The geometry of surface-by-free groups

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August 29, 2000

Abstract

We show that every word hyperbolic, surface-by-(noncyclic) free group Γ is as rigid as possible: the quasi-isometry group of Γ equals the abstract commensurator group $\mathrm{Comm}(\Gamma)$, which in turn contains Γ as a finite-index subgroup. As a corollary, two such groups are quasi-isometric if and only if they are commensurable, and any finitely-generated group quasi-isometric to Γ must be weakly commensurable with Γ. We use quasi-isometries to compute $\mathrm{Comm}(\Gamma)$ explicitly, an example of how quasi-isometries can actually detect finite-index information. The proofs of these theorems involve ideas from coarse topology, Teichmüller geometry, pseudo-Anosov dynamics, and singular SOLV geometry.

1 Introduction

Let Σ_g be a closed surface of genus $g \geq 2$, and let $\mathcal{M}(\Sigma_g) = \pi_0(\mathrm{Homeo}(\Sigma_g))$ denote the mapping class group of Σ_g. A *Schottky subgroup* H of $\mathcal{M}(\Sigma_g)$ is a free group of pseudo-Anosov mapping classes whose action on the Teichmüller space $\mathcal{T}(\Sigma_g)$ is "weak convex cocompact"—the group H has a limit set in Thurston's boundary of $\mathcal{T}(\Sigma_g)$, when $\mathrm{rank}(H) \geq 2$ this limit set is a Cantor set, and the action of H on the "weak convex hull" of the limit set is cocompact. Schottky subgroups of $\mathcal{M}(\Sigma_g)$ exist in abundance: given any collection $\{\phi_1, \ldots, \phi_r\}$ of pairwise independent pseudo-Anosov mapping classes of Σ_g, for any sufficiently large positive integers $a_1, \ldots, a_r$ the elements $\{\phi_1^{a_1}, \ldots, \phi_r^{a_r}\}$ freely generate a Schottky subgroup. See §3 for a review of Schottky groups, taken from [Mos97] and [FM00b].

Given $g \geq 2$ and a Schottky subgroup $H < \mathcal{M}(\Sigma_g) \approx \mathrm{Out}(\pi_1(\Sigma_g))$, one can construct a group Γ_H given by the split extension

$$1 \to \pi_1(\Sigma_g) \to \Gamma_H \to H \to 1$$

*Supported in part by NSF grant DMS 9704640 and by a Sloan Foundation Fellowship.
†Supported in part by NSF grant DMS 9504946.

The groups Γ_H are precisely the surface-by-free groups which are word hyperbolic [FM00b], and the construction of [Mos97] shows that they are abundant.

We are interested in studying quasi-isometries of the groups Γ_H for several reasons: the Γ_H provide basic examples of rigidity theorems for word hyperbolic groups outside the context of negatively curved manifolds (see also [Bou00], [KK98]); they are examples of groups which can be viewed as phase spaces of dynamical systems arising from hyperbolic endomorphisms of manifolds (see also [FM98], [FM99], [FM00a]); and they provide examples of groups which are as rigid as possible in a very concrete sense (see Theorem 1.3 below). In particular, each Γ_H has finite index in its own abstract commensurator $\mathrm{Comm}(\Gamma_H)$ as well as in its quasi-isometry group. The computation of $\mathrm{Comm}(\Gamma_H)$ given in Theorem 1.3 is explicit, one of the few instances outside lattices in Lie groups where this has been done.

Finally, we propose the general problem of studying the asymptotic geometry of extensions of surface groups Σ. These groups exhibit a beautiful and rich geometry which is encoded by a subgroup of $\mathcal{M}(\Sigma)$ acting on the Teichmüller space $\mathcal{T}(\Sigma)$. Some elements of the geometry of such groups can be found in [Mos96], [Mit98] and [FM00b], but our needs will require a somewhat involved account of this theory for extensions of surface groups by Schottky groups, which we give in §4.

Statement of results

Our first theorem gives a complete classification of the groups Γ_H up to quasi-isometry. Although the usual quasi-isometry invariants such as growth, ends, isoperimetric functions, etc., are the same for all of these groups, they are only quasi-isometric under the strictest algebraic conditions.

Given a Schottky subgroup H of $\mathcal{M}(\Sigma)$, we will show in §7.2 that there is a smallest 2-orbifold covered by Σ, denoted $\mathcal{O}_H$, such that H descends via the covering map $\Sigma \to \mathcal{O}_H$ to a Schottky subgroup of $\mathcal{M}(\mathcal{O}_H)$, still denoted H. The orbifold $\mathcal{O}_H$ plays an important role in describing the quasi-isometry class of the surface-by-free group Γ_H.

Theorem 1.1 (Classification Theorem). *Given $g_1, g_2 \geq 2$, let $H_1 < \mathcal{M}(\Sigma_{g_1})$ and $H_2 < \mathcal{M}(\Sigma_{g_2})$ by Schottky subgroups of rank ≥ 2. The following are equivalent:*

(1) The surface-by-free groups Γ_{H_1} and Γ_{H_2} are quasi-isometric.

(2) Γ_{H_1} and Γ_{H_2} are abstractly commensurable, meaning that they have finite-index subgroups which are isomorphic.

(3) *There is an isomorphism $\mathcal{O}_{H_1} \approx \mathcal{O}_{H_2}$ such that in the group $\mathcal{M}(\mathcal{O}_{H_1}) = \mathcal{M}(\mathcal{O}_{H_2})$ the Schottky subgroups H_1 and H_2 are commensurable, meaning that $H_1 \cap H_2$ has finite index in each of H_1 and H_2.*

(4) *There is an isomorphism $\mathcal{O}_{H_1} \approx \mathcal{O}_{H_2}$ such that in the group $\mathcal{M}(\mathcal{O}_{H_1}) = \mathcal{M}(\mathcal{O}_{H_2})$ the Schottky subgroups H_1 and H_2 have the same limit set in the Thurston boundary of the Teichmüller space $\mathcal{T}(\mathcal{O}_{H_1}) = \mathcal{T}(\mathcal{O}_{H_2})$.*

The next theorem shows that each of the groups Γ_H is determined among all finitely-generated groups by its asymptotic geometry, up to finite data.

Theorem 1.2 (Quasi-isometric rigidity). *Let Γ_H be a surface-by-free group with $H < \mathcal{M}(\Sigma_g)$ a Schottky group of rank ≥ 2. If G is any finitely-generated group which is quasi-isometric to Γ_H, then there is a finite normal subgroup $F < G$ such that G/F is abstractly commensurable to a surface-by-free group. Combining with Theorem 1.1 it follows that G/F is abstractly commensurable to Γ_H.*

Remark. We emphasize that it is essential for our methods that $\mathrm{rank}(H) \geq 2$ in the statements of Theorems 1.1 and 1.2. Indeed, when H is a rank 1 Schottky subgroup, in other words an infinite cyclic subgroup generated by a pseudo-Anosov mapping class, then Γ_H is the fundamental group of a closed hyperbolic 3-manifold that fibers over the circle [Ota96], and hence all the groups Γ_H with H Schottky of rank 1 are quasi-isometric to each other and to $\mathbf{H}^3$. Moreover, the restatement of Theorem 1.2 (minus the last sentence) for H of rank 1 is equivalent to Thurston's virtual surface bundle conjecture for closed hyperbolic 3-manifolds. [1]

Theorem 1.2 and (most of) Theorem 1.1 follow from our main result, Theorem 1.3 below.

Commensurations and quasi-isometries. Recall that a *commensuration* of Γ is an isomorphism between finite index subgroups of Γ. Composition of two commensurations is defined on a further finite index subgroup. Two commensurations are equivalent if they agree on a common finite index subgroup. Composition of equivalence classes gives a well-defined group operation, and we thereby obtain the *abstract commensurator group* $\mathrm{Comm}(\Gamma)$ of the group Γ.

The *quasi-isometry group* $\mathrm{QI}(\Gamma)$ is the group of coarse equivalence classes of self quasi-isometries of Γ (endowed with any word metric), where two quasi-isometries are coarsely equivalent if they have finite distance in the sup norm.

[1] This equivalence, however, hides the following fact: Theorem 1.1 provides a commensuration from Γ_{H_1} to Γ_{H_2} coarsely taking $\pi_1(\Sigma_{g_2})$ to $\pi_1(\Sigma_{g_2})$, a situation which is often impossible for rank 1; see §8.2. Hence the proofs of Theorems 1.1 and 1.2 cannot shed light on the virtual surface bundle conjecture.

In general it is difficult to compute the abstract commensurator of a group. The computation for irreducible lattices in semisimple groups $G \neq \mathrm{PSL}(2, \mathbf{R})$ is the content of Mostow Rigidity together with theorems of Borel and Margulis (see e.g. [Zim84]). Our main result, Theorem 1.3, says that the groups $\mathrm{Comm}(\Gamma_H)$ and $\mathrm{QI}(\Gamma_H)$ are isomorphic, and gives an explicit computation of these groups. This is the first time we know of where quasi-isometries are used to compute the abstract commensurator group. While our expression for $\mathrm{Comm}(\Gamma_H)$ is purely algebraic, we do not know how to do the computation algebraically.

Recall that given groups $K < Q$, the *(relative) commensurator* of K in Q, denoted $\mathrm{Comm}_Q(K)$, is the subgroup of all $q \in Q$ such that conjugation by q takes some finite index subgroup of K to another, or equivalently $K \cap qKq^{-1}$ has finite index in both K and qKq^{-1}; this subgroup is also known as the *virtual normalizer* of K in Q. The group $\mathcal{C} = \mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H)$, the relative commensurator of H in $\mathcal{M}(\mathcal{O}_H)$, plays a key role in computing the abstract commensurator of Γ_H:

Theorem 1.3 (Computation of $\mathrm{QI}(\Gamma_H)$ and $\mathrm{Comm}(\Gamma_H)$). *Given $g \geq 2$ and a Schottky group $H < \mathcal{M}(\Sigma_g)$ of rank ≥ 2, the natural homomorphism*

$$\mathrm{Comm}(\Gamma_H) \to \mathrm{QI}(\Gamma_H)$$

is an isomorphism. Furthermore, these groups are isomorphic to the group $\Gamma_{\mathcal{C}}$ given explicitly by the short exact sequence

$$1 \to \pi_1(\mathcal{O}_H) \to \Gamma_{\mathcal{C}} \to \mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H) \to 1$$

Moreover, $\mathcal{C} = \mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H)$ contains H as a finite index subgroup. In particular, Γ_H has finite index in $\mathrm{Comm}(\Gamma_H) \approx \mathrm{QI}(\Gamma_H)$.

This theorem is proved in Sections 7.3 and 7.4.

The fact that Γ_H has finite index in $\mathrm{QI}(\Gamma_H)$ and also in $\mathrm{Comm}(\Gamma_H)$ shows that the groups Γ_H are extremely rigid. Among irreducible lattices in semisimple Lie groups, this phenomenon holds for the noncocompact, nonarithmetic lattices and for no other lattice; see [Mar91] for the commensurator statement, and [Sch96] and e.g. [Far97] for the quasi-isometry statement.

Outline of the proof of Theorem 1.3

After some preliminary material, we begin in §4 by constructing a quasi-isometric model space X_H on which Γ_H acts properly cocompactly by isometries. A Schottky subgroup $H < \mathcal{M}(\Sigma)$ gives an H-equivariant embedding of the Cayley graph T_H of H, a tree, in the Teichmüller space $\mathcal{T}(\Sigma)$. This embedding gives a Σ-bundle over T_H, each fiber carrying a hyperbolic structure representing the point of $T_H \subset \mathcal{T}(\Sigma)$

over which that fiber lies. The universal cover of this Σ-bundle is our model space X_H, an $\mathbf{H}^2$-bundle over the tree T_H.

An isometry of Γ_H acts on X_H, permuting or "respecting" various patterns of geometric objects, typically foliations. Indeed, the same patterns respected by isometries of X_H are also respected by the finite index supergroup $\Gamma_{\mathcal{C}}$ which is defined by the short exact sequence given in Theorem 1.3; see §7.3 for the precise definition of $\Gamma_{\mathcal{C}}$.

The rest of the proof is devoted to showing that an arbitrary quasi-isometry $f\colon X_H \to X_H$ coarsely respects so many patterns that it must be close to an element of $\Gamma_{\mathcal{C}}$. Using coarse topology, we show in §4 that f permutes the collection of "hyperplanes" P_w; these are the $\mathbf{H}^2$-bundles over bi-infinite lines w in T_H. The Schottky property is used to relate each line w to a Teichmüller geodesic, which in turn allows us to impose extra structure on the hyperplane P_w: a "pseudo-Anosov flow" and a singular SOLV structure. In §5 we use this structure to prove that f coarsely respects several dynamically defined foliations associated to these flows, such as the stable and unstable foliations.

In §6, we apply the above together with R. Schwartz's geodesic pattern rigidity [Sch97] to show that the quasi-isometry f actually permutes the collection of *periodic hyperplanes*, i.e. those P_w with w a bi-infinite periodic line in the tree T_H.

In §7 we study how the quasi-isometry f acts on the limit set of H in Thurston's boundary of $\mathcal{T}(\Sigma)$, the space of projective measured foliations. This limit set is a Cantor set, and the set of periodic hyperplanes gives a countable dense subset which is preserved by the action of f. This information is then used to show that f is close to an element of $\Gamma_{\mathcal{C}}$.

The final parts of the proofs of all of the main theorems are contained in §7.

Surfaces versus orbifolds. Our main theorems are stated solely for closed, oriented surfaces Σ_g of genus $g \geq 2$, and Schottky subgroups H of $\mathcal{M}(\Sigma_g)$. But the conclusions of these theorems force us to consider a wider realm: closed 2-orbifolds $\mathcal{O}$ and virtual Schottky subgroups of $\mathcal{M}(\mathcal{O})$; see e.g. Theorem 1.3 in which, as the proof will show, $\mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H)$ is a virtual Schottky subgroup of $\mathcal{M}(\mathcal{O}_H)$. Various results that we will need about (virtual) Schottky subgroups of orbifold mapping class groups are formulated in [FM00b].

This raises the question of whether the quasi-isometry classes of (orbifold)-by-(virtual Schottky) groups constitute a wider universe than the quasi-isometry classes of (surface)-by-(Schottky) groups. The answer is no: these universes are identical. The proof is given in Section 8.1.

Acknowledgements We are grateful to Kevin Whyte for crucial help in proving Theorem 7.4.

2 Preliminaries

In this section we briefly review some facts about Teichmüller space and about quasi-isometries. For details we refer to [Abi80], [FLP$^+$79], [IT92].

For most of the paper, while we concentrate on the proof of Theorem 1.3, we shall fix the genus g and denote $\Sigma = \Sigma_g$.

2.1 Teichmüller space and mapping class groups

The *Teichmüller space* of Σ, denoted $\mathcal{T} = \mathcal{T}(\Sigma)$, has two equivalent descriptions, related to each other by Riemann's uniformization theorem: $\mathcal{T}$ is the space of conformal structures on Σ modulo isotopy; or it is the space of hyperbolic structures on Σ modulo isotopy. A topology and a real analytic structure on $\mathcal{T}$ is specified by the geodesic length embedding $\mathcal{T} \to [0,\infty)^{\mathcal{C}}$ where $\mathcal{C}$ is the set of isotopy classes of nontrivial simple closed curves on Σ, and a hyperbolic structure on Σ determines an element of $[0,\infty)^{\mathcal{C}}$ by taking the length of the unique closed geodesic in each isotopy class. With respect to this topology, $\mathcal{T}$ is homeomorphic to Euclidean space of dimension $6g-6$.

The *mapping class group* of Σ, denoted $\mathcal{M} = \mathcal{M}(\Sigma)$, is the group $\pi_0(\text{Homeo}(\Sigma) = \text{Homeo}(\Sigma)/\text{Homeo}_0(\Sigma)$ where $\text{Homeo}(\Sigma)$ is the group of all self-homeomorphisms of Σ and $\text{Homeo}_0(\Sigma)$ is the normal subgroup of self-homeomorphisms isotopic to the identity.

The group $\mathcal{M}$ acts real analytically on $\mathcal{T}$. Also, $\mathcal{M}$ acts on $\mathcal{C}$ and so on $[0,\infty)^{\mathcal{C}}$, and the embedding $\mathcal{T} \to [0,\infty)^{\mathcal{C}}$ is $\mathcal{M}$-equivariant. The action of $\mathcal{M}$ on $\mathcal{T}$ is properly discontinuous. The quotient orbifold $M = M(\Sigma) = \mathcal{T}/\mathcal{M}$ is called the *moduli space* of Σ. A subset of $\mathcal{T}$ is said to be *cobounded* if its image under the quotient map $\mathcal{T} \to \mathcal{M}$ is bounded.

2.2 Measured foliations

A *measured foliation* on Σ is a foliation on the complement of a finite set of singularities, together with a positive, transverse Borel measure, such that each singularity is an *n-pronged singularity* for some $n \geq 3$, locally modelled on the horizontal measured foliation of the quadratic differential $z^{n-2}\,dz$ in the complex plane. A *saddle connection* of a measured foliation is leaf segment which connects two distinct singularities, and *Whitehead equivalence* is the equivalence relation on the set of measured foliations generated by isotopy and the collapse of saddle connections. The *measured foliation space* of Σ, denoted $\mathcal{M}F = \mathcal{M}F(\Sigma)$, is the space of Whitehead equivalence classes of measured foliations on Σ. A topology on $\mathcal{M}F$ is specified by the "transverse measure embedding" $\mathcal{M}F \to [0,\infty)^{\mathcal{C}}$, where a measured foliation determines an element of $[0,\infty)^{\mathcal{C}}$ by taking the infimum of the transverse measures of representatives of $\mathcal{C}$.

Given a measured foliation $\mathcal{F}$ and $r \in (0,\infty)$, multiplying the transverse measure on $\mathcal{F}$ by r gives a new measured foliation denoted $r\mathcal{F}$. This gives a free action of $(0,\infty)$ on $\mathcal{M}F$, whose quotient space is the space of projective measured foliations on Σ, denoted $\mathbf{P}\mathcal{M}F = \mathbf{P}\mathcal{M}F(\Sigma)$.

The embedding $\mathcal{T} \hookrightarrow [0,\infty)^{\mathcal{C}}$, composed with the projectivization map $[0,\infty)^{\mathcal{C}} \to \mathbf{P}[0,\infty)^{\mathcal{C}}$, produces an embedding $\mathcal{T} \hookrightarrow \mathbf{P}[0,\infty)^{\mathcal{C}}$. The embedding $\mathcal{M}F \hookrightarrow [0,\infty)^{\mathcal{C}}$ induces an embedding $\mathbf{P}\mathcal{M}F \hookrightarrow \mathbf{P}[0,\infty)^{\mathcal{C}}$. Thurston's Compactification Theorem [FLP+79] says that image of $\overline{\mathcal{T}} = \mathcal{T} \cup \mathbf{P}\mathcal{M}F$ in $\mathbf{P}[0,\infty)^{\mathcal{C}}$ is a closed ball of dimension $6g-6$, whose interior is $\mathcal{T}$ and whose boundary sphere is $\mathbf{P}\mathcal{M}F$.

