the symplectic representation theory used here, see [FH, §17]. For each $i \leq n$ there is a unique contraction $C\colon \bigwedge^i H \to \bigwedge^{i-2} H$, with $\ker C \approx V(\lambda_i)$ generated by $a_1 \wedge \cdots \wedge a_i$. This induces a filtration of $\bigwedge^i H$ by

$$\ker C^j \approx V(\lambda_i) \oplus \cdots \oplus V(\lambda_{i-2j+2}).$$

A complement to $\ker C$ is given by the image of $\cdot \wedge \omega_n \colon \bigwedge^{i-2} H \to \bigwedge^i H$, where

$$\omega_n = a_1 \wedge b_1 + \cdots + a_n \wedge b_n$$

spans the trivial subrepresentation of $\bigwedge^2 H$. It follows that $V(\lambda_k)_n \subset \bigwedge^i H_n$ is the $\mathrm{Sp}_{2n}\,\mathbb{Q}$–span of $v_{k,n} \coloneqq a_1 \wedge \cdots \wedge a_k \wedge (\omega_n)^j$, where $j = (i-k)/2$. Let $W \subset \bigwedge^i H_{n+1}$ be the desired representation, the $\mathrm{Sp}_{2n+2}\,\mathbb{Q}$–span of $v_{k,n}$. The contractions C commute with the inclusion $\bigwedge^i H_n \hookrightarrow \bigwedge^i H_{n+1}$. Thus since $v_{k,n}$ is contained in $\ker C^{j+1}$ but not $\ker C^j$, we know that W is contained in $V(\lambda_i)_{n+1} \oplus \cdots \oplus V(\lambda_k)_{n+1}$ but not in $V(\lambda_i)_{n+1} \oplus \cdots \oplus V(\lambda_{k+2})_{n+1}$.

Under the inclusion $\bigwedge^2 H_n \hookrightarrow \bigwedge^2 H_{n+1}$, ω_n is not mapped to ω_{n+1}. Instead we have $\omega_n = \omega_{n+1} - a_{n+1} \wedge b_{n+1}$, and so $(\omega_n)^j = (\omega_{n+1})^j - (\omega_{n+1})^{j-1} \wedge a_{n+1} \wedge b_{n+1}$. Writing

$$v_{k,n} = a_1 \wedge \cdots \wedge a_k \wedge (\omega_{n+1} - a_{n+1} \wedge b_{n+1}) \wedge (\omega_{n+1})^{j-1},$$

we see that $v_{k,n}$ is in the image of $\cdot \wedge (\omega_{n+1})^{j-1} \colon \bigwedge^{k+2} H_{n+1} \to \bigwedge^i H_{n+1}$. Combined with the above bound on W, this implies that W is contained in $V(\lambda_{k+2})_{n+1} \oplus V(\lambda_k)_{n+1}$.

Since we know that W is not contained in $V(\lambda_{k+2})_{n+1}$, it remains to show that W is not contained in $V(\lambda_k)_{n+1}$. Note that $C^j(v_{k,n}) = \frac{(n-k)!}{(n-k-j)!} a_1 \wedge \cdots \wedge a_k$. Then if $A \in \mathrm{Sp}_{2n+2}\,\mathbb{Q}$ is any element fixing $a_1 \wedge \cdots \wedge a_k$ but not $v_{n,k}$ (for example, the permutation matrix exchanging a_n with a_{n+1} and b_n with b_{n+1}), we have $C^j(A \cdot v_{k,n}) = A \cdot C^j(v_{k,n}) = C^j(v_{k,n})$. In particular, the vector $v_{k,n} - A \cdot v_{k,n}$ lies in $\ker C^j$. As explained above, this shows that $W \approx V(\lambda_{k+2})_{n+1} \oplus V(\lambda_k)_{n+1}$. □

Remark 5.12. Lemma 5.11 can serve as a substitute for the type-preserving assumption in the argument outlined before the lemma. Take any sequence of maps $f_n \colon \bigwedge^i H_n \to V_n$ commuting with the inclusions $\bigwedge^i H_n \hookrightarrow \bigwedge^i H_{n+1}$ and $V_n \hookrightarrow V_{n+1}$. If $V(\lambda_k)_n \subset \ker f_n$, Lemma 5.11 implies that $V(\lambda_{k+2})_{n+1} \oplus V(\lambda_k)_{n+1} \subset \ker f_{n+1}$. Thus the multiplicity of $V(\lambda_k)_n$ in $\ker f_n$ is nondecreasing. (In fact, by induction we see there is some $k \leq i$ so that $\ker f_n$ is always exactly $V(\lambda_i)_n \oplus \cdots \oplus V(\lambda_k)_n$ for sufficiently large n.) The same applies to $\operatorname{im} f_n$.

Heisenberg Lie algebras. As an explicit example to which Remark 5.12 applies, we consider the Heisenberg Lie algebras $\mathcal{H}_{2n+1}$, defined as follows. Given a symplectic form ω on $\mathbb{Q}^{2n}$, this is the central extension

$$0 \to \mathbb{Q} \to \mathcal{H}_{2n+1} \to \mathbb{Q}^{2n} \to 0$$

classified by $\omega \in H^2(\mathbb{Q}^{2n};\mathbb{Q}) \approx \bigwedge^2 \mathbb{Q}^{2n}$. Since the natural action of $\operatorname{Sp}_{2g}\mathbb{Q}$ on $\mathbb{Q}^{2n}$ preserves ω, it extends to an action of $\operatorname{Sp}_{2g}\mathbb{Q}$ on $\mathcal{H}_{2n+1}$. There is an obvious inclusion $\mathcal{H}_{2n+1} \hookrightarrow \mathcal{H}_{2n+3}$ which makes $\{\mathcal{H}_{2n+1}\}$ into a consistent sequence of Lie algebras. As an $\operatorname{Sp}_{2n}\mathbb{Q}$–representation $\mathcal{H}_{2n+1} \approx V(0) \oplus V(\lambda_1)$, so the sequence $\{\mathcal{H}_{2n+1}\}$ is uniformly representation stable.

Note that

$$\textstyle\bigwedge^i \mathcal{H}_{2n+1} \approx \bigwedge^i (H_n \oplus \mathbb{Q}) \approx \bigwedge^i H_n \oplus \bigwedge^{i-1} H_n$$

and that this decomposition is respected by the maps $\bigwedge^i \mathcal{H}_{2n+1} \hookrightarrow \bigwedge^i \mathcal{H}_{2n+3}$. So $\{\bigwedge^i \mathcal{H}_{2n+1}\}$ is uniformly stable for each $i \geq 0$ and we can apply Remark 5.12 to the complex

$$\cdots \longrightarrow \textstyle\bigwedge^3 \mathcal{H}_{2n+1} \xrightarrow{\partial_3} \bigwedge^2 \mathcal{H}_{2n+1} \xrightarrow{\partial_2} \mathcal{H}_{2n+1} \xrightarrow{\partial_1} \mathbb{Q},$$

to conclude that $\{\ker \partial_i\}$ and $\{\operatorname{im} \partial_i\}$ are uniformly stable for each i. This shows that the homology of $\mathcal{H}_{2n+1}$ is uniformly stable. (It turns out that in the complex above at most one differential is nonzero in each grading, so it is easy to compute by hand that $H_i(\mathcal{H}_{2n+1};\mathbb{Q}) = V(\omega_i)$ for $i \leq n$ and see uniform stability directly; see, e.g., [CT2, Theorem 4.2].) We therefore have the following.

Example 5.13. For each $i \geq 0$ the sequence of $\operatorname{Sp}_{2n}\mathbb{Q}$–representations $\{H_i(\mathcal{H}_{2n+1};\mathbb{Q})\}$ is uniformly representation stable.

To extend this argument to adjoint and exterior coefficients, all that would be needed is to duplicate Lemma 5.11 for $\bigwedge^i H_n \otimes H_n$ and $\bigwedge^k H_n \otimes \bigwedge^\ell H_n$. However, we do not do this here, since Cagliero–Tirao have already computed these homology groups. The adjoint homology $H_i(\mathcal{H}_{2n+1};\mathcal{H}_{2n+1})$ is $V(\omega_1 + \omega_i) \oplus V(\omega_{i+1})$ for $i < n$ [CT2, Corollary 4.15]. For the exterior homology, the irreducibles appearing in $H_i(\mathcal{H}_{2n+1};\bigwedge^k \mathcal{H}_{2n+1})$ always correspond to the sum of two fundamental weights $V(\omega_j + \omega_\ell)$, with multiplicity either 1 or 0, independent of n for $i \leq n$ [CT2, Theorem 4.13]. In both cases we have uniform stability.

Example 5.14 (Cagliero–Tirao)**.** For each fixed $i \geq 0$ and $k \geq 0$ the sequences of $\operatorname{Sp}_{2n}\mathbb{Q}$–representations $\{H_i(\mathcal{H}_{2n+1};\mathcal{H}_{2n+1})\}$ and $\{H_i(\mathcal{H}_{2n+1};\bigwedge^k \mathcal{H}_{2n+1})\}$ are uniformly representation stable.

5.4 The Malcev Lie algebra of the pure braid group

In this subsection we describe a conjecture which can be thought of as a "infinitesimal" version of Theorem 4.1. Let $\Gamma = P_n$, the pure braid group on n strands, and let $\mathfrak{p}_n := \mathrm{gr}(P_n)$. The Lie algebra $\mathfrak{p}_n$ occurs, among other places, in the theory of Vassiliev invariants. Drinfeld and Kohno (see [Ko]) gave a finite presentation for $\mathfrak{p}_n$, as follows. Let $\mathcal{L}(\{X_{ij}\})$ denote the free Lie algebra on the set of formal symbols $\{X_{ij} : 1 \leq i, j \leq n, i \neq j\}$. Then for $n \geq 4$,

$$\mathfrak{p}_n = \mathcal{L}(\{X_{ij}\})/R$$

where R is the ideal generated by the quadratic relations:

$$[X_{ij}, X_{kl}] \text{ with } i, j, k, l \text{ distinct}$$

$$[X_{ij}, X_{ik} + X_{jk}] \text{ with } i, j, k \text{ distinct}$$

Consider the action of S_n on $\mathfrak{p}_n$. Since the relations above are homogeneous, the grading on $\mathcal{L}(\{X_{ij}\})$ descends to a grading on $\mathfrak{p}_n$, which is clearly preserved by the action of S_n. Let $\mathfrak{p}_n^i$ denote the i^{th} graded component of $\mathfrak{p}_n$.

