

Problems on the geometry of finitely generated solvable groups

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1 Introduction

Our story begins with a theorem of Gromov, proved in 1980.

Theorem 1 (Gromov’s Polynomial Growth Theorem [Gr1]). *Let G be any finitely generated group. If G has polynomial growth then G is virtually nilpotent, i.e. G has a finite index nilpotent subgroup.*

Gromov’s theorem inspired the more general problem (see, e.g. [GH1, BW, Gr1, Gr2]) of understanding to what extent the asymptotic geometry of a finitely generated solvable group determines its algebraic structure. One way in which to pose this question precisely is via the notion of quasi-isometry.

A (coarse) *quasi-isometry* between metric spaces is a map $f : X \rightarrow Y$ such that, for some constants $K, C, C' > 0$:

1. $\frac{1}{K}d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq Kd_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.
2. The C' -neighborhood of $f(X)$ is all of Y .

X and Y are *quasi-isometric* if there exists a quasi-isometry $X \rightarrow Y$. Note that the quasi-isometry type of a metric space X does not depend on any bounded subset of X ; hence the term “asymptotic”.

Quasi-isometries are the natural maps to study when one is interested in the geometry of a group. In particular:

1. The word metric on any f.g. group is unique up to quasi-isometry.
2. Any injective homomorphism with finite index image or surjective homomorphism with finite index kernel is a quasi-isometry. The equivalence relation generated by these two types of maps can be described more compactly: two groups G, H are equivalent in this manner if and only if they are *weakly commensurable*, which means that there exists a group Q and homomorphisms $Q \rightarrow G, Q \rightarrow H$ each having finite kernel and finite index image (proof: show that “weakly commensurable” is in fact an equivalence relation). This weakens the usual notion of *commensurability*, i.e. when G and H have isomorphic finite index subgroups. Weakly commensurable groups are clearly quasi-isometric.
3. Any two cocompact, discrete subgroups of a Lie group are quasi-isometric. There are cocompact discrete subgroups of the same Lie group which are not weakly commensurable.

The Polynomial Growth Theorem was an important motivation for Gromov when he initiated in [Gr2, Gr3] the problem of classifying finitely-generated groups up to quasi-isometry.

Theorem 1, together with the fact that nilpotent groups have polynomial growth (see §3 below), implies that the property of being nilpotent is actually an asymptotic property of groups. More precisely, the class of nilpotent groups is *quasi-isometrically rigid*: any finitely-generated group quasi-isometric to a nilpotent group is weakly commensurable to some nilpotent group ¹. Sometimes this is expressed by saying that the property of being nilpotent is a *geometric property*, i.e. it is a quasi-isometry invariant (up to weak commensurability). The natural question then becomes:

Question 2 (Rigidity question). *Which subclasses of f.g. solvable groups are quasi-isometrically rigid? For example, are polycyclic groups quasi-isometrically rigid? metabelian groups? nilpotent-by-cyclic groups?*

In other words, which of these algebraic properties of a group are actually geometric, and are determined by much weaker asymptotic information? A. Dioubina [Di] has recently found examples which show that the class of finitely generated solvable groups is *not* quasi-isometrically rigid (see §2). On the other hand, at least some subclasses of solvable groups are indeed rigid (see §4).

Along with Question 2 comes the finer classification problem:

Problem 3 (Classification problem). *Classify f.g. solvable (resp. nilpotent, polycyclic, metabelian, nilpotent-by-cyclic, etc.) groups up to quasi-isometry.*

As we shall see, the classification problem is usually much more delicate than the rigidity problem; indeed the quasi-isometry classification of finitely-generated nilpotent groups remains one of the major open problems in the field. We discuss this in greater detail in §3.

The corresponding rigidity and classification problems for irreducible lattices in semisimple Lie groups have been completely solved. This is a vast result due to many people and, among other things, it generalizes and strengthens the Mostow Rigidity Theorem. We refer the reader to [Fa] for a survey of this work.

In contrast, results for finitely generated solvable groups have remained more elusive. There are several reasons for this:

¹In fact such a group must have a finite-index nilpotent subgroup. In general, weak commensurability is the most one can hope for in quasi-isometric rigidity problems.