2.3 Geodesics in $\mathcal{T}$

The Teichmüller metric and its geodesics are usually described in terms of holomorphic quadratic differentials on Riemann surfaces. Using results of Gardiner and Masur [GM91] and of Hubbard and Masur [HM79], the metric can be presented directly in terms of measured foliations.

Consider of pair of measured foliations $\mathcal{F}_x, \mathcal{F}_y$ which are *transverse*, meaning that they have the same singular set, at each singularity $\mathcal{F}_x$ and $\mathcal{F}_y$ have the same number of prongs, they are transverse in the usual sense away from the singularities, and near an n-pronged singularity they are locally modelled on the horizontal and vertical measured foliations of the quadratic differential $z^{n-2}dz$ on the complex plane. Let $|dy|, |dx|$ denote the transverse measures on $\mathcal{F}_x, \mathcal{F}_y$, respectively; the leaves of $\mathcal{F}_x$ should be visualized as horizontal lines with transverse measure $|dy|$, and the leaves of $\mathcal{F}_y$ as vertical lines with transverse measure $|dx|$. The formula $dx^2 + dy^2$ defines a singular Euclidean metric on Σ denoted $\mu(\mathcal{F}_x, \mathcal{F}_y)$, with total area

$$\text{Area}(\mu) = \int_{\Sigma} |dx|\,|dy| < \infty$$

Underlying the metric $\mu(\mathcal{F}_x, \mathcal{F}_y)$ is a conformal structure on the complement of the singularities, but the singularities are removable and so we obtain a conformal structure on Σ and a point in $\mathcal{T}$ denoted $\sigma(\mathcal{F}_x, \mathcal{F}_y)$. This gives a well-defined map from a certain subset of $\mathcal{M}F \times \mathcal{M}F$ to $\mathcal{T}$. Namely, letting $\mathcal{U} \subset \mathcal{M}F \times \mathcal{M}F$ denote the set of pairs (ξ, η) which are represented by a transverse pair $(\mathcal{F}_x, \mathcal{F}_y)$ of measured foliations, it follows that $\sigma(\xi, \eta) = \sigma(\mathcal{F}_x, \mathcal{F}_y)$ is well-defined independent of the choice of the representative pair $(\mathcal{F}_x, \mathcal{F}_y)$, defining a map $\mathcal{U} \to \mathcal{T}$ (see [GM91], Theorem 3.1).

A transverse pair $\mathcal{F}_x, \mathcal{F}_y$ is *normalized* if $\text{Area}(\mu(\mathcal{F}_x, \mathcal{F}_y)) = 1$. Let $\mathcal{U}_0 \subset \mathcal{U}$ be the subset represented by normalized transverse pairs. For each $(\mathcal{F}_x, \mathcal{F}_y) \in \mathcal{U}_0$, the map $t \mapsto \gamma(t) = \sigma(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$ is a real analytic embedding of $\mathbf{R}$ in $\mathcal{T}$; the image of this embedding depends only on the projective classes $\xi = \mathbf{P}\mathcal{F}_x$, $\eta = \mathbf{P}\mathcal{F}_y$ and is denoted $\overleftrightarrow{(\xi, \eta)} = \overleftrightarrow{(\mathbf{P}\mathcal{F}_x, \mathbf{P}\mathcal{F}_y)}$. Teichmüller's Theorem [Abi80] says that any two points $p \neq q \in \mathcal{T}$ are contained in a unique such line $\overleftrightarrow{(\mathbf{P}\mathcal{F}_x, \mathbf{P}\mathcal{F}_y)}$; moreover, if $s, t \in \mathbf{R}$ are such that $p = \sigma(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$ and $q = \sigma(e^s \mathcal{F}_x, e^{-s} \mathcal{F}_y)$, then the formula

$$d(p, q) = |s - t|$$

gives a well-defined metric on $\mathcal{T}$, known as the *Teichmüller metric*. Each line $\overleftrightarrow{(\xi, \eta)}$ then becomes a bi-infinite geodesic in $\mathcal{T}$. If we restrict the parameterization $\gamma(t) = \sigma(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$ to the half-line $t \in [0, \infty)$ then we obtain a geodesic ray in $\mathcal{T}$ and we call the point $\mathbf{P}\mathcal{F}_x \in \mathbf{P}\mathcal{M}F$ the *ending foliation* of the ray; if $\sigma = \gamma(0)$ and $\xi = \mathbf{P}\mathcal{F}_x$ then this ray is denoted $\overrightarrow{[\sigma, \xi)}$. For any Teichmüller line $\overleftrightarrow{(\xi, \eta)}$ the two points ξ, η are called the ending foliations of the line. Any two points $\sigma, \tau \in \mathcal{T}$ are the endpoints of a unique finite geodesic segment, denoted $\overline{\sigma\tau}$.

With respect to the Teichmüller metric, $\mathcal{T}$ is a complete metric space on which $\mathcal{M}$ acts by isometries. Royden's Theorem [Roy71] says, when the surface Σ is closed and oriented, that the homomorphism $\mathcal{M} \to \text{Isom}(\mathcal{T})$ is an isomorphism, except for a small kernel on certain small surfaces: on Σ_2 the single nontrivial element of the kernel being the hyperelliptic involution of Σ_2.

Given $\sigma \in \mathcal{T}$ and $\xi \in \mathbf{P}\mathcal{M}F$ there is exactly one ray in $\mathcal{T}$ with endpoint σ and with ending foliation ξ; we denote this ray $\overrightarrow{[\sigma, \xi)}$. This gives a one-to-one correspondence between $\mathcal{T} \times \mathbf{P}\mathcal{M}F$ and geodesic rays in $\mathcal{T}$. Given $\xi, \eta \in \mathbf{P}\mathcal{M}F$, there exists at most one geodesic in $\mathcal{T}$ with ending foliations ξ, η, and it exists if and only if (ξ, η) is in the image of $\mathcal{U}_0 \subset \mathcal{M}F \times \mathcal{M}F$ under the projectivization map $\mathcal{M}F \times \mathcal{M}F \to \mathbf{P}\mathcal{M}F \times \mathbf{P}\mathcal{M}F$. If this geodesic exists we denote it $\overleftrightarrow{(\xi, \eta)}$. This gives a one-to-one correspondence between a certain subset of $\mathbf{P}\mathcal{M}F \times \mathbf{P}\mathcal{M}F$ and the set of geodesics in $\mathcal{T}$.

Remark. It is not in general true that the end of the ray $\overrightarrow{[\sigma, \xi)}$ converges in $\overline{\mathcal{T}}$ to the point ξ; however, it is at least true for cobounded rays.

Let $T\mathcal{T}$ denote the tangent space of $\mathcal{T}$. There is an embedding $\mathcal{T} \times \mathbf{P}\mathcal{M}F \mapsto T\mathcal{T}$, taking (σ, ξ) to the tangent vector at σ of the ray $\overrightarrow{[\sigma, \xi)}$ in $T\mathcal{T}$, denoted $D\overrightarrow{[\sigma, \xi)}$. The image of this embedding will be denoted $T^1\mathcal{T}$, called the *unit tangent bundle* of $\mathcal{T}$, and $T^1\mathcal{T}$ is, in fact, a topological sphere bundle over $\mathcal{T}$. Moreover, the map

$\mathcal{U}_0 \to T^1\mathcal{T}$ taking (ξ, η) to $D\overrightarrow{[\sigma(\xi,\eta),\eta)}$, is a homeomorphism. See [GM91] and [HM79] for proofs.

Remark. The Teichmüller metric is not a Riemannian metric, although it is a Finsler metric. As such, the unit tangent sphere $T^1_\sigma\mathcal{T}$ at each $\sigma \in \mathcal{T}$ is not a true ellipsoid in the vector space $T_\sigma\mathcal{T}$, but instead a more general convex, centrally symmetric sphere [Roy71].

The flow on $\mathcal{U}_0$ defined by $(\xi, \eta) \cdot t = (e^t\xi, e^{-t}\eta)$ pushes forward under the homeomorphism $\mathcal{U}_0 \to T^1\mathcal{T}$ to a flow on $T^1\mathcal{T}$, namely the *geodesic flow* of $\mathcal{T}$. In other words, each tangent vector $v \in T^1\mathcal{T}$ is tangent to a unique geodesic γ, and the geodesic flow $v \cdot t$ is obtained by pushing v forward a distance t along γ.

2.4 Pseudo-Anosov homeomorphisms

A homeomorphism $h\colon \Sigma \to \Sigma$ is *pseudo-Anosov* if there exists a transverse pair of measured foliations $\mathcal{F}_x, \mathcal{F}_y$ and a $\lambda > 1$ such that $h(\mathcal{F}_x) = \lambda\mathcal{F}_x$ and $h(\mathcal{F}_y) = \lambda^{-1}\mathcal{F}_y$; the foliations $\mathcal{F}_x, \mathcal{F}_y$ are called the *stable and unstable* measured foliations of h, and λ is the *expansion factor* of h. A mapping class $H \in \mathcal{M}$ is said to be pseudo-Anosov if and only if it has a representative $h\colon \Sigma \to \Sigma$ which is pseudo-Anosov.

By construction, a mapping class $H \in \mathcal{M} = \mathrm{Isom}(\mathcal{T})$ is pseudo-Anosov if and only if there exists a geodesic γ in $\mathcal{T}$ such that $H(\gamma) = \gamma$ and the action of H on γ is a translation of nonzero length. In this case, the geodesic γ is unique and is called the *axis* of H, denoted $\mathrm{Axis}(H)$. Moreover, if h is a pseudo-Anosov homeomorphism representing H, with stable and unstable foliations $\mathcal{F}_x, \mathcal{F}_y$ and expansion factor λ, then $\mathrm{Axis}(H) = \gamma(\mathcal{F}_x, \mathcal{F}_y)$ and the translation length of H equals $\log(\lambda)$.

Note that by a theorem of Bers [Ber78], a mapping class $H \in \mathcal{M}$ is pseudo-Anosov if and only if the function $\sigma \mapsto d(H, H\sigma)$ has a positive minumum in $\mathcal{T}$; moreover this minimum is acheived precisely on $\mathrm{Axis}(H)$.

2.5 Quasi-isometries

Given $K \geq 1, C \geq 0$, a (K, C) *quasi-isometry* between metric spaces is a map $f\colon X \to Y$ such that:

(1) For all $x_1, x_2 \in X$ we have

$$\frac{1}{K}\, d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

(2) $d_Y(y, f(X)) \le C$ for each $y \in Y$.

If f satisfies (1) but not necessarily (2) then it is called a (K,C) *quasi-isometric embedding*. A quasi-isometric embedding $f : \mathbf{R} \to X$ is a *quasigeodesic* in X.

A *coarse inverse* of a quasi-isometry $f \colon X \to Y$ is a quasi-isometry $g \colon Y \to X$ such that, for some constant $C' > 0$, we have $d(g \circ f(x), x) < C'$ and $d(f \circ g(y), y) < C'$ for all $x \in X$ and $y \in Y$. Every (K,C) quasi-isometry $f \colon X \to Y$ has a K, C' coarse inverse $g \colon Y \to X$, where C' depends only on K, C: for each $y \in Y$ define $g(y)$ to be any point $x \in X$ such that $d(f(x), y) \le C$.

A fundamental fact observed by Efremovich, by Milnor [Mil68], and by Švarc, which we use repeatedly without mention, states that if a group G acts properly discontinuously and cocompactly by isometries on a proper geodesic metric space X, then G is finitely generated, and X is quasi-isometric to G equipped with the word metric.

Given a metric space X and $A \subset X$, we denote $\mathrm{Nbhd}_r(A) = \{x \in X \bigm| d(x,r) \le A\}$, and given $A, B \subset X$, we denote the *Hausdorff distance* by

$$d_{\mathcal{H}}(A,B) = \inf\{r \in [0,\infty] \bigm| A \subset \mathrm{Nbhd}_r(B) \text{ and } B \subset \mathrm{Nbhd}_r(A)\}$$

Given a metric space X, the self quasi-isometries of X are denoted $\widehat{\mathrm{QI}}(X)$. Define the *coarse equivalence* relation on $\widehat{\mathrm{QI}}(X)$ where $f, g \in \widehat{\mathrm{QI}}(X)$ are coarsely equivalent, denoted $f \underset{c}{\approx} g$, if

$$\sup_{x \in X} d(fx, gx) = C < \infty$$

We call C the *coarseness constant*. Composition of elements of $\widehat{\mathrm{QI}}(X)$ gives a well-defined binary operation on the set of coarse equivalence classes of self quasi-isometries of X, defining a group $\mathrm{QI}(X)$, the *quasi-isometry group* of X. If $h \colon X \to Y$ is a quasi-isometry of metric spaces then h induces an isomorphism $\mathrm{QI}(X) \to \mathrm{QI}(Y)$. In particular, when Γ is a finitely generated group the identity map is a quasi-isometry with respect to the word metrics of any two finite generating sets, and so the quasi-isometry group $\mathrm{QI}(\Gamma)$ is independent of choice of word metric on Γ.

A *quasi-action* of a group G on a metric space X is a map $G \times X \to X$, denoted $(g,x) \mapsto g \cdot x$, such that for some $K \ge 1, C \ge 0$ we have:

- For each $g \in G$ the map $x \mapsto g \cdot x$ is a (K,C)-quasi-isometry.
- For each $g, h \in G$, $x \in X$ we have $d(gh \cdot x, g \cdot (h \cdot x)) \le C$; in other words, $L_{gh} \underset{c}{\approx} L_g \circ L_h$ with coarseness constant independent of g, h, where L_γ means left multiplication by γ.

The quasi-action is *cobounded* if there exists a bounded subset D having nonempty intersection with every orbit of the quasi-action. The quasi-action is *proper* if for each $R > 0$ there exists an integer $m \geq 0$ such that for any $x, y \in X$ the cardinality of the set $\{g \in G \bigm| (g \cdot B(x,R)) \cap B(y,R) \neq \emptyset\}$ is at most m.

A fundamental principle of geometric group theory says that if a finitely generated group G is quasi-isometric to a metric space X, then the left action of G on itself, when conjugated by a quasi-isometry $G \to X$, defines a cobounded, proper quasi-action of G on X. To be precise, if we have coarsely inverse quasi-isometries $h \colon X \to G$, $\bar{h} \colon G \to X$, then the formula $g \cdot x = \bar{h}(gh(x))$ defines a cobounded, proper quasi-action of G on X.

3 Schottky groups on Teichmüller space

In this section we recall from [FM00b] the motivation for and definition of Schottky subgroups of mapping class groups; see that paper for details and proofs.

Recall that a *Schottky group* in $\mathrm{Isom}(\mathbf{H}^n)$ is a discrete, free use convex cocompact terminology subgroup F such that every orbit is quasiconvex in $\mathrm{Isom}(\mathbf{H}^n)$. Equivalently, letting Λ be the limit set of F and $\mathcal{H}\Lambda$ the convex hull of Λ, the action of F on $\mathcal{H}\Lambda$ is cocompact; it follows that $\mathcal{H}\Lambda$ is quasi-isometric to F, and this quasi-isometry extends continuously to an F-equivariant homeomorphism between Λ and the Gromov boundary of F. This equivalence follows from the fact that $\mathbf{H}^n$ itself is a δ-hyperbolic metric space, and in fact the same (or closely analogous) equivalence holds for free subgroups of word hyperbolic groups, thereby providing a theory of Schottky subgroups of word hyperbolic groups.

In [FM00b] we mimic this setup for free subgroups F of $\mathrm{Isom}(\mathcal{T}) = \mathcal{M}$. The tricky part is that Teichmüller space $\mathcal{T}$ is not δ-hyperbolic. Nevertheless, when the properties above for Schottky subgroups of $\mathrm{Isom}(\mathbf{H}^n)$ are carefully translated into the language of $\mathrm{Isom}(\mathcal{T})$, the results of Minsky [Min96] provide enough negative curvature in $\mathcal{T}$ to prove the equivalence of various notions of Schottkiness for F. In addition, one of the main theorems of [FM00b] is that the Schottky condition characterizes precisely those free subgroups F for which $\pi_1(\Sigma) \rtimes F$ is word hyperbolic.

Theorem 3.1 (Schottky groups: Definitions). *Let F be a finite rank free subgroup of $\mathcal{M}(\Sigma)$. The following are equivalent:*

1. **Orbit quasiconvexity** *Each orbit $\mathcal{O}$ of the action of F on $\mathcal{T}$ is quasiconvex in $\mathcal{T}$, i.e. there is a constant A such that for any $x, y \in \mathcal{O}$, the geodesic $\overline{xy}$ is contained in the A-neighborhood of $\mathcal{O}$.*

2. **Weak convex cocompactness** *There is a continuous, F-equivariant embedding of the Gromov boundary ∂F into $\mathbf{P}\mathcal{MF}$, with image denoted Λ, satisfying the following:*

   (1) For any $\xi \neq \eta \in \Lambda$ there is a geodesic $\overleftrightarrow{(\xi,\eta)}$ in $\mathcal{T}$; let

   $$\mathcal{H}\Lambda = \cup\left\{\overleftrightarrow{(\xi,\eta)} \mid \xi \neq \eta \in \Lambda\right\}$$

   be the weak convex hull *of Λ, and let $\overline{\mathcal{H}}\Lambda = \mathcal{H}\Lambda \cup \Lambda$.*

   (2) The F-equivariant homeomorphism $\Lambda \to \partial F$ extends to an F-equivariant map

   $$(\overline{\mathcal{H}}\Lambda, \Lambda, \mathcal{H}\Lambda) \to (F \cup \partial F, \partial F, F)$$

   which is continuous at each point of Λ and which restricts to a quasi-isometry $\mathcal{H}\Lambda \to F$, with respect to the Teichmüller metric on $\mathcal{H}\Lambda$ and the word metric on F.

3. **Word hyperbolic extension** *The extension group $\Gamma_F = \pi_1(\Sigma) \rtimes F$ is word hyperbolic.*

Remark. It follows from the weak convex cocompactness property that for each geodesic $\overleftrightarrow{(\xi,\eta)}$ in $\mathcal{H}\Lambda$, the image of $\overleftrightarrow{(\xi,\eta)}$ in F is a quasigeodesic whose ends converge in $F \cup \partial F$ to the images of ξ, η respectively. It also follows that each nontrivial element $f \in F$ is pseudo-Anosov, because f has an axis in the Cayley graph of F and so f has an axis in $\mathcal{T}$.