Conjecture 5.15 (Representation stability for $\mathfrak{p}_n$)**.** *For each fixed $i \geq 1$, the sequence $\{\mathfrak{p}_n^i\}$ is a uniformly representation stable sequence of S_n–representations.*

As evidence for this conjecture, we point out that P_n is pseudo-nilpotent [Ha2, Example 5.12]. Furthermore, by Theorem 4.1, for each fixed $i \geq 0$ the cohomology $H^i(\mathfrak{p}_n; \mathbb{Q}) = H^i(P_n; \mathbb{Q})$ is a uniformly stable sequence of S_n–representations. Thus Conjecture 5.15 would follow as in Corollary 5.10 if we had a version of Theorem 5.3 for representations of S_n. We remark that not all aspherical hyperplane complements have pseudo-nilpotent fundamental group (see e.g. Falk [Fa, Proposition 5.1, Example 5.3]), so we do not expect Conjecture 5.15 to extend to all such groups.

6 Homology of the Torelli subgroups of $\mathrm{Mod}(S)$ and $\mathrm{Aut}(F_n)$

In this section we discuss representation stability in the context of the homology of the Torelli groups associated with mapping class groups and automorphism groups of free groups. Most of the picture here is conjectural. However, before the idea of representation stability, even a conjectural picture of these homology groups was lacking.

6.1 Homology of the Torelli group

Let $S_{g,1}$ be a connected, compact, oriented surface of genus $g \geq 2$ with one boundary component. Let $H \coloneqq H_1(S_{g,1};\mathbb{Q})$ and let $H_{\mathbb{Z}} \coloneqq H_1(S_{g,1},\mathbb{Z})$. The *mapping class group* $\mathrm{Mod}_{g,1}$ is the group of homotopy classes of homeomorphisms of $S_{g,1}$, where both the homeomorphisms and the homotopies fix $\partial S_{g,1}$ pointwise. The action of $\mathrm{Mod}_{g,1}$ on $H_{\mathbb{Z}}$ preserves algebraic intersection number, which is a symplectic form on $H_{\mathbb{Z}}$, yielding a symplectic representation which fits into the exact sequence

$$1 \to \mathcal{I}_{g,1} \to \mathrm{Mod}_{g,1} \to \mathrm{Sp}_{2g}\,\mathbb{Z} \to 1,$$

where $\mathcal{I}_{g,1}$ is the *Torelli group*, consisting of those $f \in \mathrm{Mod}_{g,1}$ acting trivially on $H_{\mathbb{Z}}$. The conjugation action of $\mathrm{Mod}_{g,1}$ on $\mathcal{I}_{g,1}$ descends to an action of $\mathrm{Sp}_{2g}\,\mathbb{Z}$ by outer automorphisms, which gives each $H_i(\mathcal{I}_{g,1};\mathbb{Q})$ the structure of an $\mathrm{Sp}_{2g}\,\mathbb{Z}$–module. The natural inclusion of surfaces $S_{g,1} \hookrightarrow S_{g+1,1}$ induces an inclusion $\mathcal{I}_{g,1} \hookrightarrow \mathcal{I}_{g+1,1}$, by extending by the identity. For each $i \geq 0$ the induced homomorphism $H_i(\mathcal{I}_{g,1};\mathbb{Q}) \to H_i(\mathcal{I}_{g+1};\mathbb{Q})$ respects the action of $\mathrm{Sp}_{2g}\,\mathbb{Z}$.

If G is any group and V is any (perhaps infinite dimensional) G–representation, we define the *finite-dimensional part* of V, denoted V^{fd}, to be the subspace of V consisting of those vectors whose G–orbit spans a finite-dimensional subspace of V. Note that V^{fd} may itself be infinite dimensional. Our first conjecture about $H_i(\mathcal{I}_{g,1};\mathbb{Q})$ makes a prediction about its finite-dimensional part. It is a slight refinement of a conjecture we first stated in [CF].

Conjecture 6.1 (Homology of the Torelli group)**.** *For each fixed $i \geq 1$, each of the following statements holds.*

Preservation of finite-dimensionality: *The natural map*

$$H_i(\mathcal{I}_{g,1};\mathbb{Q})^{\mathrm{fd}} \to H_i(\mathcal{I}_{g+1,1};\mathbb{Q})$$

induced by the inclusion $\mathcal{I}_{g,1} \hookrightarrow \mathcal{I}_{g+1,1}$ has image contained in $H_i(\mathcal{I}_{g+1,1};\mathbb{Q})^{\mathrm{fd}}$.

Rationality: *Every irreducible $\mathrm{Sp}_{2g}\,\mathbb{Z}$–subrepresentation in $H_i(\mathcal{I}_{g,1};\mathbb{Q})^{\mathrm{fd}}$ is the restriction of an irreducible $\mathrm{Sp}_{2g}\,\mathbb{Q}$–representation.*

Stability: *The sequence of $\mathrm{Sp}_{2g}\,\mathbb{Q}$–representations $\{H_i(\mathcal{I}_{g,1};\mathbb{Q})^{\mathrm{fd}}\}$ is uniformly representation stable.*

Remarks.

1. Along with Conjecture 6.1 for $H_i(\mathcal{I}_{g,1};\mathbb{Q})$, we have a corresponding, equivalent conjecture for $H^i(\mathcal{I}_{g,1};\mathbb{Q})$, with stability in the sense of Definition 2.8.

2. A form of the Margulis Superrigidity Theorem (see [Ma, Theorem VIII.B]) gives that any finite-dimensional representation (over $\mathbb{C}$) of $\mathrm{Sp}_{2g}\,\mathbb{Z}$ virtually extends to a rational representation of $\mathrm{Sp}_{2g}\,\mathbb{Q}$.[1] Thus the Rationality part of Conjecture 6.1 is meant to ensure that we can extend to $\mathrm{Sp}_{2g}\,\mathbb{Q}$ without passing to a finite index subgroup. We also remark that, by a statement close to the Borel Density Theorem (namely Proposition 3.2 of [Bo2]), a representation of $\mathrm{Sp}_{2g}\,\mathbb{Q}$ is irreducible if and only if its restriction to $\mathrm{Sp}_{2g}\,\mathbb{Z}$ is irreducible, so we can (and will) ignore this distinction. Similar statements apply to $\mathrm{GL}_n\,\mathbb{Q}$ as well.

3. It is known that the "finite-dimensional part" of $H_i(\mathcal{I}_{g,1};\mathbb{Q})$ is not all of $H_i(\mathcal{I}_{g,1};\mathbb{Q})$; see the examples discussed after Conjecture 6.7 below.

4. The Torelli group is often defined for closed surfaces or for surfaces with punctures. In this case there are no maps connecting the Torelli groups for different g, so the strongest statement one could hope for is multiplicity stability for the homology of the corresponding Torelli groups. We conjecture this to be true.

Some evidence for Conjecture 6.1 in each dimension $i \geq 1$ is given and discussed in detail in [CF]. Further, we note that Conjecture 6.1 is true for $i = 1$, by Johnson's computation that

$$H_1(\mathcal{I}_{g,1}) \approx H_1(\mathcal{I}_{g,*}) \approx V(\omega_3) \oplus V(\omega_1) \quad \text{for each } g \geq 3.$$

Finally, we note the well-known analogy of $\mathcal{I}_{g,1}$ and $\mathrm{Sp}_{2g}\,\mathbb{Z}$ with P_n and S_n. Since representation stability holds for the latter example (Theorem 4.1), one is led to believe it holds for the former.

Malcev Lie algebra of $\mathcal{I}_{g,1}$. There is a kind of "infinitesimal" version of Conjecture 6.1, parallel to Conjecture 5.15 for the pure braid group. Let $\mathrm{gr}(\mathcal{I}_{g,1})$ denote the graded rational Lie algebra associated to the lower central series of $\mathcal{I}_{g,1}$ (see §5.3), and let $\mathrm{gr}(\mathcal{I}_{g,1})^i$ denote its i^{th} graded piece. Hain [Ha] computed $\mathrm{gr}(\mathcal{I}_g)$ in [Ha]. This was extended by Habegger–Sorger [HS] to the case of surfaces with boundary. To state their result, let $H = H_1(S_{g,1};\mathbb{Q})$ as usual, and for any vector space V let $\mathcal{L}(V)$ denote the free

[1] One can also use the solution to the congruence subgroup property for $\mathrm{Sp}_{2g}\,\mathbb{Z}$, $g > 1$ here; see [BMS].

Lie algebra on V as in §5.3. The extension by Habegger–Sorger of Hain's theorem states that, for all $g \geq 6$, the rational Lie algebra $\mathrm{gr}(\mathcal{I}_{g,1})$ has a presentation:

$$\mathrm{gr}(\mathcal{I}_{g,1}) = \mathcal{L}(\textstyle\bigwedge^3 H)/(R_1, R_2)$$

where (R_1, R_2) denotes the ideal generated by the $\mathrm{Sp}_{2g}(\mathbb{Q})$–span of the two elements

$$R_1 = (a_1 \wedge a_2 \wedge b_2) \wedge (a_3 \wedge a_4 \wedge b_4)$$
$$R_2 = (a_1 \wedge a_2 \wedge b_2) \wedge (a_g \wedge \omega)$$

where $w := \sum_{i=1}^{g} a_i \wedge b_i$. One nontrivial consequence of Hain's theorem is that the natural $\mathrm{Sp}_{2g}\,\mathbb{Z}$–action on $\mathrm{gr}(\mathcal{I}_{g,1})$ extends to an $\mathrm{Sp}_{2g}\,\mathbb{Q}$–action. As previously mentioned, $\mathrm{gr}(\mathcal{I}_{g,1})$ is the associated graded associated to the Malcev Lie algebra of $\mathcal{I}_{g,1}$, and $\mathrm{gr}(\mathcal{I}_{g,1})^i$ denotes the i^{th} graded component of $\mathrm{gr}(\mathcal{I}_{g,1})$.