1. Finitely generated solvable groups are defined algebraically, and so they do not always come equipped with an obvious or well-studied geometric model (see, e.g., item (4) below).
2. Dioubina's examples show not only that the class of finitely-generated solvable groups is not quasi-isometrically rigid; they also show (see §4 below) that the answer to Question 2 for certain subclasses of solvable groups (e.g. abelian-by-cyclic) differs in the finitely presented and finitely generated cases.
3. Solvable groups are much less rigid than irreducible lattices in semisimple Lie groups. This phenomenon is exhibited concretely by the fact that finitely generated solvable groups typically (and probably always for infinite solvable groups) have infinite-dimensional groups of self quasi-isometries². In contrast, all irreducible lattices in semisimple Lie groups $G \neq \mathrm{SO}(n, 1), \mathrm{SU}(n, 1)$ have countable or finite-dimensional quasi-isometry groups.
4. There exists a finitely presented solvable group Γ of derived length 3 with the property that Γ has unsolvable word problem (see [Kh]). Solving the word problem for a group is equivalent to giving an algorithm to build the Cayley graph of that group. Hence there are finitely presented solvable groups whose geometry cannot be understood, at least by a Turing machine³.

At this point in time, our understanding of the geometry of finitely-generated solvable groups is quite limited. In §3 we discuss what is known about the quasi-isometry classification of nilpotent groups (the rigidity being given by Gromov's Polynomial Growth Theorem). Beyond nilpotent groups, the only detailed knowledge we have is for the finitely-presented, nonpolycyclic abelian-by-cyclic groups. We discuss this in depth in §4, and give a conjectural picture of the polycyclic case in §5. One of the interesting discoveries described in these sections is a connection between finitely presented solvable groups and the theory of dynamical systems. This connection is discussed in a more general context in §6. We end by mentioning some questions which are just beyond the limits of current knowledge.

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²The set of self quasi-isometries of a space X can be made into a group $\mathrm{QI}(X)$ once one mods out by the relation $f \sim g$ if $d(f, g) < \infty$ in the sup norm.

³We will leave to the reader to decide the implications that this may have for human understanding.

2 Dioubina's examples

Recall that the *wreath product* of groups A and B , denoted $A \wr B$, is the semidirect product $(\oplus_A B) \rtimes A$, where $\oplus_A B$ is the direct sum of copies of B indexed by elements of A , and A acts via the “shift”, i.e. the left action of A on the index set A via left multiplication. Note that if A and B are finitely-generated then so is $A \wr B$.

The main result of Dioubina [Di] is that, if there is a bijective quasi-isometry between finitely-generated groups A and B , then for any finitely-generated group C the groups $C \wr A$ and $C \wr B$ are quasi-isometric. Dioubina then applies this theorem to the groups $A = C = \mathbf{Z}$, $B = \mathbf{Z} \oplus D$ where D is a finite nonsolvable group. It is easy to construct a one-to-one quasi-isometry between A and B . Hence $G = C \wr A$ and $H = C \wr B$ are quasi-isometric.

Now G is torsion-free solvable, in fact $G = \mathbf{Z} \wr \mathbf{Z}$ is an abelian-by-cyclic group of the form $\mathbf{Z}[\mathbf{Z}]$ -by- \mathbf{Z} . On the other hand H contains $\oplus_{\mathbf{Z}} D$, and so is not virtually solvable, nor even weakly commensurable with a solvable group. Hence the class of finitely-generated solvable groups is not quasi-isometrically rigid.

It is not hard to show that Dioubina's examples never have any finite presentation. In fact if $A \wr B$ is finitely presented then either A or B is finite. This leads to the following question.

Question 4. *Is the class of finitely presented solvable groups quasi-isometrically rigid?*

Note that the property of being finitely presented is a quasi-isometry invariant.

3 Nilpotent groups and Pansu's Theorem

While the Polynomial Growth Theorem shows that the class of finitely generated nilpotent groups is quasi-isometrically rigid, the following remains an important open problem.

Problem 5. *Classify finitely generated nilpotent groups up to quasi-isometry.*

The basic quasi-isometry invariants for a finitely-generated nilpotent group G are most easily computed in terms of the set $\{c_i(G)\}$ of ranks (over \mathbf{Q}) of the quotients G_i/G_{i+1} of the lower central series for G , where G_i is defined inductively by $G_1 = G$ and $G_{i+1} = [G, G_i]$.