Theorem 3.1 is proved in [FM00b]. We call a subgroup F satisfying any one of the equivalent conditions of Theorem 3.1 a *Schottky subgroup* of $\mathcal{M}(\Sigma)$, or a *Schottky group* of mapping classes. These groups exist in abundance:

Theorem 3.2 (Abundance of Schottky groups). *Let $\phi_1, \ldots, \phi_n$ be a collection of n independent pseudo-Anosov elements of $\mathcal{M}(\Sigma)$. Then for any sufficiently large natural numbers $a_1, \ldots, a_n$, the subgroup of $\mathcal{M}(\Sigma)$ generated by $\phi_1^{a_1}, \ldots, \phi_n^{a_n}$ is a Schottky subgroup.*

Proof. As noted in [FM00b], this follows from Theorem 3.1 together with the main result of [Mos97]. ♢

4 The geometry and topology of Γ_H

4.1 A geometric model for Γ_H

Let H be a Schottky subgroup of $\mathcal{M} = \text{Isom}(\mathcal{T})$. We now build a contractible, piecewise-Riemannian 3-complex X_H on which Γ_H acts freely, properly discontinuously, and cocompactly by isometries, so that Γ_H is quasi-isometric to X.

Choose a free generating set $h_1, \ldots, h_n \in \mathcal{M}$ for H. Let Δ be a graph with n edges $\delta_1, \ldots, \delta_n$, each with one end at a common vertex v_0 of valence n, and with opposite ends at valence one vertices $v_1, \ldots, v_n$, respectively. Let R be the rose with n petals obtained from Δ by identifying v_i with v_0 for each $i = 1, \ldots, n$, and identify $\pi_1(R, v_0)$ with H so that the homotopy class of the loop $[\delta_i]$ is identified with h_i.

On the product $\Sigma \times \Delta$ make the following identifications: for each i choose a homeomorphism $\eta_i \colon \Sigma \to \Sigma$ representing h_i and identify $\Sigma \times v_n$ with $\Sigma \times v_0$ by identifying $(x, v_0) \sim (\eta_i(x), v_i)$ for each $x \in \Sigma$. Let K_H be the quotient 3-complex, and so we obtain a locally trivial fiber bundle $K_H \to R$ with fiber Σ and with monodromy H. Up to a bundle isomorphism homotopic to the identity, K_H does not depend on the choices of representatives η_i. By Van Kampen's Theorem we have an isomorphism $\pi_1(K_H) \approx \Gamma_H$.

Let X_H be the universal cover of K_H. Hence Γ_H acts properly discontinuously and cocompactly on X_H, with quotient K_H. We now specify a metric on X_H for which this action is isometric.

Let T_H be the universal cover of R, an infinite, regular, $2n$-valent tree, regarded as the Cayley graph for H. The universal cover of Σ is the Poincaré disc D. The bundle $K_H \to R$ with fiber Σ lifts to a locally trivial fiber bundle $\pi \colon X_H \to T_H$ with fiber D, and it follows that X_H is homeomorphic to $D \times T_H$. In order to construct a Γ_H-equivariant metric on X_H the metrics on the fibers of π must be appropriately "twisted" by considering the action of H on $\mathcal{T}$.

We can embed the graph Δ in the tree T_H as a fundamental domain for the action of H, so that the restriction of the universal covering map $T_H \to R$ agrees with the quotient map $\Delta \to R$. Pick a base point σ_0 in the Teichmüller space $\mathcal{T}$. Choose a map $\rho \colon \Delta \to \mathcal{T}$ taking v_0 to σ_0, taking v_i to $h_i(\sigma_0)$, and taking δ_i to a smooth path between σ_0 and $h_i(\sigma_0)$, say the Teichmüller geodesic. Extend H-equivariantly to obtain a map $\rho \colon T_H \to \mathcal{T}$. Henceforth we shall fix the map ρ, and arcs in the tree T_H will be parameterized by arc length in $\mathcal{T}$ with respect to the map ρ. We use τ as a variable taking values in T_H and $d\tau$ as the arc length parameter in T_H.

Given an arc $\alpha \subset T_H$, we can impose a hyperbolic structure on leaves of $\Sigma \times \alpha$ in the form of a Riemannian metric g_τ on each $\Sigma \times \tau$, so that the conformal class of g_τ

represents the point $\tau \in \mathcal{T}$, and so that the metrics g_τ vary smoothly with $\tau \in \alpha$. This can be obtained for instance by choosing a pair-of-pants decomposition of Σ and using the associated Fenchel-Nielsen coordinates for $\mathcal{T}$ (see [FM00b] for a further discussion). Then we may extend to obtain a piecewise smooth Riemannian metric on $\Sigma \times \alpha$ by the formula

$$ds^2 = g_\tau^2 + d\tau^2$$

This metric is smooth on $\Sigma \times \alpha$ except over the vertices of T_H in α.

Applying this to each edge δ_i of Δ we obtain hyperbolic structures on the fibers of $\Sigma \times \Delta$, varying smoothly over each edge δ_i, so that for each $\tau \in \Delta$ the hyperbolic surface $\Sigma \times \tau$ represents $\rho(\tau) \in \mathcal{T}$. We therefore obtain a piecewise smooth Riemannian metric on the 3-complex $\Sigma \times \Delta$. Since $h_i(v_0) = v_i$ it follows that there is a unique isometry $\eta_i \colon \Sigma \times v_0 \to \Sigma \times v_i$ representing the mapping class h_i, and we use these choices of η_i to construct K_H (here, for each $\tau \in \Delta$, we are implicitly identifying $\Sigma \times \tau$ with Σ by projection to the first factor). Thus we have defined a piecewise smooth Riemannian metric on K_H whose restriction to each fiber of the bundle $K_H \to R$ is a hyperbolic metric on that fiber.

Lifting the metric on K_H we obtain a piecewise Riemannian metric on the universal cover X_H, whose restrictions to the fibers of the bundle $\pi \colon X_H \to T_H$ give a continuously varying family of hyperbolic metrics $\overline{g}_\tau$ on the fibers, parameterized by $\tau \in T_H$. Each fiber is isometric to $\mathbf{H}^2$. For each arc α of T_H the sub-bundle $\pi^{-1}(\alpha)$ has the form $D \times \alpha$ and the Riemannian metric has the form

$$d\overline{s}^2 = \overline{g}_\tau^2 + d\tau^2$$

where $\overline{g}_\tau$ is the lift to $D \times \tau$ of the metric g_τ on the appropriate fiber of K_H.

With respect to the piecewise Riemannian metric and the associated geodesic metric on X_H, the group Γ_H acts properly discontinuously and cocompactly by isometries.

4.2 What it means to coarsely respect a pattern

A *pattern* in a metric space is simply a collection of subsets. We will loosely use the term *foliation* to refer to a pattern forming a partition of the space into disjoint subsets called *leaves*, and in that context the pattern itself will be called the *leaf space*.

Let X, Y be metric spaces, and let $\mathcal{F}, \mathcal{G}$ be patterns in X, Y respectively. A quasi-isometry $\phi \colon X \to Y$ is said to *coarsely respect* the patterns $\mathcal{F}, \mathcal{G}$ if there exists a number $A \geq 0$ and a map $h \colon \mathcal{F} \to \mathcal{G}$ such that for each element $L \in \mathcal{F}$ we have

$$d_{\mathcal{H}}(\phi(L), h(L)) \leq A$$

and if further a similar statement holds for a coarse inverse of ϕ. When distinct elements of $\mathcal{F}$ have infinite Hausdorff distance in X, and similarly for $\mathcal{G}$, then h is a bijection and it is uniquely determined by the quasi-isometry ϕ.

An isometry of X_H respects many different patterns in X_H. The idea of the proof of Theorem 1.3 is to show that an arbitrary quasi-isometry f of Γ_H (hence of X_H) preserves more and more structure, in the sense of coarsely respecting finer and finer patterns, until so much structure is preserved that f must actually be a bounded distance from an element of the extension group $\Gamma_{\mathcal{C}}$ defined in Theorem 1.3.

4.3 Hyperplanes and the horizontal foliation

The bundle $\pi\colon X_H \to T_H$ associated to the group Γ_H determines two important patterns in X_H: the *horizontal foliation* and the pattern of *hyperplanes*.

The *horizontal foliation* of X_H is the pattern of fibers of π, subsets $D_\tau = \pi^{-1}(\tau)$, $\tau \in T_H$, each called a *horizontal leaf* of X_H. The leaf space of the horizontal foliation is identified with T_H via the bundle map π. Any two horizontal leaves have finite Hausdorff distance in X_H, in fact from the form of the metric on X_H we see easily that $d_{\mathcal{H}}(D_\tau, D_u) = d(\tau, u)$ for all $\tau, u \in T_H$. In other words, the bundle map π induces an isometry between the horizontal foliation equipped with the Hausdorff metric and the metric tree T_H. As noted earlier, the metric on X_H restricts to a metric $\overline{g}_\tau$ on each horizontal leaf D_τ making D_τ isometric to $\mathbf{H}^2$.

A *hyperplane* in X_H is any set $P_w = \pi^{-1}(w)$, where w is a bi-infinite geodesic in T_H. The metric on X_H restricts to a piecewise Riemannian metric on P_w of the form $\tilde{g}_\tau^2 + d\tau^2$, with respect to the coordinates $P_w \approx D \times w$. Any two distinct hyperplanes in X_H have infinite Hausdorff distance.

The pattern of hyperplanes in X_H is the first pattern which must be coarsely respected by a quasi-isometry.

Proposition 4.1 (Hyperplanes respected). *Any quasi-isometry $f\colon X_H \to X_H$ coarsely respects the pattern of hyperplanes in X_H, with coarseness constant depending only on the quasi-isometry constants of f.*

In other words, f must map each hyperplane in X_H a uniform Hausdorff distance from some other hyperplane in X_H. Since distinct hyperplanes have infinite Hausdorff distance in X_H it follows that f induces a bijection on the pattern of hyperplanes; when f is understood we denote this bijection $P_w \mapsto P_{w'}$.

Each hyperplane P_w is uniformly properly embedded in X_H, that is, there is a proper function $r\colon [0,\infty) \to [0,\infty)$ independent of w such that for all $x, y \in P_w$ we have:

$$d_{X_H}(x,y) \geq r\big(d_{P_w}(x,y)\big)$$

By applying Proposition 4.1 together with a simple general principle (made explicit in Lemma 2.1 of [FM00a]) we have for any quasi-isometry $f : X_H \to X_H$ the following fact: for each hyperplane P_w, the map f induces via restriction composed with nearest-point projection, a quasi-isometry $\phi\colon P_w \to P_{w'}$; the quasi-isometry constants for ϕ depend only on those for f.

The horizontal foliation on X_H restricts to a horizontal foliation on P_w. Using the coordinates $P_w \approx D \times \mathbf{R}$ as described above, the horizontal leaves in P_w are of the form $D_t = D \times t$, $t \in \mathbf{R}$. The leaf space of this foliation is $\mathbf{R}$ and we have $d_{\mathcal{H}}(D_s, D_t) = |s - t|$ using Hausdorff distance in P_w.

The fact that X_H fibers over a tree T_H with nontrivial branching restricts the behavior of ϕ as follows:

Proposition 4.2 (Horizontal foliation respected). *For any quasi-isometry $f\colon X_H \to X_H$ and any hyperplane P_w, the induced quasi-isometry $\phi\colon P_w \to P_{w'}$ uniformly coarsely respects horizontal foliations.*

In other words, there exists a constant A *independent of* w such that for any horizontal leaf $D_t \subset P_w$ there is a horizontal leaf $D_{t'} \subset P_{w'}$ with $d_{\mathcal{H}}(\phi(D_t), D_{t'}) \leq A$

This is precisely where the assumption is used that H has rank greater than one.

Proof of Propositions 4.1 and 4.2. Since H has rank greater than one, its Cayley graph T_H is a *bushy* tree, meaning that each point of T_H is within some fixed distance $\beta = 1$ of some vertex v such that $T - v$ has at least 3 unbounded components. We can thus apply the following result, which is Theorem 7.7 of [FM00a], to the metric fibration $\pi\colon X_H \to T_H$.

Lemma 4.3. *Let $\pi\colon X \to T$, $\pi'\colon X' \to T'$ be metric fibrations over bushy trees T, T', such that the fibers of π and π' are contractible n-manifolds for some n. Let $f\colon X \to X'$ be a quasi-isometry. Then there exists a constant A, depending only on the metric fibration data of π, π', the quasi-isometry data for f, and the constant β, such that:*

1. *For each hyperplane $P \subset X$ there exists a unique hyperplane $Q \subset X'$ such that $d_{\mathcal{H}}(f(P), Q) \leq A$.*
2. *For each horizontal leaf $L \subset X$ there is a horizontal leaf $L' \subset X'$ such that $d_{\mathcal{H}}(f(L), L') \leq A$.*

Proposition 4.1 is an immediate consequence. To obtain Proposition 4.2, consider a horizontal leaf D_t of P_w. By applying Lemma 4.3 we obtain a horizontal leaf D_s of X_H uniformly Hausdorff close to $f(D_t)$. But $f(P_w)$ is uniformly Hausdorff close to $P_{w'}$ and so D_s is uniformly Hausdorff close to some horizontal leaf $D_{t'} \subset P_{w'}$. Finally, the closest point projection $f(P_w) \mapsto P_{w'}$ moves points a uniformly bounded distance, and so $f(P_w)$ is a uniformly Hausdorff close to its closest point projection in $P_{w'}$. $\diamondsuit$

4.4 The singular solv metric on a hyperplane

In this subsection we find a different metric on hyperplanes P_w that will allow us to apply ideas from pseudo-Anosov dynamics.

Associated to each geodesic γ in $\mathcal{T}$ we recall the construction of a "singular SOLV metric" on $D \times \gamma$. Choose a normalized transverse pair of measured foliations $\mathcal{F}_x, \mathcal{F}_y$ on Σ, with transverse measures $|dy|, |dx|$ respectively, so that $\gamma = \gamma(\mathcal{F}_x, \mathcal{F}_y)$; we can parameterize γ by arc length:

$$\gamma(t) = \sigma(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$$

For each t we have a singular Euclidean metric $\mu_t = \mu(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$, that is,

$$d\mu_t^2 = e^{-2t} dx^2 + e^{2t} dy^2$$

Note that the transverse pair $(e^t \mathcal{F}_x, e^{-t} \mathcal{F}_y)$ is normalized for all t, that is, $\text{Area}(\mu_t) = \int_\Sigma e^t |dy| e^{-t} |dx| = \int_\Sigma |dx|\,|dy| = 1$. Lifting to the universal cover $D = \widetilde{\Sigma}$ we obtain a transverse pair of measured foliations $\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y$ with transverse measures still denoted $|dy|, |dx|$, and a one-parameter family of singular Euclidean metrics $\tilde\mu_t$ on D, given by

$$d\tilde\mu_t^2 = e^{-2t} dx^2 + e^{2t} dy^2$$

Now we can define a singular Riemannian metric on $D \times \gamma$ by the formula

$$\begin{aligned} ds^2 &= d\tilde\mu_t^2 + dt^2 \\ &= e^{-2t} dx^2 + e^{2t} dy^2 + dt^2 \end{aligned}$$

The singular locus of this metric is a family of vertical lines in $D \times \gamma$, whose intersection with the fiber $D \times t$ is precisely the singular set of $e^t \widetilde{\mathcal{F}}_x, e^{-t} \widetilde{\mathcal{F}}_y$. The Riemannian metric extends across the singular locus to a complete geodesic metric on $D \times \gamma$. The space $D \times \gamma$ with this metric is called the *singular* SOLV *space* associated to γ, denoted Q_γ. The reason for the terminology is that away from the singular locus the metric is locally modelled on 3-dimensional SOLV-geometry, and at the singular locus (each component of which is a line) the metric is modelled on some number of "half SOLV's" glued together.

Note that the singular SOLV-metric on Q_γ is uniquely determined by the fibration $Q_\gamma \to \gamma$ together with the family of xy-structures $e^t \widetilde{\mathcal{F}}_x, e^{-t} \widetilde{\mathcal{F}}_y$: in each leaf; the measured foliation $e^t \widetilde{\mathcal{F}}_x$ provides the $e^{2t} dy$ term and the measured foliation $e^{-t} \widetilde{\mathcal{F}}_y$ provides the $e^{-2t} dx$ term, and the map to γ with the arc length parameter t provides the dt^2 term.

In summary, we have a bijective correspondence

$$\{\text{geodesics } \gamma \text{ in } \mathcal{T}\} \leftrightarrow \{\pi_1(\Sigma)\text{-equivariant singular SOLV metrics on } D \times \mathbf{R}\}$$

***xy*-structures.** For convenience we shall define an *xy-structure* on a surface S to be a transverse pair of measured foliations $\mathcal{F}_x, \mathcal{F}_y$. We will work with xy-structures on Σ as well as the lifted structures on the universal cover D.

An *affine automorphism* of an xy-structure on S is a homeomorphism of S which respects the (unordered) pair of (unmeasured) foliations $\{\mathcal{F}_x, \mathcal{F}_y\}$, multiplying the transverse measure on one by $\lambda > 0$ and multiplying the transverse measure on the other by $1/\lambda$. Let $\text{Aff}(\mathcal{F}_x, \mathcal{F}_y)$ be the group of affine automorphisms of the xy-structure $\mathcal{F}_x, \mathcal{F}_y$. There is a subgroup of index ≤ 2 in $\text{Aff}(\mathcal{F}_x, \mathcal{F}_y)$, denoted $\text{Aff}_+(\mathcal{F}_x, \mathcal{F}_y)$, which respects the *ordered* pair of (unmeasured) measured foliations $(\mathcal{F}_x, \mathcal{F}_y)$, with transverse measures multiplied as described above. There is a homomorphism

$$\text{Stretch}\colon \text{Aff}(\mathcal{F}_x, \mathcal{F}_y) \to \text{SL}(2, \mathbf{R})$$

whose image lies in the subgroup of $\text{SL}(2,\mathbf{R})$ corresponding to matrices which are either zero off the diagonal or zero on the diagonal, as follows. Given $\phi \in \text{Aff}(\mathcal{F}_x, \mathcal{F}_y)$: if $\phi(\mathcal{F}_x) = \lambda \mathcal{F}_x$ and $\phi(\mathcal{F}_y) = \frac{1}{\lambda}\mathcal{F}_y$ then $\text{Stretch}(\phi)$ is the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{smallmatrix}\right)$; whereas if $\phi(\mathcal{F}_x) = \lambda \mathcal{F}_y$ and $\phi(\mathcal{F}_y) = \frac{1}{\lambda}\mathcal{F}_x$ then $\text{Stretch}(\phi)$ is the matrix $\left(\begin{smallmatrix} 0 & \lambda \\ 1/\lambda & 0 \end{smallmatrix}\right)$. In the cases of interest where $S = \Sigma$ or D, the image of the homomorphism, denoted $\text{Stretch}(\mathcal{F}_x, \mathcal{F}_y)$, is a discrete subgroup of $\text{SL}(2,\mathbf{R})$; discreteness follows easily using the fact that the singular set of the pair $\mathcal{F}_x, \mathcal{F}_y$ forms a nonempty, discrete net in S. As a consequence, $\text{Stretch}(\mathcal{F}_x, \mathcal{F}_y)$ is isomorphic to either D_∞, $\mathbf{Z}$, $\mathbf{Z}/2$, or the trivial group. The kernel of the stretch homomorphism is the group $\text{Isom}_+(\mathcal{F}_x, \mathcal{F}_y)$ of automorphisms of the *ordered* pair of *measured* foliations $\mathcal{F}_x, \mathcal{F}_y$.