Conjecture 6.2 (Stability of the Malcev Lie algebra of $\mathcal{I}_{g,1}$)**.** *For each fixed $i \geq 1$ the sequence $\{\mathrm{gr}(\mathcal{I}_{g,1})^i\}$ of $\mathrm{Sp}_{2g}\,\mathbb{Q}$-representations is uniformly representation stable.*

As evidence for Conjecture 6.2, we remark that the conjecture is true when $i = 1$ and when $i = 2$, as follows. Johnson proved [Jo2] that

$$\mathrm{gr}(\mathcal{I}_{g,1})^1 \approx \textstyle\bigwedge^3 H \approx V(1,1,1) \oplus V(1)$$

as $\mathrm{Sp}_{2g}\,\mathbb{Z}$–modules, and Habegger–Sorger [HS, Theorem 2.2] use the work of Hain [Ha] to deduce that (in our terminology) :

$$\mathrm{gr}(\mathcal{I}_{g,1})^2 \approx V(2,2) \oplus V(1,1) \oplus V(0)^{\oplus 2}$$

as $\mathrm{Sp}_{2g}\,\mathbb{Z}$–modules.

6.2 Homology of IA_n

The above discussion has an analogy in the case of free groups and their automorphisms. Let F_n denote the free group of rank n, and let $\mathrm{Aut}(F_n)$ denote its automorphism group. The action of $\mathrm{Aut}(F_n)$ on $H_1(F_n;\mathbb{Z})$ gives the well-known exact sequence

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}_n\,\mathbb{Z} \to 1.$$

The conjugation action of $\mathrm{Aut}(F_n)$ on IA_n descends to an outer action of $\mathrm{GL}_n\,\mathbb{Z}$, giving each $H_i(\mathrm{IA}_n;\mathbb{Q})$ the structure of a $\mathrm{GL}_n\,\mathbb{Z}$–module. The standard inclusion $F_n \hookrightarrow F_{n+1}$

induces an inclusion $\mathrm{IA}_n \hookrightarrow \mathrm{IA}_{n+1}$ by extending by the identity. Thus for each $i \geq 0$ we have an induced homomorphism $H_i(\mathrm{IA}_n; \mathbb{Q}) \to H_i(\mathrm{IA}_{n+1}; \mathbb{Q})$ of $\mathrm{GL}_n \mathbb{Q}$–representations.

It is natural to conjecture the analogue of Conjecture 6.1 for IA_n, and in particular that $\{H_i(\mathrm{IA}_n; \mathbb{Q})\}$ is representation stable. However such a conjecture would not capture what is going on, even in dimension 1: a computation of Andreadakis, Farb, Kawazumi, and Cohen-Pakianathan (see e.g. [Ka]) gives:

$$H_1(\mathrm{IA}_n; \mathbb{Q}) \approx \textstyle\bigwedge^2 \mathbb{Q}^n \otimes (\mathbb{Q}^n)^* \approx V(L_1 + L_2 - L_n) \oplus V(L_1) \tag{20}$$

from which we see that the decomposition of the sequence $\{H_1(\mathrm{IA}_n; \mathbb{Q})\}$ does not stabilize (except in the trivial sense, observing that *no* representation ever appears twice as the first summand). However, the notion of *mixed* tensor stability, defined in §2.4, suffices to capture the stability here. Indeed, the computation in (20) shows that the sequence $\{H_1(\mathrm{IA}_n; \mathbb{Q})\}$ is mixed representation stable, since

$$H_1(\mathrm{IA}_n; \mathbb{Q}) = V(1, 1; 1) \oplus V(1)$$

for sufficiently large n. With this alteration, we give the analogue of Conjecture 6.1 for IA_n.

Conjecture 6.3 (Homology of IA_n)**.** *For each fixed $i \geq 1$, each of the following statements hold.*

Preservation of finite-dimensionality: *The natural map*

$$H_i(\mathrm{IA}_n; \mathbb{Q})^{\mathrm{fd}} \to H_i(\mathrm{IA}_{n+1}; \mathbb{Q})$$

induced by the inclusion $\mathrm{IA}_n \hookrightarrow \mathrm{IA}_{n+1}$ has image contained in $H_i(\mathrm{IA}_{n+1}; \mathbb{Q})^{\mathrm{fd}}$.

Rationality: *Every irreducible $\mathrm{GL}_n \mathbb{Z}$–subrepresentation in $H_i(\mathrm{IA}_n; \mathbb{Q})^{\mathrm{fd}}$ is the restriction of an irreducible $\mathrm{GL}_n \mathbb{Q}$–representation.*

Stability: *The sequence of $\mathrm{GL}_n \mathbb{Q}$–representations $\{H_i(\mathrm{IA}_n; \mathbb{Q})^{\mathrm{fd}}\}$ is uniformly* mixed *representation stable.*

As for the "infinitesimal" version of Conjecture 6.3, we conjecture that each of the $\mathrm{GL}_n \mathbb{Z}$–representations $\mathrm{gr}(\mathrm{IA}_n)^i$ extend to $\mathrm{GL}_n \mathbb{Q}$–representations, and that these form a uniformly stable sequence. However, we would like to point out that the Lie algebra $\mathrm{gr}(\mathrm{IA}_n)$ is still not known.

6.3 Vanishing and finiteness conjectures for the (co)homology of $\mathcal{I}_{g,1}$ and IA_n

We now make a few other natural conjectures concerning the (co)homology of $\mathcal{I}_{g,1}$ and IA_n. Our goal is to give as much of a conjectural picture as possible where there was none before.

A Morita-type conjecture for IA_n. Let $e_i \in H^i(\mathcal{I}_{g,1};\mathbb{Q})$ denote the i^{th} Morita–Mumford–Miller class restricted to $\mathcal{I}_{g,1}$. The following is Conjecture 3.4 of [Mo1].

Conjecture 6.4 (Morita's Conjecture)**.** *The $\mathrm{Sp}_{2g}\,\mathbb{Z}$–invariant stable rational cohomology of $\mathcal{I}_{g,1}$ is generated as a $\mathbb{Q}$–algebra by $\{e_2, e_4, e_6, \ldots\}$.*

Note that all the e_i generate the stable rational cohomology of $\mathrm{Mod}_{g,1}$, by Madsen–Weiss [MW], and the odd classes $e_1, e_3, e_5, \ldots$ vanish when restricted to $\mathcal{I}_{g,1}$. Morita's Conjecture predicts the trivial representations that can occur in $H^i(\mathcal{I}_{g,1};\mathbb{Q})$. However, it is not known which of the even Morita–Mumford–Miller classes e_i, or combinations thereof, are nonzero in $H^*(\mathcal{I}_{g,1};\mathbb{Q})$. Thus even an affirmative answer to Morita's Conjecture would not imply Conjecture 6.1 for the trivial representation.

Since Galatius [Ga] has proven that $H^i(\mathrm{Aut}(F_n);\mathbb{Q}) = 0$ for $n \gg i$, it is natural to make the following conjecture.

Conjecture 6.5 (Vanishing conjecture)**.** *The $\mathrm{GL}_n\,\mathbb{Z}$–invariant part of the stable rational cohomology of IA_n vanishes.*

By the computation $H_1(\mathrm{IA}_n;\mathbb{Q}) \approx \bigwedge^2 \mathbb{Q}^n \otimes (\mathbb{Q}^n)^*$ for $n \geq 3$, which has no trivial subrepresentations, Conjecture 6.5 is true for cohomology in dimension 1.

Two finiteness conjectures. The infinite-dimensional spaces $H_1(\mathcal{I}_{2,1};\mathbb{Q})$, $H_3(\mathcal{I}_3;\mathbb{Q})$, $H_{3n-2}(\mathcal{I}_{n,1};\mathbb{Q})$ and $H_{2n-3}(\mathrm{IA}_n;\mathbb{Q})$ mentioned above are not "stable" in n. One might hope that stably such representations do not arise, and all irreducible $\mathrm{Sp}_{2n}\,\mathbb{Z}$–submodules of $H_i(\mathcal{I}_{n,1};\mathbb{Q})$ and $\mathrm{GL}_n\,\mathbb{Z}$–submodules of $H_i(\mathrm{IA}_n;\mathbb{Q})$ are finite-dimensional for $n \gg i$. The limited evidence we have seems to point to the following.

Conjecture 6.6 (Stable finite-dimensionality)**.** *For each $i \geq 1$ and each n sufficiently large (depending on i), the natural maps*

$$H_i(\mathcal{I}_{n,1};\mathbb{Q})^{\mathrm{fd}} \hookrightarrow H_i(\mathcal{I}_{n,1};\mathbb{Q})$$

and

$$H_i(\mathrm{IA}_n;\mathbb{Q})^{\mathrm{fd}} \hookrightarrow H_i(\mathrm{IA}_n;\mathbb{Q})$$

are isomorphisms.

One may even go so far as to give a conjectural picture of all of the homology of $\mathcal{I}_{n,1}$ and IA_n, including the infinite-dimensional part.