One of the first quasi-isometry invariants to be studied (by Wolf and Milnor) was the growth of a group. The *growth* of G is the number of

elements in the ball of radius r in G . This is a polynomial when G is nilpotent, as was shown by Wolf. The degree $\deg(G)$ of this polynomial is easily seen to be a quasi-isometry invariant; it is given by the following formula due to Bass [Ba]:

$$\deg(G) = \sum_{i=1}^n ic_i(G)$$

where n is the degree of nilpotency of G . Another basic invariant is that of virtual cohomological dimension $\text{vcd}(G)$. For groups G with finite classifying space (which is not difficult to check for torsion-free nilpotent groups), this number was shown by Gersten [Ge] and Block-Weinberger [BW] to be a quasi-isometry invariant. On the other hand it is easy to check that

$$\text{vcd}(G) = \sum_{i=1}^n c_i(G)$$

where n is the degree of nilpotence of G . Note that, as Gersten has pointed out, the above two formulas show that any finitely generated group Γ which is quasi-isometric to \mathbf{Z}^n must have a finite index \mathbf{Z}^n subgroup: by the Polynomial Growth Theorem such a Γ has a finite index nilpotent subgroup N ; but

$$\deg(N) = \deg(\mathbf{Z}^n) = n = \text{vcd}(\mathbf{Z}^n) = \text{vcd}(N)$$

and so

$$\sum_i ic_i(N) = \sum_i c_i(N)$$

which can only happen if $c_i(N) = 0$ for $i > 1$, in which case N is abelian.

Problem 6 ([GH2]). *Give an elementary proof (i.e. without using Gromov's Polynomial Growth Theorem) that any finitely generated group quasi-isometric to \mathbf{Z}^n has a finite index \mathbf{Z}^n subgroup \mathbf{Z}^n .*

As an exercise, the reader is invited to find nilpotent groups N_1, N_2 which are not quasi-isometric but which have the same degree of growth and the same vcd .

There are many other quasi-isometry invariants for finitely-generated nilpotent groups Γ . All known invariants are special cases of the following theorem of Pansu [Pa1]. To every nilpotent group Γ one can associate a nilpotent Lie group $\Gamma \otimes \mathbf{R}$, called the *Malcev completion* of Γ (see [Ma]), as well as the associated graded Lie group $\text{gr}(\Gamma \otimes \mathbf{R})$.

Theorem 7 (Pansu’s Theorem [Pa1]). *Let Γ_1, Γ_2 be two finitely-generated nilpotent groups. If Γ_1 is quasi-isometric to Γ_2 then $\text{gr}(\Gamma_1 \otimes \mathbf{R})$ is isomorphic to $\text{gr}(\Gamma_2 \otimes \mathbf{R})$.*

We remark that there are nilpotent groups with non-isomorphic Malcev completions where the associated gradeds are isomorphic. It is not known whether or not the Malcev completion is a quasi-isometry invariant.

Theorem 7 immediately implies:

Corollary 8. *The numbers $c_i(\Gamma)$ are quasi-isometry invariants.*

In particular we recover (as very special cases) that growth and cohomological dimension are quasi-isometry invariants of Γ .

To understand Pansu’s proof one must consider *Carnot groups*. These are graded nilpotent Lie groups N whose Lie algebra \mathcal{N} is generated (via bracket) by elements of degree one. Chow’s Theorem states that such Lie groups N have the property that the left-invariant distribution obtained from the degree one subspace \mathcal{N}^1 of \mathcal{N} is a *totally nonintegrable* distribution: any two points $x, y \in N$ can be connected by a piecewise smooth path γ in N for which the vector $d\gamma/dt(t)$ lies in the distribution. Infimizing the length of such paths between two given points gives a metric on N , called the *Carnot Carethéodory* metric d_{car} . This metric is non-Riemannian if $N \neq \mathbf{R}^n$. For example, when N is the 3-dimensional Heisenberg group then the metric space (N, d_{car}) has Hausdorff dimension 4.

One important property of Carnot groups is that they come equipped with a 1-parameter family of dilations $\{\delta_t\}$, which gives a notion of (*Carnot*) *differentiability*. Further, the differential $Df(x)$ of a map $f : N_1 \rightarrow N_2$ between Carnot groups N_1, N_2 which is (Carnot) differentiable at the point $x \in N_1$ is actually a *Lie group homomorphism* $N_1 \rightarrow N_2$.

Sketch of Pansu’s proof of Theorem 7. If (Γ, d) is a nilpotent group endowed with a word metric d , it is possible to attach to the sequence of metric spaces $\{(\Gamma, \frac{1}{n}d)\}_{n \in \mathbf{N}}$ a limit metric space:

$$(\Gamma_\infty, d_\infty) = \lim_{n \rightarrow \infty} (\Gamma, \frac{1}{n}d)$$

where the limit is taken in the sense of Gromov-Hausdorff convergence (see [Gr1, Pa2], or [DW] for a different approach). The salient features of the limit metric space $(\Gamma_\infty, d_\infty)$ are:

- (Identifying limit) It is isometric to the Carnot group $\text{gr}(\Gamma \otimes \mathbf{R})$ endowed with the Carnot metric d_{car} .