We obtain a short exact sequence

$$1 \to \text{Isom}_+(\mathcal{F}_x, \mathcal{F}_y) \to \text{Aff}(\mathcal{F}_x, \mathcal{F}_y) \to \text{Stretch}(\mathcal{F}_x, \mathcal{F}_y) \to 1$$

where $\text{Isom}_+(\mathcal{F}_x, \mathcal{F}_y)$ is the group of automorphisms of the *ordered* pair of *measured* foliations $\mathcal{F}_x, \mathcal{F}_y$.

Given a geodesic $\gamma = \gamma(\mathcal{F}_x, \mathcal{F}_y)$ in $\mathcal{T}$, the universal cover $D = \widetilde{\Sigma}$ with the lifted xy-structure $\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y$ may be identified with a certain horizontal fiber of Q_γ.

With respect to this identification, every isometry of the singular SOLV-manifold Q_γ respects the horizontal foliation, and vertical projection of an isometry onto D defines an xy-affine automorphism of $\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y$. Every xy-affine automorphism arises in this manner, leading to an isomorphism between $\mathrm{Isom}(Q_\gamma)$ and $\mathrm{Aff}(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y)$. This leads in turn to an isomorphism of short exact sequences as follows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Isom}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & \mathrm{Aff}(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & \mathrm{Stretch}(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & 1 \\ & & \wr & & \wr & & \wr & & \\ 1 & \longrightarrow & \mathrm{Isom}_h(Q_\gamma) & \longrightarrow & \mathrm{Isom}(Q_\gamma) & \longrightarrow & C_\gamma & \longrightarrow & 1 \end{array}$$

where $\mathrm{Isom}_h(Q_\gamma)$ is the subgroup of $\mathrm{Isom}(Q_\gamma)$ preserving each horizontal leaf, and $C_\gamma = \mathrm{Isom}(Q_\gamma)/\mathrm{Isom}_h(Q_\gamma)$; the latter group is isomorphic to D_∞, $\mathbf{Z}$, $\mathbf{Z}/2$, or the trivial group.

We also have an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Isom}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & \mathrm{Aff}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & \mathrm{Stretch}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) & \longrightarrow & 1 \\ & & \wr & & \wr & & \wr & & \\ 1 & \longrightarrow & \mathrm{Isom}_h(Q_\gamma) & \longrightarrow & \mathrm{Isom}_+(Q_\gamma) & \longrightarrow & C_{\gamma+} & \longrightarrow & 1 \end{array}$$

where $\mathrm{Aff}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y)$ was defined earlier, $\mathrm{Stretch}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y)$ is the intersection of $\mathrm{Stretch}(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) \subset \mathrm{SL}(2,\mathbf{R})$ with the diagonal subgroup of $\mathrm{SL}(2,\mathbf{R})$, $\mathrm{Isom}_+(Q_\gamma)$ is the subgroup of $\mathrm{Isom}(Q_\gamma)$ preserving the transverse orientation on the horizontal foliation, and $C_{\gamma+} = \mathrm{Isom}_+(Q_\gamma)/\mathrm{Isom}_h(Q_\gamma)$; in each of these four cases, the $+$ subscript induces a subgroup of index ≤ 2. The group $\mathrm{Stretch}_+(\widetilde{\mathcal{F}}_x, \widetilde{\mathcal{F}}_y) \approx C_{\gamma+}$ is either $\mathbf{Z}$ or trivial.

Note that C_γ contains the group $\mathrm{Stab}_{\mathcal{T}}(\gamma) = \{\Phi \in \mathrm{Isom}(\mathcal{T}) \bigm| \Phi(\gamma) = \gamma\}$, and we shall show in Lemma 6.2 that this containment has finite index.

Schottky groups and singular solv-manifolds. Now consider a Schottky group $H \subset \mathcal{M}$ with limit set $\Lambda \subset \mathbf{P}\mathcal{M}F$, and weak convex hull $\mathcal{H}\Lambda$. We have the Cayley graph T_H and an H-equivariant immersion $\rho\colon T_H \to \mathcal{T}$; let $\overline{T}_H = T_H \cup \Lambda$. An immediate corollary of Theorem 3.1 is that T_H and $\mathcal{H}\Lambda$ have finite Hausdorff distance in $\mathcal{T}$, and so we obtain an H-equivariant quasi-isometry $\theta\colon \mathcal{H}\Lambda \to T_H$ which moves points a uniformly bounded distance in $\mathcal{T}$. It also follows from Theorem 3.1 that the map θ extends, via the identity map on Λ, to an H-equivariant map $\overline{\theta}\colon \overline{\mathcal{H}\Lambda} \to \overline{T}_H$ which is continuous at each point of Λ. It follows immediately that for any $\xi \ne \eta \in \Lambda$, the restriction of θ to the geodesic $\overleftrightarrow{(\xi,\eta)}$ under θ is a quasigeodesic, with quasigeodesic constants independent of ξ, η, and the ends of this quasigeodesic converge in $\overline{T_H}$ to ξ, η respectively; the unique geodesic in T_H with these endpoints is denoted $\overline{\xi\eta}$. To summarize:

- The correspondence between geodesics $\gamma = \overleftrightarrow{(\xi, \eta)}$ in $\mathcal{H}\Lambda$ and geodesics $w = \overline{\xi\eta}$ in T_H is a bijection. We denote this correspondence by $w = w_\gamma$, $\gamma = \gamma_w$. Corresponding geodesics γ, w_γ are uniformly Hausdorff close in $\mathcal{T}$.

We need to be a bit more precise. By lifting each geodesic in $\mathcal{H}\Lambda$ to the unit tangent bundle $T^1\mathcal{T}$ we get a closed subset of $T^1\mathcal{T}$ invariant under the geodesic flow denoted $T\mathcal{H}\Lambda$. The space $T\mathcal{H}\Lambda$ is locally homeomorphic to $\Lambda \times \Lambda \times \mathbf{R}$, i.e. it is locally a Cantor set crossed with the line. We will often confuse a geodesic γ in $\mathcal{H}\Lambda$ with its lift to $T\mathcal{H}\Lambda$; these geodesics form a lamination of $T\mathcal{H}\Lambda$. As we have said, the map $\theta\colon \mathcal{H}\Lambda \to T_H$ lifts to a map from $T\mathcal{H}\Lambda$ to T_H, taking each geodesic γ to a quasigeodesic in T_H uniformly Hausdorff close to w_γ. But then, by moving the map $T\mathcal{H}\Lambda \to T_H$ a bounded amount, we obtain a continuous map $\Theta\colon T\mathcal{H}\Lambda \to \mathcal{T}_H$ with the property that $\Theta(\gamma) = w_\gamma$ for each γ. The restriction to γ is denoted $\Theta_\gamma\colon \gamma \to w_\gamma$, and this is a quasi-isometry with constants independent of γ, and we may take Θ_γ to be a homeomorphism.

Proposition 4.4 (Comparison of metrics). *Given corresponding geodesics w in T_H and $\gamma = \gamma_w$ in $T\mathcal{H}\Lambda$, there exists a $\pi_1\Sigma$-equivariant, horizontal respecting quasi-isometry $F_w\colon P_w \to Q_\gamma$, with quasi-isometry constants independent of w, such that F_w is a lift of the map $\Theta_\gamma^{-1}\colon w \to \gamma$.*

In other words, the natural metric on each hyperplane of the geometric model space X_H is uniformly quasi-isometric to a singular SOLV 3-manifold. This is the key place where we use the fact that H is a Schottky group, and not just any free group of pseudo-Anosovs.

Proof. Recall that we have a Γ_H-equivariant D-bundle $X_H \to T_H$, carrying a Γ_H-equivariant piecewise Riemannian metric whose restriction to each fiber is isometric to $\mathbf{H}^2$, with Γ_H acting cocompactly. Each of the spaces P_w is embedded in X_H as the inverse image of $w \subset T_H$.

We now define a Γ_H-equivariant D-bundle $\Xi \to T\mathcal{H}\Lambda$ in which each of the singular SOLV-manifolds Q_γ sits, as follows.

First, note that each point of $T^1\mathcal{T}$ corresponds to (the isotopy class of) a normalized xy-structure $(\mathcal{F}_x, \mathcal{F}_y)$ on Σ. We may assemble these structures into an $\mathcal{M}$-equivariant Σ-bundle over $T^1\mathcal{T}$, each fiber equipped with an xy-structure in the appropriate isotopy class, so that the xy-strucures vary continuously as the base point in $T^1\mathcal{T}$ varies. Restricting to $T\mathcal{H}\Lambda$ we obtain an H-equivariant xy-Σ-bundle $\Upsilon \to T\mathcal{H}\Lambda$.

The universal cover of the Σ-bundle over $T^1\mathcal{T}$ is a D-bundle over $\mathcal{T}$, with smoothly varying xy-structures on the fibers, on which the $\pi_1\Sigma$ extension of $\mathcal{M}$ acts, namely the once-punctured mapping class group $\mathcal{M}(\Sigma, p)$. Restricting to $T\mathcal{H}\Lambda$ we obtain a Γ_H-equivariant xy-D-bundle $\Xi \to T\mathcal{H}\Lambda$. Restricting to any geodesic $\gamma \subset T\mathcal{H}\Lambda$ we obtain the fibration $Q_\gamma \to \gamma$. Note that the foliation of $T\mathcal{H}\Lambda$ by geodesics lifts to a foliation of Ξ by 3-manifolds: the 3-manifold over the geodesic $\gamma \subset T\mathcal{H}\Lambda$ is Q_γ. The fiberwise xy-structures vary continuously in Ξ, and the arc length parameter on geodesics of $T\mathcal{H}\Lambda$ varies continuously; as noted earlier, these data determine singular SOLV-metrics on each Q_γ, and these metrics vary continuously in Ξ.

Now lift the H-equivariant map $\Theta\colon T\mathcal{H}\Lambda \to T_H$ to a Γ_H-equivariant continuous map of D-bundles, $\widetilde\Theta\colon \Xi \to X_H$, taking each fiber of Ξ homeomorphically to the corresponding fiber of X_H, and taking each singular SOLV manifold Q_γ to the corresponding singular Riemannian manifold P_{w_γ}.

By cocompactness of the Γ_H actions on Ξ and X_H, and by continuity of the xy-structures on fibers of Ξ and the $\mathbf{H}^2$ structures on fibers of X_H, it follows that $\widetilde\Theta$ induces quasi-isometries from Ξ fibers to X_H fibers with uniform quasi-isometry constants; in particular, we get uniform quasi-isometries from the horizontal sets of Q_γ to the horizontal sets of P_{w_γ}, over all geodesics γ in $T\mathcal{H}\Lambda$. Moreover, since the map $\Theta\colon T\mathcal{H}\Lambda \to T_H$ is uniformly quasi-isometric from a geodesic γ in $T\mathcal{H}\Lambda$ to the corresponding geodesic w_γ in T_H, it follows that $\widetilde\Theta$ induces a uniform quasi-isometry from the $|dt|$ term in the metric on Q_γ to the $d\tau$ term in the metric on P_{w_γ}. This implies that the family of maps $Q_\gamma \to P_{w_\gamma}$ is uniformly quasi-isometric. $\diamondsuit$

5 Quasi-isometries remember the dynamics

Let $H \subset \mathcal{M}$ be a Schottky subgroup and let $\phi\colon \Gamma_H \to \Gamma_H$ be a quasi-isometry. By Propositions 4.1, 4.2, and 4.4, to each bi-infinite path $w \in T_H$ there corresponds a bi-infinite path $w' \in T'_H$, and a horizontal respecting quasi-isometry $Q_{\gamma_w} \to Q_{\gamma_{w'}}$ of singular SOLV spaces. In this section we study such quasi-isometries, and show that they must coarsely respect certain dynamically defined foliations.

5.1 The vertical flow on Q_γ

Let γ be a fixed Teichmüller geodesic in $\mathcal{T}(\Sigma)$, and let Q_γ be the associated singular SOLV metric on $D \times \gamma \approx D \times \mathbf{R}$. There is a natural flow on Q_γ given by

$$\Psi_s(x,t) = (x, t+s)$$

This flow is called the *vertical flow*, and its orbits are called *vertical geodesics* or *vertical flow lines* in Q_γ. This flow is a pseudo-Anosov flow in the sense of

[FM00c]. The stable and unstable foliations of Ψ expand and contract at the uniform rate e^t. Singular orbits of Ψ are the same as singular vertical geodesics of Q_γ.

There are three naturally defined foliations of Q_γ which are invariant under the vertical flow: the codimension-2 foliation by vertical flow lines; the codimension-1 weak stable foliation; and the codimension-1 weak unstable foliation. Given a vertical flow line ℓ, the *weak stable* (resp. *unstable*) leaf containing ℓ is the union of vertical flow lines that asymptote to ℓ as $t \to \infty$ (resp. $t \to -\infty$). Each of the weak stable and unstable foliations has some nonmanifold leaves, one such leaf containing each singular vertical geodesic. Note that for each leaf L of the weak stable or unstable foliation, if L is nonsingular then L is isometric to a hyperbolic plane, whereas if L is the singular leaf through a singular vertical geodesic ℓ then L is isometric to a union of hyperbolic half-planes meeting along their common boundary ℓ; in either case, L can be expressed as a finite union of hyperbolic planes.

Our goal now is to use pseudo-Anosov dynamics to show that a horizontal-respecting quasi-isometry between two hyperplanes must also coarsely respect all of these foliations. In short: the quasi-isometry remembers the dynamics.

5.2 Coarse intersection

We need a basic notion from coarse topology.

Definition (Coarse intersection). A subset W of a metric space X is a *coarse intersection* of subsets $U, V \subset X$, denoted $W = U \underset{c}{\cap} V$, if there exists C_0 such that for every $C \geq C_0$ there exists $A = A(C) \geq 0$ so that

$$d_{\mathcal{H}}(\mathrm{Nbhd}_C(U) \cap \mathrm{Nbhd}_C(V), W) \leq A$$

Note that although such a set W may not exist, when it does exist then any two such sets are a bounded Hausdorff distance from each other. The function $A(C), C \geq C_0$ is called the *coarse intersection function.*

We will need the following fact, which is an elementary consequence of the definitions.

Lemma 5.1. *For any quasi-isometry $f \colon X \to Y$ of metric spaces, and $U, V \subset X$, if $U \underset{c}{\cap} V$ exists then $f(U \underset{c}{\cap} V)$ is a coarse intersection of $f(U)$, $f(V)$, with coarse intersection function depending only on the quasi-isometry constants for f and the coarse intersection function for U and V.* $\diamondsuit$

5.3 The dynamically defined foliations are respected

In this subsection we prove the key proposition:

Proposition 5.2 (Stable and unstable foliations respected). *Let γ, γ' be cobounded Teichmüller geodesics. Then any horizontal-respecting quasi-isometry $\phi\colon Q_\gamma \to Q_{\gamma'}$ coarsely respects the stable and unstable foliations of the vertical flows on $Q_\gamma, Q_{\gamma'}$. Also, ϕ coarsely respects the patterns of singular stable and unstable leaves.*

Every vertical flow line can be realized as the coarse intersection of a stable and unstable leaf, with uniform coarse intersection function independent of the choice of flow line. Proposition 5.2 and Lemma 5.1 therefore imply that ϕ coarsely respects the collection of vertical geodesics. A similar argument works for the collection of singular geodesics, using singular stable and unstable leaves. We record this as:

Corollary 5.3 (Vertical flow lines respected). *The quasi-isometry ϕ of Proposition 5.2 coarsely respects the foliations of vertical flow lines in Q_γ and $Q_{\gamma'}$, as well as the patterns of singular vertical lines.*

Proof of Proposition 5.2. The horizontal foliation of Q_γ is an example of a *uniform foliation*, which means that any two leaves have finite Hausdorff distance. Any map between two horizontal leaves which moves points a bounded distance in Q_γ is a quasi-isometry between those leaves. It follows that there is a canonical coarse equivalence class of quasi-isometries between any two leaves, and moreover the composition of two such quasi-isometries is another one. Each leaf is quasi-isometric to the hyperbolic plane $\mathbf{H}^2$ and its Gromov boundary is a circle, so there is a canonical identification of all the circles at infinity to a single circle which we denote SQ_γ. These facts were noted by Thurston in [Thu97].

It is well-known that a quasi-isometry $\mathbf{H}^2 \to \mathbf{H}^2$ induces a homeomorphism $\partial\mathbf{H}^2 \to \partial\mathbf{H}^2$ between the circles at infinity, and that coarsely equivalent quasi-isometries induce the same boundary map. Any horizontal respecting quasi-isometry $Q_\gamma \to Q_{\gamma'}$ therefore induces a homeomorphism $SQ_\gamma \to SQ_{\gamma'}$ between their respective circles at infinity. The underlying idea of our proof is to find additional quasi-isometrically invariant structures on SQ_γ and $SQ_{\gamma'}$ which encode the stable and unstable foliations, and use this information to prove quasi-isometric invariance of these foliations.

To proceed with the proof we need some notation. Let D_t denote the plane $D \times t \subset D \times \mathbf{R} = Q_\gamma$. The plane D_t comes equipped with an xy-structure, as explained in §4.4. For each s, t let $\phi_{st}\colon D_s \to D_t$ be the map $(p, s) \to (p, t)$, $p \in D$. In other words, ϕ_{st} flows along vertical flow segments of Q_γ from D_s to

D_t; each such flow segment has length $|s-t|$, and so ϕ_{st} is a quasi-isometry in the canonical coarse equivalence class from D_s to D_t as discussed above. In fact, the map ϕ_{st} is an xy-affine homeomorphism with stretch factor $e^{|s-t|}$, implying that ϕ_{st} is $e^{|s-t|}$-bilipschitz. Note that $\phi_{tu} \circ \phi_{st} = \phi_{su}$, for all $s, t, u \in \mathbf{R}$.

The xy-metric on D_t is a CAT(0) metric, and it is also Gromov hyperbolic because D_t is quasi-isometric to $\mathbf{H}^2$. The boundary ∂D_t is a circle, and each quasi-isometry $\phi_{st} \colon D_s \to D_t$ induces a homeomorphism $\partial\phi_{st} \colon \partial D_s \to \partial D_t$.

For any two points $\xi, \eta \in \partial D_t$ there exists an xy-geodesic with endpoints ξ, η, and any two such geodesics are the boundary of an isometric embedding of $\mathbf{R} \times [a,b]$ for some $[a,b] \in \mathbf{R}$ (see [BH99]). Since D_t is not the Euclidean plane but has a cocompact isometry group, there is an upper bound on $b-a$ independent of ξ, η. This bound is moreover independent of t, because coboundedness of the Teichmüller geodesic γ and compactness of the fibers of $T^1\mathcal{T}$ together imply that the collection of locally CAT(0) metric spaces $D_t/\pi_1(\Sigma)$ lies in a compact space of locally CAT(0) metrics (this is the one place in the proof where we use coboundedness of the Teichmüller geodesics in the weak convex hull of H). Let R_0 be a t-independent bound for $|b-a|$.