Conjecture 6.7 (Unstable finite generation)**.** *For each $i \geq 1$ and each $n \geq 1$:*

1. *The module $H_i(\mathcal{I}_{n,1}; \mathbb{Q})$ is a finitely-generated module over $\mathrm{Sp}_{2n} \mathbb{Z}$.*
2. *The module $H_i(\mathrm{IA}_n; \mathbb{Q})$ is a finitely-generated module over $\mathrm{GL}_n \mathbb{Z}$.*

Note that Conjecture 6.7 is consistent with all known computations of the homology groups of $\mathcal{I}_{n,1}$ and IA_n, including those that are known to be infinite-dimensional over $\mathbb{Q}$. Mess [Me, Corollary 1] proved that $H_1(\mathcal{I}_{2,1}; \mathbb{Q})$ contains an infinite-dimensional irreducible permutation $\mathrm{Sp}_4 \mathbb{Z}$–module, and Johnson–Millson showed that $H_3(\mathcal{I}_3; \mathbb{Q})$ contains an infinite-dimensional irreducible permutation $\mathrm{Sp}_6 \mathbb{Z}$–module [Me, Proposition 5]. The classes in $H_{2n-3}(\mathrm{IA}_n; \mathbb{Q})$ found by Bestvina–Bux–Margalit [BBM1] span an infinite-dimensional subspace, but as a $\mathrm{GL}_n \mathbb{Z}$–module this is a permutation module generated by a single element; similarly, the classes in $H_{3g-2}(\mathcal{I}_{g,1}; \mathbb{Q})$ found by Bestvina–Bux–Margalit [BBM2] span a cyclic $\mathrm{Sp}_{2g} \mathbb{Z}$–module. In particular, the action of $\mathrm{GL}_n \mathbb{Z}$ or $\mathrm{Sp}_{2g} \mathbb{Z}$ on such a subspace cannot be extended to an action of the corresponding $\mathbb{Q}$–group $\mathrm{GL}_n \mathbb{Q}$ or $\mathrm{Sp}_{2g} \mathbb{Q}$.

7 Flag varieties, Schubert varieties, and rank-selected posets

The goal of this section is to demonstrate the appearance of representation stability in the cohomology of various natural families of algebraic varieties, as well as in algebraic combinatorics. These results are used in [CEF] to compute arithmetic statistics for maximal tori in $\mathrm{GL}_n(\mathbb{F}_q)$ and Lagrangian tori in $\mathrm{Sp}_{2g}(\mathbb{F}_q)$.

7.1 Cohomology of flag varieties

Let $\mathcal{F}_n$ be the complete flag variety parametrizing complete flags in $\mathbb{C}^n$; this can be identified with G/B where $G = \mathrm{GL}_n \mathbb{C}$ and B is the Borel subgroup consisting of upper triangular matrices. The inclusion $\mathrm{GL}_n \mathbb{C} \hookrightarrow \mathrm{GL}_{n+1} \mathbb{C}$ induces an inclusion of $\mathcal{F}_n$ as a closed subvariety of $\mathcal{F}_{n+1}$. In terms of flags, this amounts to regarding a complete flag $V_1 < \cdots < V_n = \mathbb{C}^n$ as a flag in $\mathbb{C}^{n+1}$ by appending $\mathbb{C}^{n+1}$ itself. The unitary group $U(n)$ also acts on $\mathcal{F}_n$, with stabilizer a maximal torus T, giving an identification of $\mathcal{F}_n$ with $U(n)/T$. The normalizer $N(T)$ acts on $U(n)/T$ on the right, which factors through an action of the Weyl group $W = N(T)/T$. In this case W can be identified with the group

S_n of permutation matrices, so we obtain an S_n–action on $\mathcal{F}_n$, and thus an S_n–action on $H^i(\mathcal{F}_n;\mathbb{Q})$ for each $i \geq 0$.

The inclusion $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$ induces for each $i \geq 0$ a homomorphism $H^i(\mathcal{F}_{n+1};\mathbb{Q}) \to H^i(\mathcal{F}_n;\mathbb{Q})$, and the sequence $\{H^i(\mathcal{F}_n;\mathbb{Q})\}$ is easily seen to be a consistent sequence of S_n–representations. We will prove that this sequence is representation stable in the sense of Definition 2.8.

Theorem 7.1 (Stability for the cohomology of flag varieties)**.** *For each fixed $i \geq 0$, the sequence $\{H^i(\mathcal{F}_n;\mathbb{Q})\}$ of S_n–representations is representation stable.*

Proof. The cohomology $H^*(\mathcal{F}_n;\mathbb{Q})$ is described as follows. The trivial bundle $\mathcal{F}_n \times \mathbb{C}^n$ is filtered by k–dimensional subbundles U_i for $0 \leq i \leq n$, where U_i over a given flag is the ith subspace of that flag. The quotients $E_i := U_i/U_{i-1}$ are line bundles over $\mathcal{F}_n$. Let $x_i \in H^2(\mathcal{F}_n;\mathbb{Q})$ be the first Chern class $c_1(E_i)$. These classes $\{x_i\}$ generate $H^*(\mathcal{F}_n;\mathbb{Q})$, as we will see in more detail below. Note that S_n acts on $H^2(\mathcal{F}_n;\mathbb{Q})$ by permuting the generators x_i.

We are trying to prove representation stability in the sense of Definition 2.8. To prove the injectivity and surjectivity conditions, first note that $x_i \in H^2(\mathcal{F}_{n+1};\mathbb{Q})$ restricts to $x_i \in H^2(\mathcal{F}_n;\mathbb{Q})$. A basis for $H^i(\mathcal{F}_n;\mathbb{Q})$ is given by $\mathcal{B}_n = \{x_1^{j_1} \cdots x_n^{j_n} | 0 \leq j_k < k\}$ (see [Fu, Proposition 10.3]). Note that the subset of $\mathcal{B}_{n+1}$ consisting of elements with $j_{n+1} = 0$ restricts bijectively to the basis $\mathcal{B}_n$. Furthermore, as long as $n > i$, any element of $\mathcal{B}_{n+1}$ with degree i can be rearranged by a permutation in S_{n+1} to have $j_{n+1} = 0$ while still satisfying $0 \leq j_k < k$ for all k. This shows that for large enough n, the S_{n+1}–orbit of the degree i terms of this subset spans $H^i(\mathcal{F}_{n+1};\mathbb{Q})$, as desired.

Proving stability of multiplicities is more involved. A general theorem of Borel [Bo1] states that the cohomology $H^*(\mathcal{F}_n;\mathbb{Q})$ is isomorphic to the co-invariant algebra on the x_i, defined as follows. Let $\mathbb{Q}[x_1,\ldots,x_n]^{S_n}$ be the ring of symmetric polynomials, and let I_n be the ideal of $\mathbb{Q}[x_1,\ldots,x_n]$ generated by all symmetric polynomials with zero constant term. The *co-invariant algebra* $R[x_1,\ldots,x_n]$ is defined to be the quotient

$$R[x_1,\ldots,x_n] := \mathbb{Q}[x_1,\ldots,x_n]/I_n.$$

Thus $R[x_1,\ldots,x_n]$ inherits a natural grading from $\mathbb{Q}[x_1,\ldots,x_n]$, and $H^*(\mathcal{F}_n;\mathbb{Q})$ is isomorphic to $R[x_1,\ldots,x_n]$ as a graded S_n–module (see [Fu, Proposition 10.3] for a combinatorial proof). It is not hard to see that $R[x_1,\ldots,x_n]$, and thus $H^*(\mathcal{F}_n;\mathbb{Q})$, is in fact isomorphic to the regular representation $\mathbb{Q}S_n$, which is *not* representation stable. However, looking at each homogeneous piece individually, we have the following theorem of Stanley, Lusztig, and Kraskiewicz–Weyman:

Theorem 7.2 ([Re], Theorem 8.8)**.** *For any partition λ, as long as $i \le \binom{n}{2}$, the multiplicity of $V(\lambda)_n$ in $R_i[x_1, \ldots, x_n]$ equals the number of standard tableaux of shape $\lambda[n]$ with major index equal to i.*

Recall that a *standard tableau of shape* λ is a bijective labeling of the boxes of the Young diagram for λ by the numbers $1, \ldots, n$ with the property that in each row and in each column the labels are increasing. Given such a labeling, the *descent set* is the set of numbers i so that the box labeled $i+1$ is in a lower row than the box labeled i. The *major index* of a tableau is the sum of the numbers in the descent set.

Fix a partition λ and a finite set $S \subset \mathbb{N}$. Let $\mathcal{S}_n$ be the set of standard tableaux of shape $\lambda[n]$ with descent set exactly S. We will show below that for sufficiently large n, the size of $\mathcal{S}_n$ is equal to the size of $\mathcal{S}_{n+1}$. Since only finitely many $S \subset \mathbb{N}$ have $\sum_{j \in S} j = i$, applying Theorem 7.2 once n is sufficiently large will prove that the multiplicity of $V(\lambda)_n$ in $R_i[x_1, \ldots, x_n]$ is eventually constant, as desired.

First we exhibit an injection from $\mathcal{S}_n$ into $\mathcal{S}_{n+1}$. Note that the Young diagram for $\lambda[n+1]$ is obtained from that of $\lambda[n]$ by adding an additional box at the end of the first row. Our operation on tableaux will be simply to fill this newly-added box with $n+1$. Since neither n nor $n+1$ can be a descent in the resulting tableau, and whether any other j is a descent remains unchanged, the descent set is unchanged by this operation. Thus this operation, which is clearly injective, maps $\mathcal{S}_n \to \mathcal{S}_{n+1}$.

It remains to show that for sufficiently large n, the operation is also surjective. Equivalently, we must show that for sufficiently large n, any tableau of shape $\lambda[n]$ with descent set S has the label n in the top row. Let $k = \max S$. If the label n is not in the top row, then no label greater than k can be in the top row, for otherwise at least one number between k and $n-1$ would be a descent. But exactly $|\lambda|$ boxes are not contained in the first row of $\lambda[n]$. Thus taking n greater than $k + |\lambda|$, the pigeonhole principle implies that in every tableau some label greater than k appears in the top row. It follows that any tableau with descent set S has the label n in the top row, as desired.