- (Functoriality) Any quasi-isometry $f : \Gamma_1 \rightarrow \Gamma_2$ bewteen finitely-generated nilpotent groups induces a *bilipschitz homeomorphism*

$$\hat{f} : (\text{gr}(\Gamma_1 \otimes \mathbf{R}), d_{car}) \rightarrow (\text{gr}(\Gamma_2 \otimes \mathbf{R}), d_{car})$$

Note that functoriality follows immediately once we know the limit exists: the point is that if $f : (\Gamma_1, d_1) \rightarrow (\Gamma_2, d_2)$ is a (K, C) quasi-isometry of word metrics, then for each n the map

$$f : (\Gamma_1, \frac{1}{n}d_1) \rightarrow (\Gamma_2, \frac{1}{n}d_2)$$

is a $(K, C/n)$ quasi-isometry, hence the induced map

$$\hat{f} : ((\Gamma_1)_\infty, d_{car}) \rightarrow ((\Gamma_2)_\infty, d_{car})$$

is a $(K, 0)$ quasi-isometry, i.e. is a bilipschitz homeomorphism.

Given a quasi-isometry $f : \Gamma_1 \rightarrow \Gamma_2$, we thus have an induced bilipschitz homeomorphism $\hat{f} : \text{gr}(\Gamma_1 \otimes \mathbf{R}) \rightarrow \text{gr}(\Gamma_2 \otimes \mathbf{R})$ between Carnot groups endowed with Carnot-Carethéodory metrics. Pansu then proves a regularity theorem, generalizing the Rademacher-Stepanov Theorem for \mathbf{R}^n . This general regularity theorem states that a bilipschitz homeomorphism of Carnot groups (endowed with Carnot-Carethéodory metrics) is differentiable almost everywhere. Since the value of the differential $D\hat{f}$ is actually a group homomorphism, we know that for almost every point x the differential $D\hat{f} : \text{gr}(\Gamma_1 \otimes \mathbf{R}) \rightarrow \text{gr}(\Gamma_2 \otimes \mathbf{R})$ is an isomorphism.

4 Abelian-by-cyclic groups: nonpolycyclic case

The first progress on Question 2 and Problem 3 in the non-(virtually)-nilpotent case was made in [FM1] and [FM2]. These papers proved classification and rigidity for the simplest class of non-nilpotent solvable groups: the *solvable Baumslag-Solitar groups*

$$\text{BS}(1, n) = \langle a, b : aba^{-1} = b^n \rangle$$

These groups are part of the much broader class of abelian-by-cyclic groups. A group Γ is *abelian-by-cyclic* if there is an exact sequence

$$1 \rightarrow A \rightarrow \Gamma \rightarrow Z \rightarrow 1$$

where A is an abelian group and Z is an infinite cyclic group. If Γ is finitely generated, then A is a finitely generated module over the group ring $\mathbf{Z}[Z]$, although A may not be finitely generated as a group.

By a result of Bieri and Strebel [BS1], the class of finitely presented, torsion-free, abelian-by-cyclic groups may be described in another way. Consider an $n \times n$ matrix M with integral entries and $\det M \neq 0$. Let Γ_M be the ascending HNN extension of \mathbf{Z}^n given by the monomorphism ϕ_M with matrix M . Then Γ_M has a finite presentation

$$\Gamma_M = \langle t, a_1, \dots, a_n \mid [a_i, a_j] = 1, ta_it^{-1} = \phi_M(a_i), i, j = 1, \dots, n \rangle$$

where $\phi_M(a_i)$ is the word $a_1^{m_1} \cdots a_n^{m_n}$ and the vector (m_1, \dots, m_n) is the i^{th} column of the matrix M . Such groups Γ_M are precisely the class of finitely presented, torsion-free, abelian-by-cyclic groups (see [BS1] for a proof involving a precursor of the Bieri-Neumann-Strebel invariant, or [FM2] for a proof using trees). The group Γ_M is polycyclic if and only if $|\det M| = 1$; this is easy to see directly, and also follows from [BS2].