For each bi-infinite geodesic θ on D_0, the set

$$V = V_\theta = \bigcup_{t \in \mathbf{R}} \{\phi_{0t}(\theta)\}$$

is called a *vertical plane* in Q_γ. Note that if $\partial\theta = \{\xi, \eta\} \subset \partial D_0$ then $\partial\phi_{0t}(\theta) = \{\partial\phi_{0t}(\xi), \partial\phi_{0t}(\eta)\} \subset \partial D_t$, and so to V there is associated a unique pair of points in the circle SQ_γ which we call the *endpoints* of V. Moreover, the discussion in the previous paragraph shows that the Hausdorff distance between any two vertical planes of Q_γ with the same endpoints ξ, η is at most R_0.

The first structure which $\phi \colon Q_\gamma \to Q_{\gamma'}$ must coarsely respect is the collection of vertical planes.

Claim 5.4 (Vertical planes preserved). *Let $\phi \colon Q_\gamma \to Q_{\gamma'}$ be a horizontal-respecting quasi-isometry. If V is any vertical plane in Q_γ, then $\phi(V)$ is a bounded Hausdorff distance from some vertical plane V' in $Q_{\gamma'}$.*

To prove Claim 5.4, let $t \to t'$ be a bijective quasi-isometry of $\mathbf{R}$ such that $\phi(D_t)$ is Hausdorff close to $D'_{t'}$, with a uniform Hausdorff constant; we assume that the parameterizations are chosen so that $0 \to 0$ under this quasi-isometry. Composing the map $D_t \to \phi(D_t)$ with a uniformly finite distance map $\phi(D_t) \to D'_{t'}$, we obtain a quasi-isometry $\psi_t \colon D_t \to D'_{t'}$ whose quasi-isometry constants are independent of t.

Fix a vertical plane $V = V_\theta$ and for each s consider the geodesic $\theta_s = V \cap D_s$. Its image quasigeodesic $\psi_s(\theta_s)$ is uniformly Hausdorff close in $D'_{s'}$ to some geodesic

$\theta'_{s'}$, in particular $\phi_0(\theta_0)$ is close to θ'_0. We need only show that for each s, $\theta'_{s'}$ is uniformly Hausdorff close to $\phi'_{0s'}(\theta'_0)$, for then we can set

$$V' = \cup_{s'}\{\phi'_{0s'}(\theta'_0)\}$$

and it follows that $\phi(V)$ is uniformly Hausdorff close to V'.

Since any two geodesics in $D'_{s'}$ which are Hausdorff close are R_0-Hausdorff close, we need only show that the Hausdorff distance between $\theta'_{s'}$ and $\phi'_{0s'}(\theta'_0)$ is finite, in other words these two geodesics have the same endpoints in $\partial D'_{s'}$. But this is an immediate consequence of the coarse commutativity of the following diagram:

$$\begin{array}{ccc} D_0 & \xrightarrow{\psi_0} & D'_0 \\ {\scriptstyle \phi_{0s}}\downarrow & & \downarrow{\scriptstyle \phi'_{0s'}} \\ D_s & \xrightarrow[\psi_s]{} & D'_{s'} \end{array}$$

This finishes the proof of Claim 5.4. We now find finer structures which must be coarsely respected by ϕ.

Each vertical plane V comes equipped with a *horizontal foliation*, obtained by intersecting the plane with the horizontal foliation $\{D_t\}$ of Q_γ. Denote the leaves of this foliation by

$$\theta_t = D_t \cap V, \quad t \in \mathbf{R}$$

so that $\theta_t = \phi_{st}(\theta_s)$ for any $s, t \in \mathbf{R}$. The horizontal foliation $\{\theta_t\}$ has a transverse orientation, pointing in the direction of increasing t. Note that the Hausdorff distance in V between θ_t and θ_s is exactly $|t - s|$. There is a projection $\pi\colon V \to \mathbf{R}$ with $\theta_t = \pi^{-1}(t)$. Define a *quasivertical line* in V to be a subset $L \subset V$ such that the projection $\pi\colon L \to \mathbf{R}$ is a quasi-isometry.

We divide the collection of vertical planes into three types: stable, unstable, and doubly unstable. A *stable* vertical plane is one which is contained in a leaf of the stable foliation on Q_γ; thus it is either a regular leaf of the stable foliation, or it is a union of two half-planes of a singular leaf. Similarly an *unstable* vertical plane is one contained in an unstable leaf. All other vertical planes are called *doubly unstable* vertical planes.

The three types of vertical planes—stable, unstable, and doubly unstable—can be distinguished from each other by horizontal respecting quasi-isometries which respect the transverse orientation, by observing the asymptotic behavior of quasivertical lines. To make this more precise, say that two quasivertical lines $L, L' \subset P$ are *upward Hausdorff close* if the Hausdorff distance between $L \cap \pi^{-1}[0,\infty)$ and $L' \cap \pi^{-1}[0,\infty)$ is finite; *downward Hausdorff close* is similarly defined using $(-\infty, 0]$.

Claim 5.5. *Let V be a vertical plane. Then:*

Stable: *If V is a stable plane, then any two quasivertical lines in V are upward Hausdorff close but not downward Hausdorff close.*

Unstable: *If V is an unstable plane, then any two quasivertical lines in V are downward Hausdorff close but not upward Hausdorff close.*

Doubly unstable: *If V is a doubly unstable plane then there exist two quasivertical lines in V which are neither upward Hausdorff close nor downward Hausdorff close.*

To prove Claim 5.5 when V is stable or unstable, observe first that V with its horizontal foliation $\{\theta_t\}$ is isometric to the hyperbolic plane $\mathbf{H}^2$ in the upper half plane model, with the "horizontal" horocyclic foliation centered on the point ∞. If V is stable (resp. unstable) then the transverse orientation points towards ∞ (resp. away). Next observe that any quasivertical line in $\mathbf{H}^2$ is a quasigeodesic with one endpoint at ∞, and so any two quasivertical lines are Hausdorff close in the direction of ∞.

To prove Claim 5.5 when V is doubly unstable, observe that $\theta_0 = V \cap D_0$ is an xy-geodesic in D_0 which is not contained in a leaf of either the x-foliation or the y-foliation on D_0. There exists, therefore, two points $p, q \in \theta_0$ which do not lie in the same leaf of either the x-foliation or the y-foliation on D_0. The vertical flow lines $p \cdot \mathbf{R}$, $q \cdot \mathbf{R}$ in Q_γ are evidently neither upward nor downward Hausdorff close in the singular SOLV-metric on Q_γ, and so the same is true in V. This proves Claim 5.5.

The same discussion holds, of course, in $Q_{\gamma'}$. Now consider a horizontal-respecting quasi-isometry $\phi\colon Q_\gamma \to Q_{\gamma'}$. By reversing the upward orientation in $Q_{\gamma'}$ if necessary, we may assume that ϕ respects the upward orientation. Let V be any vertical plane in Q_γ. We have shown in Claim 5.4 that $\phi(V)$ is Hausdorff close to some vertical plane V' in $Q_{\gamma'}$. Composing the map $V \to \phi(V)$ with any finite distance map $\phi(V) \to V'$, we obtain therefore a quasi-isometry $\psi\colon V \to V'$. Each horizontal leaf of V (resp. V') is a coarse intersection of V with a horizontal leaf of Q_γ (resp. $Q_{\gamma'}$), and it follows that ψ coarsely respects the horizontal foliations and their transverse orientations.

Claim 5.5 shows manifestly that stable vertical planes are coarsely respected by $\phi\colon Q_\gamma \to Q_{\gamma'}$, and similarly for unstable vertical planes. To finish the proof of Proposition 5.2, given two stable (resp. unstable) vertical planes V_1, V_2 in Q_γ or in $Q_{\gamma'}$, we must give a quasi-isometrically invariant property which characterizes V_1, V_2 lying in the same weak stable (resp. unstable) leaf. Namely, V_1, V_2 lie in the same leaf if and only if, for any $t \in \mathbf{R}$, the triple coarse intersection $V_1 \underset{c}{\cap} V_2 \underset{c}{\cap} D_t$ is unbounded. $\diamondsuit$

Remark. There is an alternative "dynamical" proof of Proposition 5.2 which we worked out first, and which parallels the proof of the analogous proposition in [FM00a]. The latter is concerned with the solvable Lie groups associated to geodesics in the symmetric space of $\mathrm{GL}(n, \mathbf{R})$, and the natural Anosov flows on these solvable Lie groups; the proof in [FM00a] is an easy consequence of the shadowing lemma for Anosov flows. Unfortunately the shadowing lemma is false in pseudo-Anosov dynamics, complicating the situation drastically. There is an alternative shadowing theory for pseudo-Anosov dynamical systems, developed in [Han85], [Han88], and [Mos89], and this was used in our original proof of Proposition 5.2. The proof given above for Proposition 5.2 entirely avoids these issues by using Gromov hyperbolicity of the horizontal leaves—a fact which was not available in [FM00a], where the horizontal leaves are Euclidean.

6 Periodic hyperplanes

A hyperplane P_w in X_H is a *periodic hyperplane* if w is a periodic geodesic in T_H, that is, the subgroup C_w of H stabilizing w is infinite cyclic. Periodic hyperplanes P_w are special in that they admit *a priori* extra isometries, above and beyond the isometric action of $\pi_1(\Sigma)$: the subgroup $\pi_1(\Sigma) \rtimes C_w$ of Γ_H acts isometrically on P_w.

Our goal now is to show that horizontal-respecting quasi-isometries of periodic hyperplanes must remember the extra symmetries.

Recall that $w \leftrightarrow \gamma_w$ is a bijection between geodesics in the tree T_H and Teichmüller geodesics in $\mathcal{H}\Lambda$.

Proposition 6.1 (Periodic hyperplanes preserved). *Given w, w' geodesics in T_H, suppose that there exists a horizontal respecting quasi-isometry between hyperplanes $\phi \colon P_w \to P_{w'}$. Then:*

(1) w is periodic if and only if w' is periodic.

(2) If w, w' are periodic, and if $\gamma, \gamma' \subset \mathcal{H}\Lambda$ are the geodesics in $\mathcal{T}$ associated to w, w' respectively, then γ, γ' are periodic and there is a singular SOLV-*isometry $\Phi \colon Q_\gamma \to Q_{\gamma'}$ such that the following diagram coarsely commutes:*

$$\begin{array}{ccc} P_w & \xrightarrow{\ \phi\ } & P_{w'} \\ \downarrow{\scriptstyle F_w} & & \downarrow{\scriptstyle F_{w'}} \\ Q_\gamma & \xrightarrow{\ \Phi\ } & Q_{\gamma'} \end{array}$$

That is, $d(F_{w'}(\phi(x)), \Phi(F_w(x))) \leq A$ where A depends only on the quasi-isometry constants of ϕ, not on w or w'.

We start by reducing the proposition to a statement about singular SOLV-manifolds by using the following result, which shows that various competing notions of "periodicity" are in fact equivalent:

Lemma 6.2 (Characterizing periodicity). *Given a T_H geodesic w and the corresponding Teichmüller geodesic $\gamma = \gamma_w$, the following are equivalent:*

(1) P_w is periodic, meaning that the group C_w is infinite.

(2) γ is periodic in $\mathcal{T}$, meaning that the group $\operatorname{Stab}_{\mathcal{T}}(\gamma)$ is infinite.

(3) Q_γ is periodic, meaning that the group $C_\gamma = \operatorname{Isom}(Q_\gamma)/\operatorname{Isom}_h(Q_\gamma)$ is infinite.

Proof. We clearly have inclusions $C_w \subset \operatorname{Stab}_{\mathcal{T}}(\gamma_w) \subset C_\gamma$ and so (1) implies (2) implies (3).

To prove (2) implies (1), suppose that γ is periodic in $\mathcal{T}$, that is, γ is the axis of some pseudo-Anosov element $\phi \in \mathcal{M}$. Choose a point $p \in \gamma$, and consider the sequence $\phi^n(p)$, $n > 0$. Since H acts cocompactly on $\mathcal{H}\Lambda$, there is a sequence $\psi_n \in H$ such that $\{\psi_n \circ \phi^n(p) \bigm| n > 0\}$ is a bounded subset of $\mathcal{T}$. Since $\mathcal{M}$ acts properly on $\mathcal{T}$, there exist $m > n > 0$ such that $\psi_n \circ \phi^n = \psi_m \circ \phi^m$. It follows that $\phi^{m-n} = \psi_m^{-1}\psi_n \in H$, and so w is periodic in T_H.

To prove (3) implies (2), choose a horizontal leaf D_γ of Q_γ and so we have the split exact sequence

$$1 \to \operatorname{Isom}_+(D_\gamma) \approx \operatorname{Isom}_h(Q_\gamma) \to \operatorname{Isom}_+(Q_\gamma) \to C_{\gamma+} \to 1$$

where $C_{\gamma+}$ is an infinite cyclic subgroup of index ≤ 2 in C_γ. We also have a finite index inclusion $\pi_1(\Sigma) < \operatorname{Isom}_+(D_\gamma)$. Since $C_{\gamma+}$ acts by automorphisms of $\operatorname{Isom}_+(D_\gamma)$ it follows that $C_{\gamma+}$ has a finite index subgroup stabilizing $\pi_1(\Sigma)$; this subgroup is clearly identified with a subgroup of $\operatorname{Stab}_{\mathcal{T}}(\gamma_w)$ and so the latter is infinite. ♢

Note from the proof that each of the inclusions $C_w \subset \operatorname{Stab}_{\mathcal{T}}(\gamma_w) \subset C_\gamma$ has finite index; examples may be constructed in which any number of these inclusions is proper.

By combining Lemma 6.2 with Proposition 4.4, it follows immediately that Proposition 6.1 is reduced to the following:

Lemma 6.3. *Given geodesics γ, γ' in $\mathcal{T}$, if γ is periodic, and if there exists a horizontal respecting quasi-isometry $Q_\gamma \to Q_{\gamma'}$, then γ' is periodic and there exists a singular* SOLV *isometry $Q_\gamma \to Q_{\gamma'}$.*

Remark. Our proof of Lemma 6.3 uses both Thurston's hyperbolization theorem for mapping tori of pseudo-Anosov homeomorphisms [Ota96] as well as the geodesic pattern rigidity theorem of R. Schwartz [Sch97], and to apply these results we need to invoke periodicity of γ. Existence of a horizontal respecting quasi-isometry between Q_γ and $Q_{\gamma'}$ ought to imply existence of a singular SOLV-isometry, without assuming periodicity of γ; however, we do not know a proof.

6.1 Rigidity of periodic hyperplanes

The main step in the proof of Lemma 6.3 is Proposition 6.4 below, which shows that a periodic singular SOLV-manifold Q_γ has the rigidity property that its horizontal respecting quasi-isometry group equals its isometry group. Here is the basic setup.

For each periodic geodesic γ in $\mathcal{T}$, fix a base horizontal leaf $D_\gamma \subset Q_\gamma$. Recall that $\pi_1(\Sigma)$ acts as a deck transformation group on D_γ. The singular SOLV-metric on Q_γ induces an xy-structure on D_γ with respect to which we have an inclusion $\pi_1(\Sigma) \hookrightarrow \mathrm{Isom}_+(D_\gamma)$. There is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Isom}_+(D_\gamma) & \longrightarrow & \mathrm{Isom}(Q_\gamma) & \longrightarrow & C_\gamma & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \Gamma_\gamma & \longrightarrow & \mathrm{Stab}_{\mathcal{T}}(\gamma) & \longrightarrow & 1 \end{array}$$

where each vertical arrow is a finite index inclusion. The group C_γ is either $\mathbf{Z}$ or D_∞. The quotient $Q_\gamma / \mathrm{Isom}(Q_\gamma)$ is a closed 3-dimensional orbifold, equipped with an orbifold fibration over the circle (when $C_\gamma = \mathbf{Z}$) or the interval orbifold (when $C_\gamma = D_\infty$); in the latter case we choose D_γ so that it projects to a generic point of the interval orbifold, i.e. so that the stabilizer of D_γ in $\mathrm{Isom}(Q_\gamma)$ equals $\mathrm{Isom}_+(D_\gamma)$. It follows that the generic fiber of the quotient 3-orbifold $Q_\gamma / \mathrm{Isom}(Q_\gamma)$ is the closed 2-orbifold $\mathcal{O}_\gamma = D_\gamma / \mathrm{Isom}_+(D_\gamma)$ equipped with the quotient xy-structure.

Note that the quotient 3-orbifold Q_γ/Γ_γ also fibers over the circle or the interval orbifold, with fiber Σ. There is a finite covering map $\Sigma \to \mathcal{O}_\gamma$ and the fiber monodromy map on $\mathcal{O}_\gamma$ lifts to a pseudo-Anosov homeomorphism $\phi \colon \Sigma \to \Sigma$.

Let $\mathrm{QI}_h(Q_\gamma)$ be the subgroup of $\mathrm{QI}(Q_\gamma)$ represented by horizontal respecting quasi-isometries of Q_γ. Clearly there is a homomorphism

$$i \colon \mathrm{Isom}(Q_\gamma) \to \mathrm{QI}_h(Q_\gamma)$$

Proposition 6.4 (Rigidity of periodic hyperplanes). *If γ is periodic then the homomorphism $i \colon \mathrm{Isom}(Q_\gamma) \to \mathrm{QI}_h(Q_\gamma)$ is an isomorphism.*

Proof. A nontrivial isometry ϕ of Q_γ must map some vertical geodesic to a different vertical geodesic. Since any two vertical geodesics in Q_γ have infinite Hausdorff distance, ϕ is an infinite distance from the identity, so that i is injective (injectivity of i therefore is true regardless of periodicity).

We now prove that i is surjective: every horizontal respecting quasi-isometry of Q_γ is a bounded distance from an isometry.

Thurston's geometrization theorem for pseudo-Anosov mapping tori gives a hyperbolic structure on the 3-dimensional orbifold $Q_\gamma / \operatorname{Isom}(Q_\gamma)$. This yields a properly discontinuous, cocompact, isometric, faithful action

$$h\colon \operatorname{Isom}(Q_\gamma) \to \operatorname{Isom}(\mathbf{H}^3)$$

and a quasi-isometric, h-equivariant homeomorphism

$$q\colon Q_\gamma \to \mathbf{H}^3$$

For each singular vertical geodesic $\ell \subset Q_\gamma$, the image $q(\ell)$ is a quasigeodesic in $\mathbf{H}^3$, which by the Morse-Mostow Lemma is a bounded Hausdorff distance (not depending on ℓ) from a unique geodesic in $\mathbf{H}^3$; let L denote the set of all such geodesics in $\mathbf{H}^3$. Let $\operatorname{Isom}(\mathbf{H}^3, L)$ be the group of isometries of $\mathbf{H}^3$ which permute the collection L. Also, let F be the foliation of $\mathbf{H}^3$ obtained by pushing forward via q the horizontal foliation of Q_γ, and let $\operatorname{Isom}(\mathbf{H}^3, L, F)$ be the subgroup of $\operatorname{Isom}(\mathbf{H}^3, L)$ which coarsely respects F. Note that $\operatorname{image}(h) \subset \operatorname{Isom}(\mathbf{H}^3, L, F)$. Now we show that h factors through i.