Applying Theorem 7.2, we see that the multiplicity of $V(\lambda)_n$ in $H^i(\mathcal{F}_n; \mathbb{Q})$ is eventually independent of n, as desired. □

It can be seen from the proof of Theorem 7.1 that $H^i(\mathcal{F}_n; \mathbb{Q}) = R_i[x_1, \ldots, x_n]$ is in fact uniformly representation stable.

Lagrangian flags. Let $\mathcal{F}'_n$ be the flag variety parametrizing pairs of a Lagrangian subspace L of $\mathbb{C}^{2n}$, together with a complete flag on L. For $G = \mathrm{Sp}_{2n}\mathbb{C}$ and B a Borel subgroup, $\mathcal{F}'_n$ is identified with G/B. The Weyl group in this case is the hyperoctahedral group W_n. Borel proved in [Bo1] that $H^*(\mathcal{F}'_n;\mathbb{Q})$ is isomorphic to the co-invariant algebra for W_n.

Theorem 7.3. *For each fixed $i \geq 0$, the sequence $\{H^i(\mathcal{F}'_n;\mathbb{Q})\}$ of W_n–representations is representation stable (in the sense of Definition 2.8).*

Proof. Given a double partition $\lambda = (\lambda^+, \lambda^-)$, Stembridge [Ste, Theorem 5.3] generalized Stanley's theorem and proved that the multiplicity of $V(\lambda)_n$ in the i^{th} graded piece of the co-invariant algebra for W_n is the number of double standard Young tableaux of shape $\lambda[n]$ whose flag major index is i, as long as $n^2 \geq i$. We now summarize the necessary terminology. If $|\lambda^-| = k$, recall that $\lambda[n] = (\lambda^+[n-k], \lambda^-)$. A *double standard Young tableau* is a bijective labeling by the labels $1, \ldots, n$ of the diagrams for $\lambda^+[n-k]$ and λ^- together, which within each diagram is increasing on each row and column. The *flag descent set* can be described as follows. Place the diagram for λ^- above the diagram for $\lambda^+[n-k]$. Then the flag descent set consists of those j for which $j+1$ appears below j in the tableau, together with n if and only if n appears in the diagram for λ^-. Finally, the *flag major index* is

$$2\sum j + |\lambda^-|,$$

where the sum is over those j in the flag descent set.

As in the proof of Theorem 7.1, it will suffice to prove that for each double partition λ and each finite set $S \subset \mathbb{N}$, the number of double standard tableaux of shape $\lambda[n]$ with flag descent set S is eventually constant. Passing from double tableaux of shape $\lambda[n]$ to $\lambda[n+1]$ requires adding a box to the first row of $\lambda^+[n-k]$; we always fill that box with $n+1$. Call this the *main row* of the diagram. Note that the definition of flag descent set is such that this operation does not change the descent set. Thus it suffices to show that for sufficiently large n, every double standard Young tableau of shape $\lambda[n]$ having flag descent set S has n in the main row. When n is larger than $\max S$ it cannot appear in the diagram for λ^- above the main row. But there are exactly $|\lambda^+|$ boxes below the main row. So once $n \geq |\lambda^+| + \max S$, if n were below the main row, some number larger than $\max S$ would appear in the descent set. Thus for sufficiently large n, the label n must appear in the main row, as desired.

Since only finitely many descent sets $S \subset N$ have associated flag major index i, we conclude that for each double partition λ, the multiplicity of $V(\lambda)_n$ in $H^i(\mathcal{F}'_n;\mathbb{Q})$ is eventually constant. Injectivity and surjectivity follow as in the proof of Theorem 7.1, so we conclude that $\{H^i(\mathcal{F}'_n;\mathbb{Q})\}$ is representation stable. □

7.2 Cohomology of Schubert varieties

Recall from above that $\mathcal{F}_n = G/B$ is the variety of complete flags in $\mathbb{C}^n$, where $G = \mathrm{GL}_n \mathbb{C}$ and B is a Borel subgroup; G naturally acts on $\mathcal{F}_n = G/B$ by left multiplication. Choosing the standard flag in $\mathbb{C}^n$ as a basepoint, each permutation $w \in S_n$ determines a flag, which can be identified with $[w] \in G/B$. The orbits of the flags $[w]$ under the Borel subgroup B are the Bruhat cells BwB. The *Schubert variety* X_w associated to w is the closure $\overline{B[w]}$ in G/B of the Bruhat cell BwB.

Let T be a maximal torus in G. Then the G–action on G/B restricts to a T–action and this T–action preserves X_w. We denote by $H^*_T(X_w;\mathbb{Q})$ the equivariant cohomology with respect to T. There is an action of S_n on $H^*_T(X_w;\mathbb{Q})$, which is somewhat involved to describe; it is given in Tymoczko [Ty].

Given $w \in S_n$, we can view it as an element of S_{n+1} by the usual inclusion; let $X_w[n+1]$ be the corresponding Schubert variety in $\mathcal{F}_{n+1}$, and so on. Then the equivariant cohomology $\{H^*_T(X_w[n];\mathbb{Q})\}$ is a consistent sequence of S_n–representations.

Theorem 7.4 (Stability for the cohomology of Schubert varieties)**.** *Let w be any permutation. Then for each fixed $i \geq 0$ the sequence $\{H^i_T(X_w;\mathbb{Q})\}$ of S_n–representations is multiplicity stable.*

Proof of Theorem 7.4. For $v \in S_n$, let $\ell(v)$ denote the length of v with respect to the standard Coxeter generators. For a graded ring M let $M[n]$ denote the shift in grading by n. Tymoczko proved [Ty, Theorem 1.1] that

$$H^*_T(X_w;\mathbb{Q}) = \bigoplus_{[v]\in X_w} \mathbb{Q}[t_1,\ldots,t_n][\ell(v)]$$

as graded S_n–modules. Here the sum is over those permutations $v \in S_n$ whose image $[v]$ lies in X_w. It is standard (see, e.g., [Fu, Proposition 10.7]) that these are exactly the v for which $v \leq w$ in the Bruhat partial order. The Bruhat order has the property that $v \leq w$ in S_n if and only if $v \leq w$ when considered as elements of S_{n+1}. Thus for fixed w the collection of v in the sum is independent of n; similarly the lengths $\ell(v)$ do not change. Denote the degree i homogeneous polynomials over $\mathbb{Q}$ by $P_i[x_1,\ldots,x_n]$. Since

$$H^i_T(X_w;\mathbb{Q}) = \bigoplus_{[v]\in X_w} P_{i-\ell(v)}[t_1,\ldots,t_n],$$

it suffices to prove that the homogeneous polynomials $\{P_i[x_1,\ldots,x_n]\}$ are representation stable for each $i \geq 0$.

As an aside, we remark that the surjection $H^i_T(X_w;\mathbb{Q}) \to H^i(X_w;\mathbb{Q})$ is given by mapping each $t_i \mapsto 0$, so

$$H^i(X_w;\mathbb{Q}) = \bigoplus_{\substack{[v]\in X_w,\\ \ell(v)=i}} \mathbb{Q}.$$

Combining this with the preceding discussion, we see that classical homological stability holds for the ordinary cohomology $\{H^i(X_w;\mathbb{Q})\}$ of Schubert varieties.

Note that $P_i[x_1,\ldots,x_{n+1}]$ is spanned by monomials which involve at most i variables; thus for $n \geq i$ any such monomial is the image under S_{n+1} of a monomial in $P_i[x_1,\ldots,x_n]$. This verifies surjectivity, and injectivity is immediate. Let

$$\Lambda[x_1,\ldots,x_n] \coloneqq \mathbb{Q}[x_1,\ldots,x_n]^{S_n}$$

be the ring of symmetric polynomials; $\mathbb{Q}[x_1,\ldots,x_n]$ is a free $\Lambda[x_1,\ldots,x_n]$–module, and in fact

$$\mathbb{Q}[x_1,\ldots,x_n] \approx R[x_1,\ldots,x_n] \otimes_{\mathbb{Q}} \Lambda[x_1,\ldots,x_n]$$

as graded S_n–modules (see, e.g., the proof of [Re, Theorem 8.8]). It follows that

$$P_i[x_1,\ldots,x_n] \approx \bigoplus_{j+k=i} R_j[x_1,\ldots,x_n]^{\oplus \dim \Lambda_k[x_1,\ldots,x_n]}$$

as S_n–representations. We can see that the dimension $\dim \Lambda_k[x_1,\ldots,x_n]$ is eventually constant as follows. It is classical that the ring of symmetric functions is a polynomial algebra $\mathbb{Q}[e_1,\ldots,e_n]$ on the elementary symmetric polynomials $\{e_j\}$. Since the degree of e_j is j, we see that once n is larger than i, the dimension of $\Lambda_i[x_1,\ldots,x_n]$ is the number of partitions of i and thus does not depend on n.

For any λ the multiplicity of $V(\lambda)_n$ in $R_j[x_1,\ldots,x_n]$ is eventually constant by Theorem 7.1. Since there are finitely many solutions to $j+k=i$ once i is fixed, we may assume all these multiplicities have stabilized for n large enough. We conclude that the multiplicity of $V(\lambda)_n$ in $P_i[x_1,\ldots,x_n]$ is eventually constant, as desired. □

Remark 7.5. The results of Tymoczko quoted in the proof of Theorem 7.4 hold more generally for other semisimple groups G, replacing the polynomial algebra with the W–algebra induced by the coadjoint action on the root system [Ty, Theorem 4.10]; here W is the Weyl group of G. We believe that it should be possible to prove representation stability for the equivariant cohomology of the corresponding Schubert varieties.

7.3 Rank-selected posets

The poset Z_n of subsets of the finite set $\{1, \ldots, n\}$, ordered by inclusion, is a basic object of study in combinatorics. The group S_n acts on $\{1, \ldots, n\}$, inducing an action on Z_n. One can view this action as an analogue of the S_n–action on the flag variety $\mathcal{F}_n$. In this subsection we prove some stability results for some refinements of these actions on the associated cohomology groups.