The results of [FM1] and [FM2] are generalized in [FM3], which gives the complete classification of the finitely presented, nonpolycyclic abelian-by-cyclic groups among all f.g. groups, as given by the following two theorems.

The first theorem in [FM3] gives a classification of all finitely-presented, nonpolycyclic, abelian-by-cyclic groups up to quasi-isometry. It is easy to see that any such group has a torsion-free subgroup of finite index, so is commensurable (hence quasi-isometric) to some Γ_M . The classification of these groups is actually quite delicate—the standard quasi-isometry invariants (ends, growth, isoperimetric inequalities, etc.) do not distinguish any of these groups from each other, except that the size of the matrix M can be detected by large scale cohomological invariants of Γ_M .

Given $M \in \text{GL}(n, \mathbf{R})$, the *absolute Jordan form* of M is the matrix obtained from the Jordan form for M over \mathbf{C} by replacing each diagonal entry with its absolute value, and rearranging the Jordan blocks in some canonical order.

Theorem 9 (Nonpolycyclic, abelian-by-cyclic groups: Classification).

Let M_1 and M_2 be integral matrices with $|\det M_i| > 1$ for $i = 1, 2$. Then Γ_{M_1} is quasi-isometric to Γ_{M_2} if and only if there are positive integers r_1, r_2 such that $M_1^{r_1}$ and $M_2^{r_2}$ have the same absolute Jordan form.

Remark. Theorem 9 generalizes the main result of [FM1], which is the case when M_1, M_2 are positive 1×1 matrices; in that case the result of [FM1] says even more, namely that Γ_{M_1} and Γ_{M_2} are quasi-isometric if and only if they are commensurable. When $n \geq 2$, however, it's not hard to find $n \times n$ matrices M_1, M_2 such that $\Gamma_{M_1}, \Gamma_{M_2}$ are quasi-isometric but not commensurable. Polycyclic examples are given in [BG]; similar ideas can be used to produce nonpolycyclic examples.

The following theorem shows that the algebraic property of being a finitely presented, nonpolycyclic, abelian-by-cyclic group is in fact a geometric property.

Theorem 10 (Nonpolycyclic, abelian-by-cyclic groups: Rigidity).

Let $\Gamma = \Gamma_M$ be a finitely presented abelian-by-cyclic group, determined by an $n \times n$ integer matrix M with $|\det M| > 1$. Let G be any finitely generated group quasi-isometric to Γ . Then there is a finite normal subgroup $N \subset G$ such that G/N is commensurable to Γ_N , for some $n \times n$ integer matrix N with $|\det N| > 1$.

Remark. Theorem 10 generalizes the main result of [FM2], which covers the case when M is a positive 1×1 matrix. The latter result is given a new proof in [MSW], which is adapted to prove Theorem 10 in [FM3].

Remark. The “finitely presented” hypothesis in Theorem 10 cannot be weakened to “finitely generated”, since Diuobina’s example (discussed in §2) is abelian-by-cyclic, namely $\mathbf{Z}[\mathbf{Z}]$ -by- \mathbf{Z} .

One new discovery in [FM3] is that there is a strong connection between the geometry of solvable groups and the theory of dynamical systems. Assuming here for simplicity that the matrix M lies on a 1-parameter subgroup M^t in $\mathrm{GL}(n, \mathbf{R})$, let G_M be the semi-direct product $\mathbf{R}^n \rtimes_M \mathbf{R}$, where \mathbf{R} acts on \mathbf{R}^n by the 1-parameter subgroup M^t . We endow the solvable Lie group G_M with a left-invariant metric. The group G_M admits a *vertical flow*:

$$\Psi_s(x, t) = (x, t + s)$$

There is a natural *horizontal foliation* of G_M whose leaves are the level sets $P_t = \mathbf{R}^n \times \{t\}$ of time. A quasi-isometry $f : G_M \rightarrow G_N$ is *horizontal respecting* if it coarsely permutes the leaves of this foliation; that is, if there is a constant $C \geq 0$ so that

$$d_{\mathcal{H}}(f(P_t), P_{h(t)}) \leq C$$

where $d_{\mathcal{H}}$ denotes Hausdorff distance and $h : \mathbf{R} \rightarrow \mathbf{R}$ is some function, which we think of as a *time change* between the flows.