For each horizontal respecting quasi-isometry ψ of Q_γ, we obtain a quasi-isometry $q\psi q^{-1}$ of $\mathbf{H}^3$, inducing a homomorphism $\hat{q}\colon \operatorname{QI}_h(Q_\gamma) \to \operatorname{QI}(\mathbf{H}^3)$. Since ψ coarsely respects singular vertical geodesics and horizontal leaves in Q_γ, it follows that $q\psi q^{-1}$ coarsely respects L and F.

Since $q\psi q^{-1}$ coarsely respects L, since L is invariant under the cocompact isometry group $\operatorname{image}(h)$, and since there are only finitely many orbits of the action of $\operatorname{image}(h)$ on L, we may directly apply the main theorem of [Sch97] which says in this setting that $q\psi q^{-1}$ is a bounded distance (not uniformly so) from a unique isometry of $\mathbf{H}^3$ which strictly respects L.

It follows that the image of $\hat{q}$ is contained in $\operatorname{Isom}(\mathbf{H}^3, L)$, and in fact in $\operatorname{Isom}(\mathbf{H}^3, L, F)$. It's evident that the composition

$$\operatorname{Isom}(Q_\gamma) \xrightarrow{i} \operatorname{QI}_h(Q_\gamma) \xrightarrow{\hat{q}} \operatorname{Isom}(\mathbf{H}^3, L, F) \subset \operatorname{Isom}(\mathbf{H}^3)$$

is identical with the homomorphism h. The homomorphism $\hat{q}$ is obviously injective, and so to prove surjectivity of i it suffices to show that $\operatorname{image}(h) = \operatorname{Isom}(\mathbf{H}^3, L, F)$.

Consider the short exact sequence

$$1 \to \operatorname{Isom}_+(D_\gamma) \to \operatorname{Isom}(Q_\gamma) \to C_\gamma \to 1$$

Since $\operatorname{Isom}_+(D_\gamma)$ is normal in $\operatorname{Isom}(Q_\gamma)$, and since $\operatorname{Isom}(Q_\gamma)$ is identified via h with a finite index subgroup of $\operatorname{Isom}(\mathbf{H}^3, L, F)$ (both being discrete and cocompact on $\mathbf{H}^3$), it follows that $\operatorname{Isom}_+(D_\gamma)$ has a normalizer of finite index in $\operatorname{Isom}(\mathbf{H}^3, L, F)$; choose coset representatives $g_1, \ldots, g_n$ of the normalizer. Each leaf of F is coarsely equivalent to $\operatorname{Isom}_+(D_\gamma)$, and $g_1, \ldots, g_n$ coarsely respects F, and so each conjugate subgroup

$$g_1 \operatorname{Isom}_+(D_\gamma) g_1^{-1}, \ldots, g_n \operatorname{Isom}_+(D_\gamma) g_n^{-1}$$

is coarsely equivalent to $\operatorname{Isom}_+(D_\gamma)$.

Now we apply an elementary lemma of [MSW00] which says that for a finite collection of subgroups in a finitely generated group, the coarse intersection of those subgroups is coarsely equivalent to their intersection. The intersection of the above conjugates of $\operatorname{Isom}_+(D_\gamma)$ is therefore coarsely equivalent to D_γ.

Another elementary lemma of [MSW00] says that in a finitely generated group, given subgroups $A \subset B$, if A, B are coarsely equivalent, then A has finite index in B. It follows that the intersection of the conjugates of $\operatorname{Isom}_+(D_\gamma)$ in $\operatorname{Isom}(\mathbf{H}^3, L, F)$ has finite index in $\operatorname{Isom}_+(D_\gamma)$. Thus we obtain a normal subgroup $N \subset \operatorname{Isom}(\mathbf{H}^3, L, F)$ of finite index in $\operatorname{Isom}_+(D_\gamma)$. The quotient group $\operatorname{Isom}(\mathbf{H}^3, L, F)/N$ is a finite index supergroup of $C_\gamma = \operatorname{Isom}(Q_\gamma)/\operatorname{Isom}_+(D_\gamma)$, and so the quotient is virtually cyclic. Since $\operatorname{Isom}(\mathbf{H}^3, L, F)$ is a 3-orbifold group it follows that the quotient $\operatorname{Isom}(\mathbf{H}^3, L, F)/N$ is either $\mathbf{Z}$ or D_∞.

The orbifold $\mathbf{H}^3/\operatorname{Isom}(\mathbf{H}^3, L, F)$ therefore fibers over the circle or the interval orbifold, with generic fiber $\mathcal{O}$, and with $\pi_1(\mathcal{O})$ identified with N. Lifting this fibration to $\mathbf{H}^3$ we obtain an $\operatorname{Isom}(\mathbf{H}^3, L, F)$ equivariant fibration coarsely equivalent to F. We may therefore replace F with this fibration, and so F is strictly invariant under $\operatorname{Isom}(\mathbf{H}^3, L, F)$. The monodromy map on $\mathcal{O}$ is pseudo-Anosov. There is a finite index covering map $Q_\gamma/\operatorname{Isom}(Q_\gamma) \to \mathbf{H}^3/\operatorname{Isom}(\mathbf{H}^3, L, F)$, taking fibration to fibration. By uniqueness of pseudo-Anosov homeomorphisms in their isotopy classes [FLP+79], it follows that the stable and unstable measured foliations for the monodromy map of $\mathcal{O}$ lift to the stable and unstable measured foliations for the monodromy map on the generic fiber $\mathcal{O}_\gamma$ of $Q_\gamma/\operatorname{Isom}(Q_\gamma)$. But this shows that $\operatorname{Isom}(\mathbf{H}^3, L, F)$ acts isometrically on Q_γ, proving that $\operatorname{image}(h) = \operatorname{Isom}(\mathbf{H}^3, L, F)$.

This proves Proposition 6.4. ♢

6.2 Proof of Lemma 6.3

Assume Q_γ is periodic and $\phi\colon Q_\gamma \to Q_{\gamma'}$ is a horizontal respecting quasi-isometry. We'll construct an isometry $\Phi\colon Q_\gamma \to Q_{\gamma'}$, the existence of which implies that $Q_{\gamma'}$

is periodic.

Applying Proposition 5.2, we may move ϕ a bounded distance so that ϕ is a homeomorphism, respecting the horizontal foliations and the weak stable and unstable foliations. We may assume that the base horizontal leaves $D_\gamma \subset Q_\gamma$, $D_{\gamma'} \subset Q_{\gamma'}$ are chosen so that $\phi(D_\gamma) = D_{\gamma'}$.

Let $A_\gamma \colon \pi_1(\Sigma) \to \text{Isom}(Q_\gamma)$ and $A_{\gamma'} \colon \pi_1(\Sigma) \to \text{Isom}(Q_{\gamma'})$ be the standard isometric actions, preserving each horizontal leaf. Let $\widehat{\text{QI}}_h(Q_\gamma)$ be the subsemigroup of $\widehat{\text{QI}}(Q_\gamma)$ that coarsely respects the horizontal foliation of Q_γ, and let

$$B_\gamma = \phi^{-1} \circ A_{\gamma'} \circ \phi \colon \pi_1(\Sigma) \to \widehat{\text{QI}}_h(Q_\gamma)$$

be the conjugated action (its really an action, not just a quasi-action, because ϕ is a homeomorphism). Note that B_γ preserves each horizontal leaf of Q_γ as well as the strong stable and unstable foliations in that leaf; however, B_γ does not a priori preserve the invariant measures on those foliations.

Applying Proposition 6.4, we conclude that B_γ is a bounded distance from an isometric action, that is, there exists a homeomorphism $\xi \colon Q_\gamma \to Q_\gamma$ which moves each point a uniformly bounded distance, such that $\xi^{-1} \circ B_\gamma \circ \xi$ is an isometric action of $\pi_1(\Sigma)$ on Q_γ. The action $\xi^{-1} \circ B_\gamma \circ \xi$ also preserves each horizontal leaf of Q_γ and the strong stable and unstable foliations in that leaf.

We claim that $B_\gamma = \xi^{-1} \circ B_\gamma \circ \xi$. Consider a horizontal leaf L of Q_γ and a point $x \in L$. Let ℓ^s, ℓ^u be the strong stable and unstable leaves in L passing through x, and so $x = \ell^s \cap \ell^u$. Consider an element $\beta \in B_\gamma(\pi_1(\Sigma))$. We know that $\beta(L) = \xi^{-1}\beta\xi(L) = L$. We also know that $\beta(\ell^s)$ and $\xi^{-1}\beta\xi(\ell^s)$ are both strong stable leaves in L, and they are a bounded distance from each other, implying that $\beta(\ell^s) = \xi^{-1}\beta\xi(\ell^s)$. Similarly, $\beta(\ell^u) = \xi^{-1}\beta\xi(\ell^u)$. Therefore, $\beta(x) = \xi^{-1}\beta\xi(x)$, proving the claim.

It follows from the claim that B_γ does, in fact, preserve the invariant measures on the strong stable and unstable foliations in each horizontal leaf of Q_γ.

Fix a horizontal leaf $L \subset Q_\gamma$ and let $L' = \phi(L) \subset Q_{\gamma'}$. So ϕ takes the strong stable and unstable foliations in L to those in L'. Let f^s, f^u be the strong stable and unstable measured foliations in L, and let f'^s, f'^u the strong stable and unstable measured foliations in L'. The map ϕ pushes the transverse measures on f^s, f^u forward to new transverse measures on f'^s, f'^u.

Since $B_\gamma = \phi^{-1} A_{\gamma'} \phi$ acts isometrically on the transverse measures of f^s, f^u, it follows that $A_{\gamma'}$ acts isometrically on both the old and the new transverse measures on f'^s, f'^u. However, f'^s, f'^u are *uniquely ergodic* with respect to the $A_{\gamma'}$ action, i.e. they have projectively unique transverse measures invariant under the $A_{\gamma'}$ action. This follows from the fact that the Teichmüller geodesic γ' is cobounded, together with a theorem of H. Masur that if $\xi \in \mathbf{P}\mathcal{M}F$ is not uniquely ergodic then any Teichmüller ray with ending foliation ξ is not cobounded [Mas80]. Hence the old and new transverse measures on f'^s, f'^u differ by multiplicative constants. These multiplicative constants are inverses to each other, because the action $A_{\gamma'}$ has cofinite area. It follows that ϕ restricts to an xy-affine isomorphism from D_γ to $D_{\gamma'}$. This implies in turn that the quasi-isometry $\phi\colon Q_\gamma \to Q_{\gamma'}$ may be altered a bounded amount, moving each point a fixed amount up or down in its vertical geodesic, to obtain an isometry $\Phi\colon Q_\gamma \to Q_{\gamma'}$.

It follows that $D_{\gamma'}$ has an extra, nonisometric, affine symmetry, and so γ' is periodic.

This completes the proof of Lemma 6.3 and therefore also of Proposition 6.1.

7 The endgame: computing $\mathrm{QI}(\Gamma_H)$ and $\mathrm{Comm}(\Gamma_H)$

7.1 An injection $\mathrm{QI}(X_H) \to \mathrm{QSym}(S^1)$

Let $\mathrm{QSym}(S^1)$ denote the group of quasisymmetric homeomorphisms of the circle S^1. It is well-known that the extension of a quasi-isometry of $\mathbf{H}^2$ to S^1 induces an isomorphism $\mathrm{QI}(\mathbf{H}^2) \approx \mathrm{QSym}(S^1)$: boundary extension of quasi-isometries defines an injection from the quasi-isometry group of any word hyperbolic group to the homeomorphism group of its boundary; and quasi-symmetric homeomorphisms of S^1 are exactly the extensions of quasi-isometries of $\mathbf{H}^2$. The key to all of this is the theorem of Ahlfors and Beurling [AB56] that quasisymmetric homeomorphisms of S^1 are exactly the extensions of quasiconformal homeomorphisms of the unit disc.

Fix once and for all a base fiber $D_0 = \pi^{-1}(\tau_0)$ of X_H, $\tau_0 \in T_H$. Identify D_0 with $\mathbf{H}^2$, so ∂D_0 is identified with S^1. Then there is a map $\Psi\colon \mathrm{QI}(X_H) \to \mathrm{QSym}(S^1)$, defined as follows. Given a quasi-isometry $f\colon X_H \to X_H$, the image $f(D_0)$ is Hausdorff close to some fiber D'. Consider the composition

$$D_0 \to f(D_0) \to D' \to D_0$$

where the first map is f and the other maps are closest point projections. Then the composition $D_0 \to D_0$ is clearly a quasi-isometry, and so induces an element $\mathrm{QSym}(S^1)$. Hence we have a map $\Psi\colon \mathrm{QI}(X_H) \to \mathrm{QSym}(S^1)$, which is easily seen to be well-defined, and in fact a homomorphism.

Proposition 7.1. *The homomorphism $\Psi\colon \mathrm{QI}(X_H) \to \mathrm{QSym}(S^1)$ is injective.*

Proof. By Propositions 4.1 and 4.2, and the fact that the projection $X_H \to T_H$ induces an isometry between the space of horizontal leaves of X_H with the Hausdorff metric and the quotient tree T_H, it follows that any quasi-isometry $f\colon X_H \to X_H$ induces a quasi-isometry of T_H and so f induces a homeomorphism $\Xi(f)\colon \Lambda \to \Lambda$ of the Cantor set $\Lambda = \partial T_H \subset \mathbf{P}\mathcal{MF}$.

Suppose that $\Psi(f) \in \mathrm{QSym}(S^1)$ is the identity map on S^1.

We claim that the induced map $\Xi(f) \colon \Lambda \to \Lambda$ is the identity. It follows that the induced quasi-isometry $T_H \to T_H$ is a bounded distance from the identity, and so f takes each horizontal leaf of X_H a bounded distance from itself; and since the induced boundary map of that leaf is the identity, it follows that f takes each point a bounded distance from itself, proving the proposition.

For proving the claim (and for later purposes) we review the well-known embedding of $\mathcal{MF}$ into the space of $\pi_1(\Sigma)$-invariant measures on the "Möbius band beyond infinity" of the hyperbolic plane. That is, consider the *double set* of the circle, $DS^1 = \{\{x,y\} \subset S^1 \bigm| x \ne y\}$; with respect to the Klein model $D_0 = \mathbf{H}^2 \subset \mathbf{RP}^2$, the usual duality gives a bijection between DS^1 and the Möbius band beyond infinity $\mathbf{RP}^2 - \overline{\mathbf{H}}^2$.

Let $M(DS^1)$ be the space of Borel measures on DS^1 with the weak* topology, and let $\mathbf{P}M(DS^1)$ be the space of projective classes of elements of $M(DS^1)$. The space $\mathcal{MF}$ embeds into the space of $\pi_1(\Sigma)$-invariant elements of $M(DS^1)$, by lifting a measured foliation on Σ to a $\pi_1(\Sigma)$-invariant measured foliation on D_0, and then identifying each leaf of the lifted foliation with the correspond pair of endpoints in DS^1. The space $\mathbf{P}\mathcal{MF}$ therefore embeds as $\pi_1(\Sigma)$-invariant elements of $\mathbf{P}M(DS^1)$. Given $b \in \mathbf{P}\mathcal{MF}$ let $\mu_b \in \mathbf{P}M(DS^1)$ be the corresponding projective class of measures, let $\mathrm{supp}(\mu_b) \subset DS^1$ be the support, and let $E_b = |\mathrm{supp}(\mu_b)|$ be the union of all the pairs in $\mathrm{supp}(\mu_b)$. Note that E_b may also be described as the *endpoint set* of b, the set of endpoints in S^1 of the leaves of the measured foliation $\widetilde{\mathcal{F}}_b$ on D_0 obtained by lifting any measured foliation $\mathcal{F}_b$ on Σ that represents b.

By Proposition 5.2, for all $b \in \Lambda$ and all quasi-isometries ϕ of X_H we have

$$\Psi(\phi)(E_b) = E_{\Xi(\phi)(b)}$$

From our assumption that $\Psi(f)$ is the identity on S^1 it follows that $\Psi(f)(E_b) = E_b$, for all $b \in \Lambda$. Note however that if $b \ne b' \in \Lambda$ then $E_b \cap E_{b'} = \emptyset$, because the projective classes of $\mathcal{F}_b, \mathcal{F}_{b'}$ in $\mathbf{P}\mathcal{MF}$ are connected by a Teichmüller geodesic in $\mathcal{H}\Lambda$, and so the measured foliations $\mathcal{F}_b, \mathcal{F}_{b'}$ can be chosen to be transverse in Σ; the lifted foliations $\widetilde{\mathcal{F}}_b, \widetilde{\mathcal{F}}_{b'}$ are therefore transverse in D_0 and so their endpoint sets $E_b, E_{b'}$ in S^1 are disjoint. It follows that $\Xi(f)(b) = b$ for all $b \in \Lambda$. ◇

7.2 The orbifold $\mathcal{O}_H$ associated to a Schottky group H

Our goal in this subsection and the next is to compute the quasi-isometry group $\mathrm{QI}(\Gamma_H)$. Using the injection $\Psi \colon \mathrm{QI}(\Gamma_H) \to \mathrm{QSym}(S^1)$ provided by Proposition 7.1, our computation will consist of an explicit description of the subgroup $\Psi(\mathrm{QI}(\Gamma_H)) < \mathrm{QSym}(S^1)$; without further mention we shall identify $\mathrm{QI}(\Gamma_H)$ with this subgroup.

The first step in the computation of $\mathrm{QI}(\Gamma_H)$ is to find a natural orbifold subcover $\Sigma \to \mathcal{O}_H$ associated to a Schottky subgroup $H < \mathcal{M}(\Sigma)$. The orbifold $\mathcal{O}_H$ is an important invariant of H; it is the smallest subcover of Σ to which H descends as a subgroup of $\mathcal{M}(\mathcal{O}_H)$.

Recall we have fixed a base fiber D_0 of X_H over a base point $\tau_0 \in T_H$, and we identify D_0 with the universal cover $\widetilde{\Sigma}$.