Suppose G is a group acting on an n–dimensional space X. The *Lefschetz representation* associated to this action is the virtual G–representation

$$\sum_{i=0}^{n}(-1)^i H_i(X;\mathbb{Q}),$$

meaning the formal linear combination of the representations $H_i(X;\mathbb{Q})$. The name reflects the observation that for each $g \in G$, the associated *virtual character* is the Lefschetz number

$$\sum_{i=0}^{n}(-1)^i \operatorname{tr}\big(g_*\colon H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})\big).$$

For any finite set $S \subset \mathbb{N}$ we may consider the *rank-selected* poset $Z_n(S)$. This is the poset consisting of $\emptyset$ and $\{1, \ldots, n\}$, together with those subsets of $\{1, \ldots, n\}$ whose cardinality lies in S. Let $|Z_n(S)|$ be the geometric realization of this poset. The natural action of the symmetric group S_n on Z_n preserves the subposet $Z_n(S)$, yielding an action of S_n on the geometric realization $|Z_n(S)|$. Let $L_n(S)$ be the associated Lefschetz representation

$$L_n(S) := \sum_i (-1)^i H_i(|Z_n(S)|;\mathbb{Q}).$$

Theorem 7.6 (Stability for Lefschetz representations of rank-selected posets)**.** *Let $S \subset \mathbb{N}$ be any finite set. Then the sequence $\{L_n(S)\}$ of virtual S_n–representations is multiplicity stable.*

Proof. Consider the related virtual representation

$$L'_n(S) := (-1)^{|S|-1}(L_n(S) \oplus \mathbb{Q}).$$

Clearly $\{L'_n(S)\}$ is multiplicity stable if and only if $\{L_n(S)\}$ is multiplicity stable. Given a partition λ, Stanley [Sta, Theorem 4.3] proves that the multiplicity of $V(\lambda)_n$ in $L'_n(S)$ equals the number of standard Young tableaux with shape $\lambda[n]$ whose descent set is exactly $S \cap \{1, \ldots, n-1\}$. As we saw in the proof of Theorem 7.1, this implies that the multiplicity of $V(\lambda)_n$ is constant for sufficiently large n, as desired. □

Let C_n be the n–dimensional *cross-polytope*, i.e. the convex hull of the set of unit coordinate vectors $\{\pm e_1, \ldots, \pm e_n\}$ in $\mathbb{R}^n$. Let Q_n be the poset of *faces* of C_n, meaning convex hulls of subsets of vertices. For $S \subset \mathbb{N}$, let $Q_n(S)$ be the rank-selected poset consisting of faces whose dimension lies in S, together with $\emptyset$ and C_n. The hyperoctahedral group W_n naturally acts on C_n, and thus on the poset $Q_n(S)$ and its geometric realization $|Q_n(S)|$. Let $L_n^C(S)$ be the associated Lefschetz representation

$$L_n^C(S) := \sum_i (-1)^i H_i(|Q_n(S)|; \mathbb{Q}).$$

Theorem 7.7 (Stability for Lefschetz representations of rank-selected cross-polytopes)**.** *Let $S \subset \mathbb{N}$ be any finite set. Then the sequence $\{L_n^C(S)\}$ of virtual W_n–representations is multiplicity stable.*

Proof. Given a double partition $\lambda = (\lambda^+, \lambda^-)$, Stanley [Sta, Theorem 6.4] shows that the multiplicity of $V(\lambda)_n$ in $(-1)^{|S|-1}(L_n^C(S) \oplus \mathbb{Q})$ is the number of double standard Young tableaux of shape $\lambda[n]$ whose flag descent set is exactly $S \cap \{1, \ldots, n-1\}$. As we showed in the proof of Theorem 7.3, this implies that the multiplicity of $V(\lambda)_n$ is constant for sufficiently large n, as desired. □

7.4 The $(n+1)^{n-1}$ conjecture

There is a variation of the co-invariant algebra (discussed in the proof of Theorem 7.1 above) that has been intensely studied by combinatorialists. The symmetric group S_n acts on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ diagonally, permuting the $x_\bullet$ and the $y_\bullet$ separately. The *diagonal co-invariant algebra* is the $\mathbb{Q}$–algebra defined by:

$$R_n := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I_n$$

where I_n denotes the ideal generated by the S_n–invariant polynomials without constant term. The bigrading of $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by total degree in $\{x_\bullet\}$ and total degree in $\{y_\bullet\}$ descends to a bigrading $(R_n)_{i,j}$ of the algebra R_n. This bigrading is preserved by the action of S_n on R_n. The $(n+1)^{n-1}$ *conjecture* was the conjecture that

$$\dim(R_n) = (n+1)^{n-1}.$$

This conjecture was proved by Haiman (see, e.g., the survey [Hai]), using a connection between this problem and the geometry of the Hilbert scheme of configurations of n points in $\mathbb{C}^2$. Just as with the classical co-invariant algebra, the structure of R_n as an

S_n–representation has been determined [Hai, Theorem 4.24]. However, the following seems to be unknown. It can be viewed as an "asymptotic refinement" of the $(n+1)^{n-1}$ conjecture.

Question 7.8. *Is the sequence of S_n–representations $\{(R_n)_{i,j}\}$ representation stable for each fixed $i, j \geq 1$?*

This question has a natural generalization to the "k–diagonal co-invariant algebra" $R_n^{(k)}$ for $k \geq 3$, by which we mean the algebra defined by the same construction as above, with kn variables partitioned into k subcollections and S_n acting diagonally on each subcollection separately. In this case the dimension of $R_n^{(k)}$ is not known. It would be especially interesting if representation stability as in Question 7.8 could be proved without knowing the irreducible decomposition, or even the dimension, of $R_n^{(k)}$.

8 Congruence subgroups, modular representations and stable periodicity

Recall that a *modular representation* of a finite group G is an action of G on a vector space over a field of positive characteristic dividing the order of G. Such representations need not decompose as a direct sum of irreducible representations and in general are very difficult to analyze. For finite groups of Lie type, for example $G = \mathrm{SL}_n(\mathbb{F}_p)$, the modular representation theory is significantly better understood in the *defining characteristic* of G, meaning in this case over a field of characteristic p. There are a number of important examples of groups Γ whose cohomology $H^i(\Gamma; \mathbb{F}_p)$ is naturally a modular representation of a finite group of Lie type. Examples of such Γ include various congruence subgroups of arithmetic groups as well as congruence subgroups of mapping class groups.

After explaining in detail a key motivating example, we briefly review the modular representation theory that will be needed to formulate representation stability in this context. One new phenomenon here is that natural sequences of representations arise that do not satisfy representation stability, but instead exhibit a form of "stable periodicity" as representations. After defining this precisely, we present several results and conjectures using this concept.

8.1 A motivating example

Consider the following fundamental example from arithmetic. For any prime p the *level p congruence subgroup* $\Gamma_n(p) < \mathrm{SL}_n \mathbb{Z}$ is the kernel

$$\Gamma_n(p) := \ker(\pi\colon \mathrm{SL}_n \mathbb{Z} \twoheadrightarrow \mathrm{SL}_n(\mathbb{F}_p))$$

where π is the map reducing the entries of a matrix modulo p. Charney proved in [Ch] that over $\mathbb{Q}$ (indeed even over $\mathbb{Z}[1/p]$) the sequence of groups $\{\Gamma_n(p)\}$ satisfy classical homological stability. Furthermore, she proved that this is equivalent to the claim that the natural action of $\mathrm{SL}_n(\mathbb{F}_p)$ on $H^i(\Gamma_n(p); \mathbb{Q})$ is trivial for large enough n, so that

$$H^i(\Gamma_n(p); \mathbb{Q})^{\mathrm{SL}_n(\mathbb{F}_p)} = H^i(\Gamma_n(p); \mathbb{Q}) = H^i(\mathrm{SL}_n \mathbb{Z}; \mathbb{Q}).$$

Replacing the coefficient field $\mathbb{Q}$ with $\mathbb{F}_p$ or its algebraic closure $\overline{\mathbb{F}}_p$, the situation becomes more interesting, and the cohomology is much richer (see, e.g., [Ad, As, CF2]). First note that Charney's result is not true in this case: the action of $\mathrm{SL}_n(\mathbb{F}_p)$ on $H^i(\Gamma_n(p); \overline{\mathbb{F}}_p)$ is certainly not trivial. We can work this out for $H_1(\Gamma_n(p); \mathbb{F}_p)$ explicity. Each $B \in \Gamma_n(p)$ can be written as $B = I + pA$ for some A. It is easy to check that the map $B \mapsto A \pmod p$ gives a surjective homomorphism

$$\psi\colon \Gamma_n(p) \to \mathfrak{sl}_n(\mathbb{F}_p) \tag{21}$$

where $\mathfrak{sl}_n(\mathbb{F}_p)$ is the abelian group of traceless $n \times n$ matrices with entries in $\mathbb{F}_p$. Lee–Szczarba [LSz] observed that the proof of the Congruence Subgroup Property implies that ψ yields an isomorphism

$$H_1(\Gamma_n(p); \mathbb{Z}) \approx H_1(\Gamma_n(p); \mathbb{F}_p) \approx \mathfrak{sl}_n(\mathbb{F}_p).$$

We thus see that, since the dimension of $H_1(\Gamma(n,p); \mathbb{F}_p)$ increases with n, the sequence of groups $\{\Gamma_n(\mathbb{F}_p)\}$ does not satisfy homological stability over $\mathbb{F}_p$ in the classical sense. However, it is clear from the construction that the $\mathrm{SL}_n(\mathbb{F}_p)$–action on $H_1(\Gamma_n(p); \mathbb{F}_p) \approx \mathfrak{sl}_n(\mathbb{F}_p)$ is just the usual (modular) *adjoint representation*; this is a modular representation because $\mathfrak{sl}_n(\mathbb{F}_p)$ is a vector space over $\mathbb{F}_p$, and p divides the order of $\mathrm{SL}_n(\mathbb{F}_p)$. We can thus hope to use the modular representation theory of $\mathrm{SL}_n(\mathbb{F}_p)$ to define and study a version of representation stability for each sequence $\{H^i(\Gamma_n(p), \mathbb{F}_p)\}$ of $\mathrm{SL}_n(\mathbb{F}_p)$–representations. For example, $\mathfrak{sl}_n(\mathbb{F}_p)$ is an irreducible $\mathrm{SL}_n(\mathbb{F}_p)$–representation, and so an appropriate form of representation stability holds for $\{H_1(\Gamma_n(p); \mathbb{F}_p)\}$.