A key technical result of [FM3] is the phenomenon of *time rigidity*: the time change h must actually be *affine*, so taking a real power of M allows one to assume $h(t) = t$. It is then shown that “quasi-isometries remember the dynamics”; that is, f also respects several dynamically defined foliations associated with the flow Ψ , such as the weak stable, weak unstable, and center-leaf foliations. An inductive scheme on divergence rates and flags of foliations then completes the proof of:

Theorem 11 (Horizontal respecting quasi-isometries). *If there is a horizontal-respecting quasi-isometry $f : G_M \rightarrow G_N$ then there exist nonzero $a, b \in \mathbf{R}$ so that M^a and M^b have the same absolute Jordan form.*

The “nonpolycyclic” hypothesis (i.e. $|\det M| > 1$) in Theorem 9 is used in two ways. First, the group Γ_M has a model space which is topologically a product of \mathbf{R}^n and a regular tree of valence $|\det M|+1$, and when this valence is greater than 2 we can use coarse algebraic topology (as developed in [FS], [EF], and [FM1]) to show that any quasi-isometry $\Gamma_M \rightarrow \Gamma_N$ induces a quasi-isometry $G_M \rightarrow G_N$ satisfying the hypothesis of Theorem 11. Second, we are able to pick off *integral* a, b by developing a “boundary theory” for Γ_M ; in case $|\det M| > 1$ this boundary is a self-similar Cantor set whose bilipschitz geometry detects the primitive integral power of $|\det M|$, which is easily seen to finish the proof of Theorem 9.

Problem 12 (Nilpotent-by-cyclic groups). *Extend Theorem 9 and Theorem 10 to the class of finitely-presented nilpotent-by-cyclic groups.*

Of course, as the classification of finitely-generated nilpotent groups is still open, Problem 12 is meant in the sense of reducing the nilpotent-by-cyclic case to the nilpotent case, together with another invariant. This second invariant for a nilpotent-by-cyclic group G will perhaps be the absolute Jordan form of the matrix which is given by the action of the generator of the cyclic quotient of G on the nilpotent kernel of G .

5 Abelian-by-cyclic groups: polycyclic case

The polycyclic, abelian-by-cyclic groups are those Γ_M for which $|\det M| = 1$, so that Γ_M is cocompact and discrete in G_M , hence quasi-isometric to G_M . In this case the proof of Theorem 9 outlined above breaks down, but this is so in part because the answer is quite different: the quasi-isometry classes of polycyclic Γ_M are much coarser than in the nonpolycyclic case, as the former are (conjecturally) determined by the absolute Jordan form up to *real*, as opposed to integral, powers. The key conjecture is:

Conjecture 13 (Horizontal preserving). *Suppose that $|\det M|, |\det N| = 1$, and that (for simplicity) M and N have no eigenvalues on the unit circle. Then every quasi-isometry of $G_M \rightarrow G_N$ is horizontal-respecting.*

The general case of Conjecture 13 (which is slightly more complicated to state) together with Theorem 11 easily imply:

Conjecture 14 (Classification of polycyclic, abelian-by-cyclic groups).

Suppose that $|\det M|, |\det N| = 1$. Then Γ_M is quasi-isometric to Γ_N if and only if there exist nonzero $a, b \in \mathbf{R}$ so that M^a and N^b have the same absolute Jordan form.

Here by M^a we mean $\phi(a)$, where $\phi : \mathbf{R} \rightarrow \mathrm{GL}(n, \mathbf{R})$ is a 1-parameter subgroup with $\phi(1) = M$ (we are assuming that M lies on such a subgroup, which can be assumed after squaring M).

Now let us concentrate on the simplest non-nilpotent example, which is also considered by most people to be one of the central open problems in the field. Thurston's 3-dimensional geometry SOLV is the Lie group G_M where $M \in \mathrm{SL}(2, \mathbf{Z})$ is any matrix with 2 distinct real eigenvalues (up to scaling, it doesn't matter which such M is chosen).

Conjecture 15 (Rigidity of solv). *The 3-dimensional Lie group SOLV is quasi-isometrically rigid: any f.g. group G quasi-isometric to SOLV is weakly commensurable with a cocompact, discrete subgroup of SOLV.*

There is a natural boundary for SOLV which decomposes into two pieces $\partial^s \mathrm{SOLV}$ and $\partial^u \mathrm{SOLV}$; these are the leaf spaces of the weak stable and weak unstable foliations, respectively, of the vertical flow on SOLV, and are both homeomorphic to \mathbf{R} .

