Real-analytic, volume-preserving actions of lattices on 4-manifolds

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January 17, 2002

Abstract

We prove that any real-analytic, volume-preserving action of a lattice Γ in a simple Lie group with $\mathbf{Q}$-rank$(\Gamma) \geq 7$ on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action.

1 Results

Zimmer conjectured in [Zi1] that the standard action of $\mathrm{SL}(n, Z)$ on the n-torus is minimal in the following sense:

Conjecture 1 (Zimmer). *Any smooth, volume-preserving action of any finite-index subgroup $\Gamma < \mathrm{SL}(n, \mathbf{Z})$ on a closed r-manifold factors through a finite group action if $n > r$.*

While Conjecture 1 has been proved for actions which also preserve an extra geometric structure such as a pseudo-Riemannian metric (see, e.g. [Zi1]), almost nothing is known in the general case. For $r = 2$ and $n > 4$, the conjecture was proved for real-analytic actions in [FS1]. Quite recently, Polterovich [Po] has brought ideas from symplectic topology to the problem, using these to give a proof of Conjecture 1 for orientable surfaces of genus > 1. In [FS2] we will point out how his methods actually prove Conjecture 1 for the torus as well. For $r = 3$, Conjecture 1 is known only in some special cases where Γ contains some torsion and the action is real-analytic (see [FS1]).

In this note we prove the following result.

*Both authors are supported in part by the NSF.

Theorem 2 (Actions on 4-manifolds with $\chi(M) \neq 0$). *Let Γ be a lattice in a simple Lie group G such that $\mathbf{Q}$-rank$(\Gamma) \geq 7$. Then any real-analytic, volume-preserving action of Γ on a closed 4-manifold of nonzero Euler characteristic factors through a finite group action.*

In particular, Theorem 2 applies to any finite-index subgroup of $\mathrm{SL}(n, \mathbf{Z})$, $n > 7$.

The main ingredient in the proof of Theorem 2 is Theorem 7.1 of [FS1] on real-analytic actions which preserve a volume form. This theorem, which is the most difficult result in [FS1], gives a *codimension-two* invariant submanifold for centralizers of elements with fixed-points. One can then apply results of [FS1] and [Re], which show that real-analytic (not necessarily area-preserving) actions of certain lattices on 2-dimensional manifolds must factor through finite groups.

For the case of symplectic actions, some further progress on Conjecture 1 can be found in [Po] and [FS2].

2 Proof of Theorem 2

Before giving the proof of Theorem 2, we will need two algebraic properties of lattices with large $\mathbf{Q}$-rank.

Proposition 3 (Some algebraic properties of lattices). *Let Γ be a lattice in a simple algebraic group over $\mathbf{Q}$. Then the following hold:*

1. *If $d = \mathbf{Q}$-rank$(\Gamma) \geq 7$ then Γ contains commuting subgroups A and B which are isomorphic to irreducible lattices with $\mathbf{Q}$-ranks 2 and $d - 3$ respectively.*

2. *If $\mathbf{Q}$-rank$(\Gamma) \geq 4$ then Γ contains a torsion-free nilpotent subgroup which is not metabelian.*

Proof. The proof of the first statement is similar to that of Proposition 2.1 of [FS1]. By Margulis's Arithmeticity Theorem (see, e.g., [Zi2], Theorem 6.1.2), Γ is commensurate with the group of $\mathbf{Z}$-points of a simple algebraic group G defined over $\mathbf{Q}$. Hence without loss of generality we can assume that Γ itself is the group of $\mathbf{Z}$-points in such a group G.

Since G is simple, the root system Φ of G is irreducible, and the Dynkin diagram determined Φ therefore appears in the list given in Section 11.4 of [Hu]. By going through this list, one sees that in every case where $d \geq 7$, one may "erase a vertex v" of the diagram to obtain a a graph with 2 components:

one with two vertices and another which is a Dynkin diagram with at least $d-3$ vertices. Let G_1 and G_2 be the root subgroups corresponding to these two components of the Dynkin diagram. Then the group of $\mathbf{Q}$-points of G_1 has $\mathbf{Q}$-rank at least 2, the group of $\mathbf{Q}$-points of G_2 has $\mathbf{Q}$-rank at least $d-3$, and G_1 commutes with G_2.