One can do all of the above for level p congruence subgroups $\Gamma_{2g}^{\mathrm{Sp}}(p)$ of $\mathrm{Sp}(2g,\mathbb{Z})$. As we will explain in §8.3, something new happens here: the sequence of $\mathrm{Sp}_{2g}\,\mathbb{F}_2$–representations $\{H_1(\Gamma_{2g}^{\mathrm{Sp}}(2);\mathbb{F}_2)\}$ is only representation stable when restricted to even g, or to odd g. Indeed, for each $p \geq 2$ we will see below natural examples of sequences that are "stably periodic" with period p.

8.2 Modular representations of finite groups of Lie type

In order to formalize the notion of representation stability in the modular case, we need to review the pertinent representation theory.

Representations of $\mathbf{SL}_n(\overline{\mathbb{F}}_p)$ and $\mathbf{Sp}_{2n}(\overline{\mathbb{F}}_p)$ in their defining characteristic. Before restricting to the finite group $\mathrm{SL}_n(\mathbb{F}_p)$, we consider representations of the algebraic group $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ in the defining characteristic p. While it is not true in this context that every representation is completely reducible, irreducible representations of $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ over $\overline{\mathbb{F}}_p$ are still classified by highest weights, as follows. We give the details for the case of SL_n, but all claims hold for Sp_{2n} as well. A nice reference for these assertions is Humphreys [Hu, Chapters 2 and 3].

Let $T < \mathrm{SL}_n(\overline{\mathbb{F}}_p)$ be the maximal torus consisting of diagonal matrices. Let $U < \mathrm{SL}_n(\overline{\mathbb{F}}_p)$ be the subgroup of strictly upper-triangular matrices. Any representation V of $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ decomposes into eigenspaces for T. A vector $v \in V$ is called a *highest weight vector* if v is an eigenvector for T and is invariant under U, in which case its *weight* is the corresponding eigenvalue $\lambda \in T^*$. Writing T^* additively, we identify T^* with $\mathbb{Z}[L_1,\ldots,L_n]/(L_1+\cdots+L_n)$. The same applies to $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_p)$, with $T^* = \mathbb{Z}[L_1,\ldots,L_n]$. In either case, a weight is called *dominant* if it can be written as a nonegative integral combination of the fundamental weights $\omega_i = L_1+\cdots+L_i$.

The basics of the classification of irreducible $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$–representations are the same as in the characteristic 0 case: every irreducible representation contains a unique highest weight vector; the highest weight λ determines the irreducible representation; and every dominant weight occurs as the highest weight of an irreducible representation. Thus we may unambiguously denote by $V(\lambda)_n$ the irreducible representation of $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ or $\mathrm{Sp}_{2n}(\overline{\mathbb{F}}_p)$ with highest weight λ. However, much less is known about these irreducible representations than in the characteristic 0 case, and there is no known way to uniformly construct all irreducible representations. Even the dimensions of the irreducible representations are not known in general.

One approach to the construction of irreducible $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$–representations $V(\lambda)$ is through Weyl modules. This process starts with the irreducible representation $V(\lambda)_{\mathbb{Q}}$ of $\mathrm{SL}_n \mathbb{Q}$ with weight λ. There is then a special $\mathbb{Z}$–form $V(\lambda)_{\mathbb{Z}} \subset V(\lambda)_{\mathbb{Q}}$ so that $\mathrm{SL}_n \overline{\mathbb{F}}_p$ acts on the *Weyl module* $W(\lambda) := V(\lambda)_{\mathbb{Z}} \otimes \overline{\mathbb{F}}_p$. The Weyl module $W(\lambda)$ is generated by a single highest weight vector with weight λ, but in general $W(\lambda)$ will not be irreducible. However, $W(\lambda)$ always admits a unique simple quotient, which must be the irreducible representation $V(\lambda)$. We will see below that for fixed λ, the question of whether $W(\lambda)$ is irreducible can depend on the residue of n modulo p.

Restriction to finite groups of Lie type. Given any representation of $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$, we may "twist" it by precomposing with the Frobenius map $\mathrm{SL}_n(\overline{\mathbb{F}}_p) \to \mathrm{SL}_n(\overline{\mathbb{F}}_p)$. This twisted representation clearly remains irreducible; in fact for any λ the twist of $V(\lambda)_n$ by the Frobenius is $V(p\lambda)_n$. A dominant weight λ is called *p–restricted* if it can be written as $\lambda = \sum c_i \omega_i$ with $0 \leq c_i < p$. If λ is p–restricted, then the restriction of the irreducible representation $V(\lambda)_n$ from $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ to $\mathrm{SL}_n(\mathbb{F}_p)$ remains irreducible. Every irreducible representation of $\mathrm{SL}_n(\mathbb{F}_p)$ is of this form. Thus we have found all p^{n-1} irreducible representations of $\mathrm{SL}_n(\mathbb{F}_p)$ and all p^n irreducible representations of $\mathrm{Sp}_{2n}(\mathbb{F}_p)$.

Uniqueness of composition factors. In the modular case we cannot decompose a representation into a direct sum of irreducibles. However, by the Jordan–Hölder theorem, the irreducible representations that occur as the composition factors in any Jordan–Hölder decomposition of any representation are indeed unique.

8.3 Stable periodicity and congruence subgroups

The definition of representation stability in the modular case needs to be altered in a fundamental way in order to apply to several natural examples. One of these examples is the *level p symplectic congruence subgroup* $\Gamma_{2g}^{\mathrm{Sp}}(p) < \mathrm{Sp}_{2g} \mathbb{Z}$ defined as the kernel

$$\Gamma_{2g}^{\mathrm{Sp}}(p) := \ker(\pi\colon \mathrm{Sp}_{2g} \mathbb{Z} \twoheadrightarrow \mathrm{Sp}_{2g} \mathbb{F}_p)$$

where π is the map reducing the entries of a matrix modulo the prime p. Building on work of Sato, Putman [Pu] has shown, among many other things, that for $g \geq 3$ and p odd there is an $\mathrm{Sp}_{2g} \mathbb{Z}$–equivariant isomorphism:

$$H_1(\Gamma_{2g}^{\mathrm{Sp}}(p), \mathbb{Z}) \approx H_1(\Gamma_{2g}^{\mathrm{Sp}}(p), \mathbb{F}_p) \approx \mathfrak{sp}_{2g}(\mathbb{F}_p)$$

where $\mathfrak{sp}_{2g}(\mathbb{F}_p)$ is the adjoint representation of $\mathrm{Sp}_{2g}(\mathbb{F}_p)$ on its Lie algebra. Putman also proved that the group $H_1(\Gamma_{2g}^{\mathrm{Sp}}(2), \mathbb{F}_2)$ is an extension of $\mathfrak{sp}_{2g}(\mathbb{F}_2)$ by $H := H_1(S_g; \mathbb{F}_2)$.

Note that $\mathfrak{sp}_{2g}\mathbb{F}_p$ sits inside $\mathfrak{gl}_{2g}\mathbb{F}_p \approx H^* \otimes H \approx H \otimes H$ as $\mathfrak{sp}_{2g}\mathbb{F}_p \approx \mathrm{Sym}^2 H$. When p is odd, $\mathfrak{sp}_{2g}\mathbb{F}_p \approx \mathrm{Sym}^2 H$ is irreducible with highest weight vector $a_1 \cdot a_1$ and highest weight $2\omega_1$ (see Hogeweij [Ho, Corollary 2.7]). The situation is different for $p = 2$: the representation $\mathfrak{sp}_{2g}\mathbb{F}_2 \approx \mathrm{Sym}^2 H$ is no longer irreducible. Indeed since

$$(x+y)^2 = x^2 + 2x \cdot y + y^2 = x^2 + y^2$$

there is an embedding $H \hookrightarrow \mathrm{Sym}^2 H$ defined by $x \mapsto x \cdot x$; this is a map of $\mathrm{Sp}_{2g}(\mathbb{F}_2)$–representations since $a^2 = a$ in $\mathbb{F}_2$. Recalling that $a_1 \cdot a_1$ has highest weight $2\omega_1$, over $\overline{\mathbb{F}}_2$ we see here the isomorphism between $V(2\omega_1)$ and the twist of $V(\omega_1) \approx H$ by the Frobenius map $a \mapsto a^2$. Since $x \cdot y = y \cdot x = -y \cdot x$, the quotient $\mathrm{Sym}^2 H/H$ is isomorphic to $\bigwedge^2 H$. This has an invariant contraction $\bigwedge^2 H \to \mathbb{F}_2$ (represented by the symplectic form) and an invariant vector $\omega = a_1 \cdot b_1 + \cdots a_g \cdot b_g$ (representing the symplectic form). These are independent when g is odd, but not when g is even. Thus $\mathfrak{sp}_{2g}\mathbb{F}_2$ has composition factors $V(0), V(\omega_1), V(\omega_2)$ if g is odd, and $V(0)^2, V(\omega_1), V(\omega_2)$ if g is even (see [Ho, Lemma 2.10]).

In order to take situations like this into account, we must build periodicity into the definition of stability.