To each periodic hyperplane P_w we associate a finite-index supergroup of $\pi_1(\Sigma)$ in $\mathrm{QSym}(S^1)$, as follows. In the associated singular SOLV-manifold Q_{γ_w} pick any horizontal leaf D_w, which has an induced xy structure with xy-affine automorphism group denoted $\mathrm{Aff}(D_w)$. Note that $\mathrm{Aff}(D_w)$ acts by quasi-isometries of the xy-metric on D_w. There is a canonical quasi-isometry from D_w to D_0: restrict the canonical horizontal respecting quasi-isometry $Q_{\gamma_w} \to P_w$ to the horizontal leaf D_w, giving a map $D_w \to X_H$, and then take a closest point map to D_0. In this way we obtain an inclusion

$$\mathrm{Aff}(D_w) \subset \mathrm{QI}(D_0) = \mathrm{QSym}(S^1)$$

Note that the image of this inclusion is independent of the choice of horizontal leaf D_w in Q_{γ_w}, because if we chose another horizontal leaf D'_w then the vertical flow on Q_{γ_w} induces an xy-affine homeomorphism $D'_w \to D_w$ in the correct quasi-isometry class.

Recall that we have a short exact sequence

$$1 \to \mathrm{Isom}_+(D_w) \to \mathrm{Aff}(D_w) \to C_{\gamma_w} \to 1$$

with $C_{\gamma_w} \equiv \mathbf{Z}$ or D_∞. Under the injection $\mathrm{Aff}(D_w) \hookrightarrow \mathrm{QSym}(S^1)$, the group $\mathrm{Isom}_+(D_w)$ is a finite index supergroup of $\pi_1(\Sigma)$. We have a quotient orbifold $\mathcal{O}_w = D_w / \mathrm{Isom}_+(D_w)$ with fundamental group $\pi_1(\mathcal{O}_w) \approx \mathrm{Isom}_+(D_w)$, and associated to the inclusion $\pi_1(\mathcal{O}_w) \subset \pi_1(\Sigma)$ there is a finite orbifold covering map $\Sigma \to \mathcal{O}_w$.

In the group $\mathrm{QSym}(S^1)$, take the infinite intersection of the groups $\mathrm{Isom}_+(D_w)$ over all periodic lines w in the tree T_H, and note that this group must be a finite index supergroup of $\pi_1(\Sigma)$, in fact it is the fundamental group $\pi_1(\mathcal{O}_H)$ of an orbifold $\mathcal{O}_H$ which Σ finitely covers, the smallest orbifold covered by Σ which in turn covers each orbifold $\mathcal{O}_w$.

$$\pi_1(\mathcal{O}_H) = \bigcap_{\substack{w \subset T_H \\ w \text{ periodic}}} \mathrm{Isom}_+(D_w)$$

Note that $\mathcal{O}_H$ is the smallest subcover of Σ such that the Schottky group $H \subset \mathcal{M}(\Sigma)$ descends via the covering map $\Sigma \to \mathcal{O}_H$ to a free subgroup of $\mathcal{M}(\mathcal{O}_H)$. To be precise, since H is free we may choose a section $\sigma\colon H \to \mathcal{M}(\Sigma, p) \subset \mathrm{QSym}(S^1)$, and the image group σH acting by conjugation on subgroups of $\mathrm{QSym}(S^1)$ permutes the collection of subgroups $\mathrm{Isom}_+(D_w)$ and so $\sigma H \subset \mathrm{Aut}(\pi_1(\mathcal{O}_H)) = \mathcal{M}(\mathcal{O}_H, p)$. Projecting to $\mathcal{M}(\mathcal{O}_H)$ we obtain the free subgroup H', and a section $\sigma'\colon H' \to \mathcal{M}(\mathcal{O}_H, p) < \mathrm{QSym}(S^1)$ so that $\sigma H = \sigma' H'$.

Remark. Orbifold mapping class groups obey many of the properties of surface mapping class groups. The group $\mathcal{M}(\mathcal{O}_H)$ is defined as the group of orbifold homeomorphisms of $\mathcal{O}_H$ modulo those which are isotopic to the identity through orbifold homeomorphisms. Just as with mapping class groups of surfaces, choosing a generic point $p \in \mathcal{O}_H$ we obtain an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathcal{O}_H) & \longrightarrow & \mathcal{M}(\mathcal{O}_H, p) & \xrightarrow{\pi} & \mathcal{M}(\mathcal{O}_H) & \longrightarrow & 1 \\ & & \| & & \wr & & \wr & & \\ 1 & \longrightarrow & \pi_1(\mathcal{O}_H) & \longrightarrow & \mathrm{Aut}(\pi_1(\mathcal{O}_H)) & \xrightarrow{\pi} & \mathrm{Out}(\pi_1(\mathcal{O}_H)) & \longrightarrow & 1 \end{array}$$

The proof that this isomorphism exists follows the same lines as the proof for surfaces, using the fact that $\pi_1(\mathcal{O}_H)$ is centerless.

By choosing a lift of p to the universal cover of $D_0 = \widetilde{\mathcal{O}}_H$ and lifting each element of $\mathcal{M}(\mathcal{O}_H, p)$, we identify $\mathcal{M}(\mathcal{O}_H, p)$ with a subgroup of $\mathrm{QI}(D_0) = \mathrm{QSym}(S^1)$.

In fact $H' < \mathcal{M}(\mathcal{O}_H)$ is a Schottky subgroup. To see why, consider the limit set $\Lambda(H) \subset \mathbf{P}\mathcal{M}F(\Sigma)$. Moving into S^1, the corresponding subset $\{\mu_b \bigm| b \in \Lambda(H)\}$ of $\mathbf{P}M(DS^1)$ is invariant under σH. Notice that for each $b \in \Lambda(H)$, the element $\mu_b \in \mathbf{P}M(S^1)$, which is invariant under $\pi_1(\Sigma)$, is also invariant under the larger group $\pi_1(\mathcal{O}_H)$. When b is an endpoint of a periodic geodesic w in T_H this is obvious, because μ_b is invariant under $\pi_1(\mathcal{O}_w) \supset \pi_1(\mathcal{O}_H)$. But periodic endpoints are dense in $\Lambda(H)$, and so each μ_b, $b \in \Lambda(H)$ is invariant under $\pi_1(\mathcal{O}_H)$. We therefore obtain a continuous embedding of $\Lambda(H)$ in $\mathbf{P}\mathcal{M}F(\mathcal{O}_H)$, whose image we denote $\Lambda(H')$; this embedding has the property that

$$\{\mu_b \bigm| b \in \Lambda(H)\} = \{\mu_{b'} \bigm| b' \in \Lambda(H')\} \quad \text{in} \quad \mathbf{P}M(DS^1)$$

Moreover, since $\sigma' H' = \sigma H$, it follows that $\sigma' H'$ permutes the elements of the above set, and so $\Lambda(H')$ is in fact invariant under the action of H'. For any pair $\xi \ne \eta \in \Lambda(H)$ and corresponding pair $\xi', \eta' \in \Lambda(H')$, each xy-structure on Σ corresponding to a point along the Teichmüller geodesic $\overleftrightarrow{(\xi, \eta)} \subset \mathcal{T}(\Sigma)$ lifts to an xy-structures on D_0 which is invariant under $\pi_1(\mathcal{O}_H)$, and we therefore obtain a Teichmüller geodesic $\overleftrightarrow{(\xi', \eta')}$ in $\mathcal{T}(\mathcal{O}_H)$; the $\pi_1(\mathcal{O}_H)$-invariance follows by an argument similar to the one just above where we showed invariance for elements μ_b, $b \in \Lambda(H)$. The union of these geodesics over all pairs $\xi' \neq \eta' \in \Lambda(H')$ is denoted $\mathcal{H}\Lambda(H')$ as usual. From this construction we see that the action of H' on $\Lambda(H') \cup \mathcal{H}\Lambda(H')$ agrees with the action of H on $\Lambda(H) \cup \mathcal{H}\Lambda(H)$. In particular, all the requirements in Theorem 3.1 for H' to be convex cocompact with limit set $\Lambda(H')$ are satisfied, and so H' is a Schottky subgroup of $\mathcal{M}(\mathcal{O}_H)$.

Remark. The fact that the orbifold $\mathcal{O}_H$ supports a pseudo-Anosov mapping class puts some restrictions on its topology: the underlying surface of $\mathcal{O}_H$ must have empty boundary, because otherwise the collection of peripheral curves would be invariant under any mapping class, violating the existence of pseudo-Anosov mapping classes. It follows that $\mathcal{O}_H$ is a closed surface with cone singularities.

7.3 Computing the quasi-isometry group

In this section we compute $\mathrm{QI}(\Gamma_H)$. Let $\mathcal{C}$ denote the relative commensurator of H' in $\mathcal{M}(\mathcal{O}_H)$:

$$\mathcal{C} = \mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H')$$

Form the extension group $\Gamma_{\mathcal{C}}$, a subgroup of $\mathcal{M}(\mathcal{O}_H, p)$ as the following diagram shows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathcal{O}_H) & \longrightarrow & \Gamma_{\mathcal{C}} & \longrightarrow & \mathcal{C} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(\mathcal{O}_H) & \longrightarrow & \mathcal{M}(\mathcal{O}_H, p) & \xrightarrow{\pi} & \mathcal{M}(\mathcal{O}_H) & \longrightarrow & 1
\end{array}$$

The group $\Gamma_{\mathcal{C}}$ may therefore be regarded as a subgroup of $\mathrm{QSym}(S^1)$.

Here is our computation of $\mathrm{QI}(\Gamma_H)$:

Theorem 7.2. *In* $\mathrm{QSym}(S^1)$ *we have*

$$\mathrm{QI}(\Gamma_H) = \Gamma_{\mathcal{C}}.$$

Moreover, the subgroup $\pi_1(\mathcal{O}_H)$ *consists of those classes of quasi-isometries of* X_H *which coarsely preserve each horizontal leaf of* X_H*.*

Proof. To justify this computation, first we show $\mathrm{QI}(\Gamma_H) \subset \Gamma_{\mathcal{C}}$. Consider a quasi-isometry $f\colon X_H \to X_H$, regarded as an element of $\mathrm{QSym}(S^1)$; we must show that $f \in \Gamma_{\mathcal{C}}$.

By Lemma 4.3 the quasi-isometry f coarsely respects the horizontal foliation of X_H and so f lies over a quasi-isometry of T_H, also denoted f. From Proposition 6.1 it follows that f induces a permutation on the set of periodic hyperplanes of

X: given a periodic geodesic w in T_H, if we let w' be the geodesic in T_H coarsely equivalent to $f(w)$, then f takes P_w to $P_{w'}$ and Q_{γ_w} to $Q_{\gamma_{w'}}$. It follows that the conjugation action of f on subgroups of $\mathrm{QSym}(S^1)$ takes $\mathrm{Isom}(Q_{\gamma_w}) = \mathrm{Aff}(D_w)$ to $\mathrm{Isom}(Q_{\gamma_{w'}}) = \mathrm{Aff}(D_{w'})$. Any isometry of Q_{γ_w} which preserves each leaf of the horizontal foliation is conjugated by f to a similar isometry of $Q_{\gamma_{w'}}$, and so conjugation by f takes $\mathrm{Isom}_h(Q_{\gamma_w}) = \mathrm{Isom}_+(D_w)$ to $\mathrm{Isom}_h(Q_{\gamma_{w'}}) = \mathrm{Isom}_+(D_{w'})$. In other words, conjugation by f preserves the collection of subgroups

$$\{\mathrm{Isom}_+(D_w) \bigm| w \subset T_H \text{ is periodic}\}$$

Intersecting this collection it follows that conjugation by f in the group $\mathrm{QSym}(S^1)$ preserves the subgroup $\pi_1(\mathcal{O}_H)$, acting as an automorphism of that subgroup. In other words, in $\mathrm{QSym}(S^1)$ we have $f \in \mathrm{Aut}(\pi_1(\mathcal{O}_H)) = \mathcal{M}(\mathcal{O}_H, p)$.

Now we show that the image $\pi f \in \mathcal{M}(\mathcal{O}_H)$ lies in $\mathcal{C} = \mathrm{Comm}_{\mathcal{M}(\mathcal{O}_H)}(H')$. We need the following fact:

Theorem 7.3 (Commensurators of Schottky groups). *Let $\mathcal{O}$ be a closed orbifold and H' a Schottky subgroup of $\mathcal{M}(\mathcal{O})$. Then the relative commensurator $\mathrm{Comm}_{\mathcal{M}(\mathcal{O})}(H')$ of H' in $\mathcal{M}(\mathcal{O})$ is equal to the subgroup of $\mathcal{M}(\mathcal{O})$ stabilizing the limit set $\Lambda(H')$, and H' has finite index in $\mathrm{Comm}_{\mathcal{M}(\mathcal{O})}(H')$.*

Proof. Let I be the subgroup of $\mathcal{M}(\mathcal{O})$ stabilizing $\Lambda(H')$, and so $H' < I$. Let $\mathcal{H}\Lambda(H')$ be the weak convex hull of $\Lambda(H')$ (see Theorem 3.1). Clearly I is also the subgroup of $\mathcal{M}(\mathcal{O})$ stabilizing $\mathcal{H}\Lambda(H')$. By Theorem 3.1 the group H' acts cocompactly on $\mathcal{H}\Lambda(H')$, and so I acts cocompactly on $\mathcal{H}\Lambda(H')$. Clearly I acts properly on $\mathcal{H}\Lambda(H')$. It follows that H' has finite index in I, which immediately implies $I < \mathrm{Comm}_{\mathcal{M}(\mathcal{O})}(H')$.

For the opposite inclusion, suppose $\Phi \in \mathcal{M}(\mathcal{O}) - I$, and so $\Phi(\Lambda(H')) \neq \Lambda(H')$. The limit set of the Schottky subgroup $\Phi H' \Phi^{-1}$ is $\Phi(\Lambda(H'))$. Since $\Lambda(H')$ is closed, $\Phi(\Lambda(H')) - \Lambda(H')$ is open in $\Phi(\Lambda(H'))$. Since fixed points of pseudo-Anosov elements of $\Phi H' \Phi^{-1}$ are dense in $\Phi(\Lambda(H'))$, there exists a pseudo-Anosov element $\Psi \in \Phi H' \Phi^{-1}$ having a fixed point not in $\Lambda(H')$. Infinitely many powers of Ψ are therefore not in H', and so $H' \cap \Phi H' \Phi^{-1}$ has infinite index in $\Phi H' \Phi^{-1}$. $\diamondsuit$

We showed above that H' is a Schottky subgroup of $\mathcal{M}(\mathcal{O}_H)$ and so by Theorem 7.3 it remains to show that $\pi f \in \mathcal{M}(\mathcal{O}_H)$ acting on $\mathbf{P}\mathcal{M}F(\mathcal{O}_H)$ leaves the set $\Lambda(H')$ invariant. For this purpose it suffices to show that the action of $f \in \mathrm{QSym}(S^1)$ leaves the set $\{\mu_{b'} \bigm| b' \in \Lambda(H')\}$ invariant. However, we know that this set equals $\{\mu_b \bigm| b \in \Lambda(H)\}$, and we also know that f leaves this set invariant: f permutes the elements of this set which are endpoints of periodic geodesics in T_H, by Proposition 6.1, but the endpoints of periodic geodesics are dense.

This completes the proof of the inclusion $\mathrm{QI}(\Gamma_H) \subset \Gamma_{\mathcal{C}}$.

For the opposite inclusion, consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\Sigma) & \longrightarrow & \Gamma_H & \longrightarrow & H & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \pi_1(\mathcal{O}_H) & \longrightarrow & \Gamma_{\mathcal{C}} & \longrightarrow & \mathcal{C} & \longrightarrow & 1 \end{array} \tag{7.1}$$

The left vertical arrow is a finite index injection because $\Sigma \to \mathcal{O}_H$ is a finite covering. The right vertical arrow is a finite index injection by Theorem 7.3 using the isomorphism $H \approx H'$. It follows that the middle vertical arrow is a finite index injection. But this shows that $\Gamma_{\mathcal{C}}$ is a finite index supergroup of Γ_H in $\mathrm{QSym}(S^1)$, and so $\Gamma_{\mathcal{C}} \subset \mathrm{QI}(\Gamma_H)$.

This completes the computation of $\mathrm{QI}(\Gamma_H)$.

To complete the proof of Theorem 7.2, it remains to prove that an element of $\mathrm{QI}(\Gamma_H) = \Gamma_{\mathcal{C}}$ is in the normal subgroup $\pi_1(\mathcal{O}_H)$ if and only if it coarsely preserves each horizontal leaf of X_H. Evidently $\pi_1(\mathcal{O}_H)$ coarsely preserves each horizontal leaf. Suppose conversely that the quasi-isometry $f \colon X_H \to X_H$ coarsely preserves each horizontal leaf; let $[f] \in \mathrm{QI}(\Gamma_H)$ be the coarse equivalence class of f. It follows that f coarsely respects each hyperplane of X_H, in particular each periodic hyperplane P_w, and so we have $[f] \in \mathrm{Aff}(D_w)$ for each periodic line w in T_H. But we can say more, as follows. The inclusion map $\Gamma_H \to \mathrm{QI}(\Gamma_H)$, being injective and with finite index image, is a quasi-isometry, and we therefore obtain a quasi-isometry $\mathrm{QI}(\Gamma_H) \to X_H$. Using this quasi-isometry we may conjugate the left action of $\mathrm{QI}(\Gamma_H)$ on itself to obtain a quasi-action of $\mathrm{QI}(\Gamma_H)$ on X_H. In particular, we obtain a sequence of uniform quasi-isometries $f_n \colon X_H \to X_H$, with $[f_n] = [f]^n$. Each of the f_n coarsely preserves each horizontal leaf of X_H, and note that the coarseness constant is uniform *independent of n and of the leaf*, by application of Proposition 4.1 using uniformity of the quasi-isometry constants of f_n. It follows that each f_n coarsely preserves each periodic hyperplane P_w, and f_n coarsely preserves each horizontal leaf of P_w, again with uniform coarseness constants independent of n. This implies that $[f] \in \mathrm{Isom}_+(D_w)$. We therefore have

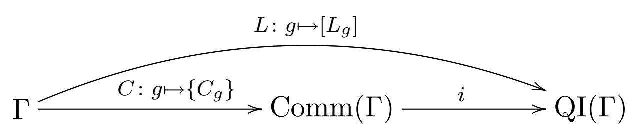
$$[f] \in \bigcap_w \mathrm{Isom}_+(D_w) = \pi_1(\mathcal{O}_H)$$

$\diamondsuit$

7.4 Computing the commensurator group

In this section we use our knowledge of quasi-isometries of Γ_H to compute the commensurator of Γ_H and complete the proof of Theorem 1.3. This is the first instance we know of where this technique of computing a commensurator group is used. We prove, in Theorem 7.4, that when the left multiplication homomorphism $\Gamma \to \mathrm{QI}(\Gamma)$ is an injection with finite index image, the natural homomorphism $i\colon \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$ is an isomorphism. Combining with Theorem 7.2 we obtain the desired computation of $\mathrm{Comm}(\Gamma_H)$. We do not have a proof of the computation of $\mathrm{Comm}(\Gamma_H)$ without going through all of the work required to understand quasi-isometries; a purely algebraic computation of $\mathrm{Comm}(\Gamma_H)$ would be interesting.