The isometry group $\mathrm{Isom}(\mathrm{SOLV})$ acts on the pair $(\partial^s \mathrm{SOLV}, \partial^u \mathrm{SOLV})$ affinely and induces a faithful representation $\mathrm{SOLV} \rightarrow \mathrm{Aff}(\mathbf{R}) \times \mathrm{Aff}(\mathbf{R})$ whose image consists of the pairs

$$(ax + b, a^{-1}x + c), \quad a \in \mathbf{R}^+, b, c \in \mathbf{R}$$

Just as quasi-isometries of hyperbolic space \mathbf{H}^n are characterized by their quasiconformal action on $\partial \mathbf{H}^n$ (a fact proved by Mostow), giving the formula $\mathrm{QI}(H^n) = \mathrm{QC}(\partial \mathbf{H}^n)$, we conjecture:

Conjecture 16 (QI group of solv).

$$\mathrm{QI}(\mathrm{SOLV}) = (\mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{R})) \rtimes \mathbf{Z}/2$$

where $\mathrm{Bilip}(\mathbf{R})$ denotes the group of bilipschitz homeomorphisms of \mathbf{R} , and $\mathbf{Z}/2$ acts by switching factors.

There is evidence for Conjecture 16: the direction \supseteq is not hard to check (see [FS]), and the analogous theorem $\mathrm{QI}(\mathrm{BS}(1, n)) = \mathrm{Bilip}(\mathbf{R}) \times \mathrm{Bilip}(\mathbf{Q}_n)$ was proved in [FM1]. By using convergence groups techniques and a theorem of Hinkkanen on uniformly quasisymmetric groups, we have been able to show:

Conjecture 13 (in the 2×2 case) \Rightarrow Conjecture 16 \Rightarrow Conjecture 15

Here is a restatement of Conjecture 13 in the 2×2 case:

Conjecture 17. *Every quasi-isometry $f: \text{SOLV} \rightarrow \text{SOLV}$ is horizontal respecting.*

Here is one way *not* to prove Conjecture 17.

One of the major steps of [FM1] in studying $\text{BS}(1, n)$ was to construct a model space X_n for the group $\text{BS}(1, n)$, study the collection of isometrically embedded hyperbolic planes in X_n , and prove that for any quasi-isometric embedding of the hyperbolic plane into X_n , the image has finite Hausdorff distance from some isometrically embedded hyperbolic plane.

However, SOLV has quasi-isometrically embedded hyperbolic planes which are *not* Hausdorff close to isometrically embedded ones. The natural left invariant metric on SOLV has the form

$$e^{2t} dx^2 + e^{-2t} dy^2 + dt^2$$

from which it follows that the xt -planes and yt -planes are the isometrically embedded hyperbolic planes. But none of these planes is Hausdorff close to the set

$$\{(x, y, t) \in \text{SOLV} : x \geq 0 \text{ and } y = 0\} \cup \{(x, y, t) \in \text{SOLV} : y \geq 0 \text{ and } x = 0\}$$

which is a quasi-isometrically embedded hyperbolic plane. An even stranger example is shown in Figure 1.

These strange quasi-isometric embeddings from \mathbf{H}^2 to SOLV do share an interesting property with the standard isometric embeddings, which may point the way to understanding quasi-isometric rigidity of SOLV . We say that a quasi-isometric embedding $\phi: \mathbf{H}^2 \rightarrow \text{SOLV}$ is *A-quasivertical* if for each $x \in \mathbf{H}^2$ there exists a vertical line $\ell \subset \text{SOLV}$ such that $\phi(x)$ is contained in the A -neighborhood of ℓ , and ℓ is contained in the A -neighborhood of $\phi(\mathbf{H}^2)$.

In order to study SOLV , it therefore becomes important to understand whether every quasi-isometrically embedded hyperbolic plane is quasi-vertical. Specifically:

Problem 18. *Show that for all K, C there exists A such that each K, C -quasi-isometrically embedded hyperbolic plane in SOLV is A -quasivertical.*