Definition 8.1 (Stable periodicity)**.** Let $G_n = \mathrm{SL}_n(\mathbb{F}_p)$ or $\mathrm{Sp}_{2n}(\mathbb{F}_p)$. Let $\{V_n\}$ be a consistent (c.f. §2.3) sequence of modular G_n–representations, i.e. representations of vector spaces over $\mathbb{F}_p$. The sequence $\{V_n\}$ is *stably representation periodic*, or just *stably periodic*, if Condition I (Injectivity) and Condition II (Surjectivity) of Definition 2.3 hold, together with the following:

PMIII. (Stable periodicity of multiplicities): For each highest weight vector λ, the multiplicity of $V(\lambda)$ as a composition factor in the Jordan–Hölder series for V_n as a G_n–representation is *stably periodic*: there exists $C = C_\lambda$ so that for all sufficiently large n, this multiplicity is periodic in n with period C.

Similarly we have the corresponding notion of *uniformly stably periodic*, where we additionally require that the eventual period C does not depend on λ, and also *mixed tensor stably periodic*. We note that a representation stable sequence is also stably periodic with period C for any $C \geq 1$.

We will apply the above definition to give a conjectural picture of the cohomology of congruence groups.

Conjecture 8.2 (Modular periodic stability for congruence groups)**.** *Fix any $i \geq 0$ and any prime p. Then*

1. *The sequence of $\mathrm{SL}_n(\mathbb{F}_p)$–representations $\{H_i(\Gamma_n(p); \mathbb{F}_p)\}$ is uniformly mixed tensor stably periodic with period p.*

2. *The sequence of $\mathrm{Sp}_{2n}(\mathbb{F}_p)$–representations $\{H_i(\Gamma_{2n}^{\mathrm{Sp}}(p); \mathbb{F}_p)\}$ is uniformly stably periodic with period p.*

We note that mixed tensor representations are really needed in Part 1 of Conjecture 8.2, since for example

$$H_1(\Gamma_n(p)) = \mathfrak{sl}_n \mathbb{F}_p = V(L_1 - L_n) = V(\omega_1 + \omega_{n-1}) = V(1;1)_n$$

is not representation stable, but is mixed representation stable. We also remark that periodicity is also needed in the conjecture. For example, by the discussion above, the sequence $\{H_1(\Gamma_{2n}^{\mathrm{Sp}}(2); \mathbb{F}_2)\}$ is a stably periodic sequence of $\mathrm{Sp}_{2n} \mathbb{F}_2$–representations with stable period 2. These examples also verify that Conjecture 8.2 is true for $i = 1$.

8.4 The abelianization of the Torelli group

Dennis Johnson computed that the abelianization of the Torelli group $\mathcal{I}_{g,1}$ comes from two sources. The first is the so-called Johnson homomorphism, which is purely algebraically defined, and captures the action of $\mathcal{I}_{g,1}$ on the universal two-step nilpotent quotient of $\pi_1(S_{g,1})$ (but see [CF] for a geometric perspective); its image is $\bigwedge^3 H_1(S_{g,1}; \mathbb{Z})$. The second is the Birman–Craggs–Johnson homomorphism, which views the Torelli group as gluing maps for Heegard splittings and bundles together the Rokhlin invariants of the resulting homology 3–spheres. Its image is 2–torsion and is isomorphic to the space B_3 of Boolean polynomials on $H_1(S_{g,1}; \mathbb{F}_2)$ of degree at most 3. Johnson showed that these quotients exhaust the homology of the Torelli group, but with some overlap. He concludes in [Jo2] that for $g \geq 3$ there is an isomorphism of abelian groups:

$$H_1(\mathcal{I}_{g,1}, \mathbb{Z}) \approx \bigwedge^3 H_1(S_{g,1}; \mathbb{Z}) \oplus B_2,$$

where B_2 is the space of Boolean polynomials of degree at most 2.

The action of $\mathrm{Sp}_{2g} \mathbb{Z}$ on $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$ descends to an action of $\mathrm{Sp}_{2g}(\mathbb{Z}/2\mathbb{Z})$ on the torsion subgroup $\mathrm{Tor}(H_1(\mathcal{I}_{g,1}; \mathbb{Z})) \approx B_2$. Shvartsman [Sh] has recently determined the structure of $\mathrm{Tor}(H_1(\mathcal{I}_{g,1}; \mathbb{Z}))$ as an $\mathrm{Sp}_{2g}(\mathbb{F}_2)$–module. From his calculation we deduce the following.

Theorem 8.3. *The torsion subgroup* $\mathrm{Tor}(H_1(\mathcal{I}_{g,1};\mathbb{Z}))$ *of the abelianization of* $\mathcal{I}_{g,1}$ *is uniformly stably periodic with period 2. The subsequence for even* g *is uniformly representation stable, and the subsequence for odd* g *is uniformly representation stable.*

Proof. The results of Shvartsman [Sh] give the following list of the simple $\mathrm{Sp}_{2g}(\mathbb{F}_2)$–modules appearing in a composition series for $\mathrm{Tor}(H_1(\mathcal{I}_{g,1};\mathbb{Z}))$ for $g \geq 3$. We list modules by their highest weight.

$$V(0), V(\omega_1), V(0), V(\omega_2) \qquad \text{for } g \text{ odd}$$

$$V(0), V(\omega_1), V(0), V(\omega_2), V(0) \qquad \text{for } g \text{ even}$$

The discrepancy between g even and g odd arises from the same source as the corresponding discrepancy for $\bigwedge^2 H_1(S_{g,1};\mathbb{F}_2)$ discussed above. □

8.5 Level p mapping class groups

The *level p mapping class group* $\mathrm{Mod}_{g,1}(p)$ is the kernel of the composition

$$\mathrm{Mod}_{g,1} \twoheadrightarrow \mathrm{Sp}_{2g}\mathbb{Z} \twoheadrightarrow \mathrm{Sp}_{2g}\mathbb{F}_p\,.$$

The group $\mathrm{Mod}_{g,1}(p)$ is the "mod p" analogue of the Torelli group $\mathcal{I}_{g,1}$, since it is the subgroup of $\mathrm{Mod}_{g,1}$ acting trivially on $H_1(S_{g,1};\mathbb{F}_p)$. Hain [Ha3, Proposition 5.1] proved that for $g \geq 3$ the group $H^1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ is trivial, so the abelianization $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ consists entirely of torsion elements.

Putman [Pu], building on work of Sato, recently proved that elements of $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ come from three sources. The first is the abelianization of the congruence subgroup $\mathfrak{sp}_{2g}(\mathbb{F}_p)$, which we discussed above. The second source is a "mod p" version of the Johnson homomorphism, which has image $\bigwedge^3 H_1(S_{g,1};\mathbb{F}_p)$. The third source contributes only when $p = 2$, and is a quotient $B_2/\mathbb{F}_2$ coming from the Birman–Craggs–Johnson homomorphism. The quotient $\mathrm{Sp}_{2g}\mathbb{F}_p$ naturally acts on $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$, and it follows from Putman's characterization that $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ is in fact an $\mathbb{F}_p$–representation of $\mathrm{Sp}_{2g}\mathbb{F}_p$.

Theorem 8.4. *Fix a prime* p. *Then the sequence* $\{H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})\}$ *of* $\mathrm{Sp}_{2g}\mathbb{F}_p$*–representations is periodically uniformly representation stable with period* p.

Proof. Let $H := H_1(S_{g,1};\mathbb{F}_p)$ be the standard representation of $\mathrm{Sp}_{2g}\mathbb{F}_p$. For any prime p, the representation $\bigwedge^3 H$ has as composition factors the simple $\mathrm{Sp}_{2g}\mathbb{F}_p$–modules:

$$V(\omega_1), V(\omega_3) \qquad \text{for } g \equiv 1 \bmod p$$

$$V(\omega_1), V(\omega_3), V(\omega_1) \qquad \text{for } g \not\equiv 1 \bmod p$$

Putman proves in [Pu, Theorem 7.8] that for p odd and $g \geq 5$, the group $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ is an extension of $\mathfrak{sp}_{2g}\,\mathbb{F}_p$ by $\bigwedge^3 H$. Thus $H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})$ has composition factors

$$\begin{aligned} &V(\omega_1), V(2\omega_1), V(\omega_3), && \text{for } g \equiv 1 \bmod p \\ &V(\omega_1)^2, V(2\omega_1), V(\omega_3) && \text{for } g \not\equiv 1 \bmod p. \end{aligned}$$

For $p = 2$, Putman proves that $H_1(\mathrm{Mod}_{g,1}(2);\mathbb{Z})$ is an extension of $H_1(\Gamma_{2g}^{\mathrm{Sp}}(p);\mathbb{F}_p)$ by $\bigwedge^3 H \oplus B_2/\,\mathbb{F}_2$. The former has composition factors $\mathfrak{sp}_{2g}\,\mathbb{F}_2$ and $V(\omega_1)$, and Shvartsman describes B_2 as in Theorem 8.3. We conclude that for $g \geq 5$, the group $H_1(\mathrm{Mod}_{g,1}(2);\mathbb{Z})$ has the following composition factors as an $\mathrm{Sp}_{2g}\,\mathbb{F}_2$–module:

$$\begin{aligned} &V(0)^2, V(\omega_1)^4, V(\omega_2)^2, V(\omega_3) && \text{for } g \text{ odd} \\ &V(0)^4, V(\omega_1)^5, V(\omega_2)^2, V(\omega_3) && \text{for } g \text{ even} \end{aligned}$$

Thus in both cases we see that the abelianization is periodic and uniformly multiplicity stable with period p. □

Given Theorem 8.4, it is natural to make the following conjecture.

Conjecture 8.5 (Modular periodic stability for $\mathrm{Mod}_{g,1}(p)$). *Fix any $i \geq 0$ and a prime p. Then the sequence of $\mathrm{Sp}_{2g}\,\mathbb{F}_p$–representations $\{H_1(\mathrm{Mod}_{g,1}(p);\mathbb{Z})\}$ is uniformly stably periodic with period p.*

We believe that all of the material in this section can be extended to corresponding "level p congruence subgroups" of IA_n.

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