For any finitely-generated group Γ, a commensuration $\phi\colon G_1 \to G_2$ extends to a quasi-isometry of Γ by precomposing ϕ with a closest point projection $\Gamma \to G_1$ and postcomposing with inclusion $G_2 \to \Gamma$. The coarse equivalence class of this extension is well-defined, giving a natural homomorphism $i\colon \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$. Note that we have a commutative triangle



where L_g is the left multiplication map $x \to gx$ with coarse equivalence class $[L_g] \in \mathrm{QI}(\Gamma)$, and C_g is the conjugation automorphism $x \to gxg^{-1}$ with equivalence class $\{C_g\} \in \mathrm{Comm}(\Gamma)$; commutativity follows because right multiplication by g^{-1} moves each point in Γ a bounded amount.

Theorem 7.4. *Given a finitely generated group Γ, if $L\colon \Gamma \to \mathrm{QI}(\Gamma)$ is an injection with finite index image, then $i\colon \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$ is an isomorphism.*

This theorem, proved in collaboration Kevin Whyte, is broken into three steps:

(1) injectivity of $i\colon \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$;

(2) construction of an injective map $\Psi\colon \mathrm{QI}(\Gamma) \to \mathrm{Comm}(\Gamma)$;

(3) the proof that $\Psi \circ i$ is the identity on $\mathrm{Comm}(\Gamma)$.

Step 1: Injectivity of i.

Proposition 7.5 (K. Whyte). *If Γ is any finitely generated group then the natural map $i\colon \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$ is injective.*

Proof. Suppose $\phi\colon G_1 \to G_2$ is a commensuration of Γ such that $i(\phi)$ equals the identity in $\mathrm{QI}(\Gamma)$. Then there is a bounded function $\delta\colon \Gamma \to \Gamma$ so that for all $g \in G_1$ we have

$$\phi(g) = g\,\delta(g)$$

Since δ is bounded, the cardinality $M = \#\,\mathrm{image}(\delta)$ is finite.

Plugging the above equation into $\phi(gh) = \phi(g)\phi(h)$ gives

$$h^{-1}\delta(g)h = \delta(gh)\delta(h)^{-1}$$

Note that this is true for all $h \in G_1$, and the right hand side takes on at most M^2 values. This implies that the centralizer of $\delta(g)$ in G_1 has index at most M^2. Intersecting all subgroups of G_1 of index $\leq M^2$ gives a finite index subgroup $H < G_1$ which commutes with each $\delta(g)$. From the above equation it follows that δ is a homomorphism on H. Since $\mathrm{image}(\delta)$ is finite it follows that δ has finite index kernel $\ker(\delta)$ in H, and so $\ker(\delta)$ has finite index in G_1 and in Γ. In other words, $\phi(g) = g$ on the finite index subgroup $\ker(\delta)$ of Γ, and so ϕ represents the identity element of $\mathrm{Comm}(\Gamma)$. $\diamondsuit$

For any finitely generated group Γ the kernel of $C\colon \Gamma \to \mathrm{Comm}(\Gamma)$ is the *virtual center* $\mathrm{VZ}(\Gamma)$ consisting of all $g \in \Gamma$ whose centralizer has finite index in Γ. Together with Proposition 7.5 it follows that injectivity of $L\colon \Gamma \to \mathrm{QI}(\Gamma)$ is equivalent to the triviality of $\mathrm{VZ}(\Gamma)$.

Step 2: An injection $\Psi\colon \mathrm{QI}(\Gamma) \to \mathrm{Comm}(\Gamma)$. Identifying Γ with its image $L(\Gamma) < \mathrm{QI}(\Gamma)$, a finite index subgroup, it follows that any automorphism of $\mathrm{QI}(\Gamma)$ restricts to a commensuration of Γ. In particular, given $F \in \mathrm{QI}(\Gamma)$, the inner automorphism $G \mapsto FGF^{-1}$ of $\mathrm{QI}(\Gamma)$ restricts to a commensuration Ψ_F of Γ, giving a well-defined homomorphism $\Psi\colon \mathrm{QI}(\Gamma) \to \mathrm{Comm}(\Gamma)$. Keeping in mind the definition of $L\colon \Gamma \to \mathrm{QI}(\Gamma)$, this means that for each $F \in \mathrm{QI}(\Gamma)$ and each $x \in \Gamma$ we have the following equation in $\mathrm{QI}(\Gamma)$:

$$F \cdot [L_x] \cdot F^{-1} = [L_{\Psi_F(x)}]$$

To prove that Ψ is an injection, given $F = [f] \in \mathrm{QI}(\Gamma)$ suppose that Ψ_F is the identity map when restricted to a finite index subgroup G_1 of Γ. Then for all $x \in G_1$ we have

$$\begin{aligned} F \cdot [L_x] \cdot F^{-1} &= [L_x] \\ F \cdot [L_x] &= [L_x] \cdot F \\ [f \circ L_x] &= [L_x \circ f] \end{aligned}$$

which means that there exists a bounded function $\delta\colon \Gamma \to \Gamma$ such that

$$f(xy) = xf(y)\delta(y), \quad x, y \in G_1$$

Plugging in $y = 1$ we get

$$f(x) = xf(1)\delta(1), \quad x \in G_1$$

This shows that the function f is a bounded distance from the identity map on G_1, and so F is the identity element in $\mathrm{QI}(\Gamma)$.

Step 3: $\Psi \circ i\colon$ **Comm(Γ) → Comm(Γ) is the identity.** Given a commensuration ϕ of Γ, we obtain a commensuration $\phi' = \Psi_{i(\phi)}$ satisfying the formulas

$$[\phi \circ L_x \circ \phi^{-1}] = [L_{\phi'(x)}]$$
$$[\phi \circ L_x \circ \phi^{-1} \circ L_{\phi'(x^{-1})}] = [\mathrm{Id}]$$

for all x in a certain finite index subgroup of Γ. We must prove that ϕ and ϕ' agree on a further finite index subgroup of Γ.

From the above equation it follows that there is a bounded function $\delta\colon \Gamma \to \Gamma$ such that for all h in a certain finite index subgroup H of Γ we have:

$$\phi(x \cdot \phi^{-1}(\phi'(x^{-1}) \cdot h) = h \cdot \delta(h)$$
$$\phi(x) \cdot \phi'(x^{-1}) \cdot h = h \cdot \delta(h)$$
$$h^{-1} \cdot \big(\phi(x) \cdot \phi'(x^{-1})\big) \cdot h = \delta(h)$$

This shows that $\phi(x) \cdot \phi'(x^{-1})$ has only finitely many conjugates by elements of H, and so the centralizer of $\phi(x) \cdot \phi'(x^{-1})$ in H has finite index in H. It follows that the centralizer of $\phi(x) \cdot \phi'(x^{-1})$ in Γ has finite index in Γ, which by definition of $\mathrm{VZ}(\Gamma)$ gives $\phi(x) \cdot \phi'(x^{-1}) \in \mathrm{VZ}(\Gamma)$. But since $L\colon \Gamma \to \mathrm{QI}(\Gamma)$ is injective, the virtual center $\mathrm{VZ}(\Gamma)$ is trivial, showing that $\phi(x) = \phi'(x)$ for all x in a finite index subgroup of Γ.

Combining steps 1–3 it follows that $\Psi\colon\ \mathrm{QI}(\Gamma) \to \mathrm{Comm}(\Gamma)$ is surjective, and so Ψ is an isomorphism with inverse $i\colon\ \mathrm{Comm}(\Gamma) \to \mathrm{QI}(\Gamma)$, finishing the proof of Theorem 7.4.

Remark. The proof of Theorem 7.4 yields a more general conclusion: if Γ is a finitely generated group whose virtual center is trivial, then $\mathrm{Comm}(\Gamma)$ is isomorphic to the relative commensurator of $L(\Gamma)$ in $\mathrm{QI}(\Gamma)$. The condition that the virtual center be trivial cannot be dropped: for example, a finite group has trivial quasi-isometry group but rarely is its abstract commensurator group trivial.

7.5 Proving Theorem 1.1 and Theorem 1.2

Theorem 1.2 follows immediately from Theorem 1.3 by a standard technique (see, e.g. [Sch96]). The basic observation we need says that if Γ is a finitely generated group and if the homomorphism $\Gamma \to \mathrm{QI}(\Gamma)$ has finite cokernel and kernel, then for any finitely generated group H and any quasi-isometry $\phi\colon H \to \Gamma$ the induced homomorphism $\phi_*\colon H \to \mathrm{QI}(\Gamma)$ has finite index kernel and cokernel. As a consequence, the groups H and Γ are weakly commensurable, because their images are commensurable in $\mathrm{QI}(\Gamma)$.

We now prove Theorem 1.1. Let $H_i < \mathcal{M}(\Sigma_{g_i})$, $i = 1, 2$, be Schottky groups of rank ≥ 2 with $g_i \geq 2$. We must prove the equivalence of the following four statements:

(1) Γ_{H_1} and Γ_{H_2} are quasi-isometric.

(2) Γ_{H_1} and Γ_{H_2} are abstractly commensurable.

(3) There is an isomorphism $\mathcal{O}_{H_1} \approx \mathcal{O}_{H_2}$ such that in the group $\mathcal{M}(\mathcal{O}_{H_1}) = \mathcal{M}(\mathcal{O}_{H_2})$ the Schottky subgroups H_1 and H_2 are commensurable, meaning that $H_1 \cap H_2$ has finite index in each of H_1 and H_2.

(4) There is an isomorphism $\mathcal{O}_1 \approx \mathcal{O}_2$ such that in the group $\mathcal{M}(\mathcal{O}_1) = \mathcal{M}(\mathcal{O}_2)$ the Schottky groups H_1 and H_2 have the same limit set in the Thurston boundary of the Teichmüller space $\mathcal{T}(\mathcal{O}_1) = \mathcal{T}(\mathcal{O}_2)$.

We also add in a fifth equivalent statement:

(5) The groups $\mathrm{QI}(\Gamma_{H_1})$ and $\mathrm{QI}(\Gamma_{H_2})$ are isomorphic.

The equivalence of statements (1) and (2) and (5) follows immediately from Theorem 1.3 using the fact that a quasi-isometry between two groups induces an isomorphism between their quasi-isometry groups.

The fact that (3) implies (2) is an immediate consequence of the commutative diagram 7.1 applied to Γ_{H_1} and to Γ_{H_2}.

To prove that (1) implies (3), suppose that there is a quasi-isometry $\Gamma_{H_1} \to \Gamma_{H_2}$, which induces an isomorphism of quasi-isometry groups $\mathrm{QI}(\Gamma_{H_1}) \approx \mathrm{QI}(\Gamma_{H_2})$. Consider, for each $i = 1, 2$, the short exact sequence

$$1 \to \pi_1(\mathcal{O}_i) \to \mathrm{QI}(\Gamma_{H_i}) \to \mathcal{C}_i \to 1$$

where as usual $\mathcal{O}_i = \mathcal{O}_{H_i}$ is the smallest orbifold subcover of Σ_{g_i} to which H_i descends, the subgroup $\mathcal{C}_i < \mathcal{M}(\mathcal{O}_i)$ is the relative commensurator of H_i in $\mathcal{M}(\mathcal{O}_i)$, and H_i has finite index in $\mathcal{C}_i$.

We claim that the isomorphism between $\mathrm{QI}(\Gamma_{H_1})$ and $\mathrm{QI}(\Gamma_{H_2})$ must take $\pi_1(\mathcal{O}_1)$ to $\pi_1(\mathcal{O}_2)$. This provides an isomorphism $\mathcal{O}_1 \approx \mathcal{O}_2$ such that the induced isomorphism $\mathcal{M}(\mathcal{O}_1) \approx \mathcal{M}(\mathcal{O}_2)$ takes $\mathcal{C}_1$ to $\mathcal{C}_2$, and statement (3) immediately follows.

To prove the claim, consider the model space X_{H_i} with its horizontal foliation. By Theorem 7.2 the subgroup $\pi_1(\mathcal{O}_i)$ of $\mathrm{QI}(\Gamma_{H_i}) \approx \mathrm{QI}(X_{H_i})$ consists of quasi-isometries which coarsely preserve each leaf of the horizontal foliation. The quasi-isometry $X_{H_1} \to X_{H_2}$ coarsely respects horizontal foliations by Lemma 4.3, and so the coarse leaf preserving elements of $\mathrm{QI}(X_{H_1})$ are taken bijectively by the isomorphism $\mathrm{QI}(X_{H_1}) \leftrightarrow \mathrm{QI}(X_{H_2})$ to the coarse leaf respecting quasi-isometries of X_{H_2}. In other words, $\pi_1(\mathcal{O}_1)$ is taken to $\pi_1(\mathcal{O}_2)$.

Finally, the equivalence of (3) and (4) follows immediately from Theorem 7.3, completing the proof of Theorem 1.1. $\diamondsuit$

8 Closing remarks

8.1 Surfaces versus orbifolds

As remarked in the introduction, the universe of groups quasi-isometric to (orbifold)-by-(virtual Schottky) groups is exactly the same as the universe of groups quasi-isometric to (surface)-by-(Schottky) groups:

Proposition 8.1. *Given a closed orbifold $\mathcal{O}$ and a virtual Schottky subgroup $N < \mathcal{M}(\mathcal{O})$, consider the extension Γ_N defined by*

$$1 \to \pi_1(\mathcal{O}) \to \Gamma_N \to N \to 1$$

There exists a closed oriented surface Σ of genus $g \geq 2$, and a Schottky subgroup $H < \mathcal{M}(\Sigma)$, such that the group $\Gamma_H = \pi_1(\Sigma) \rtimes H$ has finite index in Γ_N, and so the two groups are quasi-isometric.

Proof. There is a Schottky subgroup $H' < N$ of finite index; it follows that $\Gamma_{H'}$ has finite index in Γ_N. Choose a splitting $H' \to \mathcal{M}(\mathcal{O}, p)$, consider the action of H' on $\pi_1(\mathcal{O})$ by automorphisms. Choose a finite surface cover Σ' with corresponding subgroup $\pi_1(\Sigma') < \pi_1(\mathcal{O})$, and consider the orbit of $\pi_1(\Sigma')$ under the action of H'. This orbit consists of a finite collection of finite index subgroups of $\pi_1(\mathcal{O})$, whose intersection is a finite index subgroup corresponding to a surface group Σ which is a cover of Σ'. The group $H' < \mathcal{M}(\mathcal{O}, p)$ lifts to a subgroup $H < \mathcal{M}(\Sigma, p)$, which projects to a Schottky subgroup of $\mathcal{M}(\Sigma)$. The group $\Gamma_H = \pi_1(\Sigma) \rtimes H$ is therefore a (surface)-by-(Schottky) group with finite index in $\Gamma_{H'}$, and so also in the original group Γ_N. $\diamondsuit$

8.2 Fibered hyperbolic 3-manifold groups

As mentioned in the footnote on page 3, the method of proof of Theorem 1.1 shows that if two word hyperbolic surface-by-free groups are quasi-isometric then they are horizontal respecting quasi-isometric, indeed they are horizontal respecting commensurable, as long as the free group has rank ≥ 2. When the free group is infinite cyclic this fails: all hyperbolic 3-manifolds fibering over the circle have quasi-isometric fundamental groups; but there exist fibered hyperbolic 3-manifolds which are not abstract commensurable—take, for example, an arithmetic example and a non-arithmetic example.

Nevertheless, we do obtain a classification of fundamental groups of fibered hyperbolic 3-manifold groups up to horizontal respecting quasi-isometry. Namely, let Σ be a closed surface of genus $g \geq 2$, let $\psi \in \mathcal{M}(\Sigma)$ be a pseudo-Anosov mapping class generating an infinite cyclic subgroup $\langle\psi\rangle$ of $\mathcal{M}(\Sigma)$, and let $\Gamma_{\langle\psi\rangle}$ be the associated extension group of $\pi_1(\Sigma)$ by $\langle\psi\rangle$. Let $\mathcal{O}_{\langle\psi\rangle}$ be the smallest subcover of Σ to which $\langle\psi\rangle$ descends, let $\mathcal{C}_{\langle\psi\rangle}$ be the relative commensurator of $\langle\psi\rangle$ in $\mathcal{M}(\mathcal{O}_{\langle\psi\rangle})$, and let $\Gamma_{\mathcal{C}_{\langle\psi\rangle}}$ be the extension of $\pi_1(\mathcal{O}_{\langle\psi\rangle})$ by $\mathcal{C}_{\langle\psi\rangle}$. Note that $\mathcal{C}_{\langle\psi\rangle}$ contains $\langle\psi\rangle$ with finite index by [BLM83], and so $\Gamma_{C_{\langle\psi\rangle}}$ contains $\Gamma_{\langle\psi\rangle}$ with finite index. The proofs of §7 now apply directly to obtain the following classification theorem, which parallels Theorem 1.1:

Theorem 8.2. *Given closed surfaces Σ_i, $i = 1, 2$, and pseudo-Anosov mapping classes ψ_i of $\mathcal{M}(\Sigma_i)$, the following are equivalent, where $\mathcal{O}_i = \mathcal{O}_{\langle\psi_i\rangle}$, etc.:*

(1) There exists a horizontal respecting quasi-isometry $\Gamma_{\langle\psi_1\rangle} \to \Gamma_{\langle\psi_2\rangle}$.

(2) The groups $\Gamma_{\langle\psi_1\rangle}, \Gamma_{\langle\psi_2\rangle}$ are horizontal respecting commensurable, meaning that there is an isomorphism from a finite index subgroup $H_1 < \Gamma_{\langle\psi_1\rangle}$ to a finite index subgroup $H_2 < \Gamma_{\langle\psi_2\rangle}$, taking $H_1 \cap \pi_1(\mathcal{O}_1)$ to $H_2 \cap \pi_1(\mathcal{O}_2)$.

(3) There is an isomorphism $\mathcal{O}_1 \approx \mathcal{O}_2$ such that in the group $\mathcal{M}(\mathcal{O}_1) = \mathcal{M}(\mathcal{O}_2)$ the mapping classes ψ_1 and ψ_2 have equal powers.

(4) There is an isomorphism $\mathcal{O}_1 \approx \mathcal{O}_2$ such that in the group $\mathcal{M}(\mathcal{O}_1) = \mathcal{M}(\mathcal{O}_2)$ the mapping classes ψ_1 and ψ_2 have the same limit set in the Thurston boundary of the Teichmüller space $\mathcal{T}(\mathcal{O}_1) = \mathcal{T}(\mathcal{O}_2)$, i.e. they have the same stable/unstable measured foliation pairs.

(5) There is an isomorphism $\Gamma_{\mathcal{C}_1} \approx \Gamma_{\mathcal{C}_2}$ taking $\pi_1(\mathcal{O}_1)$ to $\pi_1(\mathcal{O}_2)$.

$\diamondsuit$

It is easy to use this theorem to obtain infinitely many distinct horizontal respecting quasi-isometry classes of groups Γ_H, for cyclic pseudo-Anosov groups H. This should be contrasted with the fact that *all* of the groups Γ_H are quasi-isometric to each other and to $\mathbf{H}^3$, by Thurston's hyperbolization theorem for fibered 3-manifolds [Ota96].

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