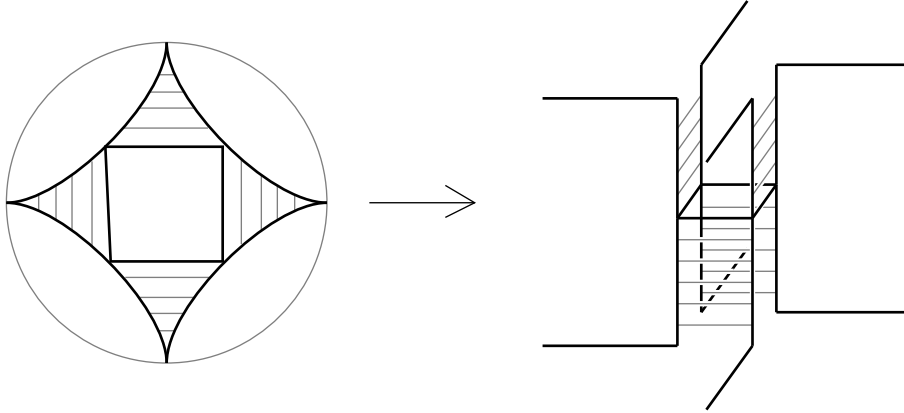


Figure 1: To map \mathbf{H}^2 quasi-isometrically into SOLV, take a regular ideal quadrilateral in \mathbf{H}^2 divided into a regular inscribed square and four triangles each with one ideal vertex. Map the square isometrically to a square in the xy -plane with sides parallel to the axes. Map each triangle isometrically to a triangle in either an xt -plane or a yt -plane, alternating around the four sides of the square. Finally, map each complementary half-plane of the quadrilateral isometrically to a half-plane of either an xt -plane or a yt -plane.

Arguing by contradiction, if Problem 18 is impossible then, fixing K, C and taking a sequence of examples whose quasi-vertical constant A goes to infinity, one can pass to a subsequence and take a renormalized limit to produce a quasi-isometric embedding $\mathbf{H}^2 \rightarrow \text{SOLV}$ whose image is entirely contained in the upper half $t \geq 0$ of SOLV. But we conjecture that this is impossible:

Conjecture 19. *There does not exist a quasi-isometric embedding $\mathbf{H}^2 \rightarrow \text{SOLV}$ whose image is entirely contained in the upper half.*

6 Next steps

While we have already seen that there is a somewhat fine classification of finitely presented, nonpolycyclic abelian-by-cyclic groups up to quasi-isometry, this class of groups is but a very special class of finitely generated solvable groups. We have only exposed the tip of a huge iceberg. An important next layer is:

Problem 20 (Metabelian groups). *Classify the finitely presented (non-*

polycyclic) metabelian groups up to quasi-isometry.

The first step in attacking this problem is to find a workable method of describing the geometry of the natural geometric model of such groups G . Such a model should fiber over \mathbf{R}^n , where n is the rank of the maximal abelian quotient of G ; inverse images under this projection of (translates of) the coordinate axes should be copies of the geometric models of abelian-by-cyclic groups.

Polycyclic versus nonpolycyclic. We've seen the difference, at least in the abelian-by-cyclic case, between polycyclic and nonpolycyclic groups. Geometrically these two classes can be distinguished by the trees on which they act: such trees are lines in the former case and infinite-ended in the latter. It is this branching behavior which should combine with coarse topology to make the nonpolycyclic groups more amenable to attack.

Note that a (virtually) polycyclic group is never quasi-isometric to a (virtually) nonpolycyclic solvable group. This follows from the theorem of Bieri that polycyclic groups are precisely those solvable groups satisfying Poincare duality, together with the quasi-isometric invariance of the latter property (proved by Gersten [Ge] and Block-Weinberger [BW]).

Solvable groups as dynamical systems. The connection of nilpotent groups with dynamical systems was made evident in [Gr1], where Gromov's Polynomial Growth Theorem was the final ingredient (combining with earlier work of Franks and Shub) in the positive solution of the *Expanding Maps Conjecture*: every locally distance expanding map on a closed manifold M is topologically conjugate to an expanding algebraic endomorphism on an infranil manifold (see [Gr1]).

In §4 and §5 we saw how invariants from dynamics give quasi-isometry invariants for abelian-by-cyclic groups. This should be no big surprise: after all, abelian-by-cyclic groups are describable as ascending HNN extensions Γ_M over finitely-generated abelian groups \mathbf{Z}^n . Such HNN extensions can be realized (up to commensurability) as fundamental groups of mapping tori of endomorphisms of n -dimensional tori. Here we see the first examples of how (the geometric model of) a solvable group is actually the phase space of a dynamical system, in this case the suspension semiflow of an endomorphism of a torus (suspension flow of an automorphism when Γ_M is polycyclic).

But Bieri-Strebel [BS1] have shown that *every* finitely presented solvable group is, up to commensurability, an ascending HNN extension with base group a finitely generated solvable group. In this way every finitely presented

solvable group is the phase space of a dynamical system, probably realizable geometrically as in the torus case.

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