Now $\Gamma_i = \Gamma \cap G_i$ is an arithmetic lattice in G_i for $i = 1, 2$, since by a theorem of Borel-Harish-Chandra (see, e.g. [Zi2]) the $\mathbf{Z}$-points of an algebraic group defined over $\mathbf{Q}$ form a lattice in the group of $\mathbf{R}$-points. Then $A = \Gamma_1$ and $B = \Gamma_2$ have the required properties.

To prove the second statement, note that since $\mathbf{Q}\text{-rank}(\Gamma) \geq 4$, we can find a connected, nilpotent Lie subgroup N which is defined over $\mathbf{Q}$ and has derived length 3, i.e. is not metabelian. As $\Gamma \cap N$ is the group of $\mathbf{Z}$-points of the $\mathbf{Q}$-group N, it is a lattice in N, and in particular is Zariski-dense in N. Hence $\Gamma \cap N$ is nilpotent and has no metabelian subgroup of finite index. As $\Gamma \cap N$ must have a tosrion-free subgroup of finite index, the assertion follows. $\diamond$

We now turn to the proof of Theorem 2. Let M be a closed 4-manifold with nonzero Euler characteristic. Let Γ be an irreducible lattice in a simple Lie group G, and assume $d = \mathbf{Q}\text{-rank}(\Gamma) \geq 7$. By part (1) of Proposition 3, Γ contains commuting subgroups A and B which are isomorphic to irreducible lattices with $\mathbf{Q}$-ranks 2 and $d - 3 \geq 4$ respectively.

Let γ_0 be any infinite order element of A. By an old theorem of Fuller [Fu], any homeomorphism of a closed manifold of nonzero Euler characteristic has a periodic point; the proof is an application of the Lefschetz fixed-point theorem and basic number theory. Hence some positive power γ of γ_0 has a fixed point.

We will also need the following two facts. First, since $\mathbf{Q}\text{-rank}(B) \geq d - 3 \geq 4$, it follows from Margulis's Superrigidity Theorem that any representation of B into $\mathrm{GL}(4, \mathbf{R})$ has finite image. Second, since Γ is a lattice in a simple Lie group G with $\mathbf{R}\text{-rank}(G) \geq 2$, the Margulis Finiteness Theorem (see, e.g., Theorem 8.1 of [Zi2]) gives that Γ is *almost simple* in the sense that any normal subgroup of Γ must be finite or of finite index.

We are now in a position to apply Theorem 7.1 of [FS1]. For the reader's convenience we recall the statement here. We say that a group action $\rho : \Gamma \to \mathrm{Diff(M)}$ is *infinite* if ρ has infinite image.

Theorem 7.1 of [FS1]: *Let Γ be an almost simple group. Suppose we are given an infinite, volume-preserving, real-analytic action of Γ on a closed,* *connected n-manifold M. Suppose further that Γ contains commuting subgroups A and B with the following properties:*

- *There exists an element $\gamma \in A$, noncentral in Γ, having a fixed point in M.*
- *A is isomorphic to an irreducible lattice of $\mathbf{Q}$-rank ≥ 2.*
- *B is noncentral in Γ.*
- *Any representation of any finite-index subgroup of B in $\mathrm{GL}(n, \mathbf{R})$ has finite image.*

Then there is a nonempty, connected, real-analytic submanifold $W \subset M$ of codimension at least 2 which is invariant under a finite-index subgroup B' of B. Furthermore, the action of this subgroup on W is infinite.

Remark. The action of B' on the surface W produced by this theorem is NOT necessarily area preserving.

We now conclude the proof of Theorem 2. Since B' is a lattice in a simple Lie group and $\mathbf{Q}$-rank$(B') \geq 4$, it follows from part (2) of Proposition 3 that B' contains a torsion-free nilpotent subgroup H which is not metabelian. But Rebelo [Re] showed that any nilpotent group of real-analytic diffeomorphisms of a compact, oriented surface must be metabelian. It follows that the action of H on W is not effective.

Since H is torsion-free, there is an infinite-order element of $H \leq B'$ which acts trivially on W. Since B' has finite index in the almost simple group B, and hence is almost simple, some finite index subgroup C of B' acts trivially on W; in particular C has a global fixed point in M. Since C is isomorphic to a lattice of $\mathbf{Q}$-rank at least 4, by Lemma 3.2 of [FS1] we have that a finite index subgroup D of C acts trivially on M. Since Γ is almost simple, it follows that some finite index normal subgroup of Γ acts trivially on M, and we are done. $\diamond$

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