Commensurability invariants for nonuniform tree lattices

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1 Introduction

Let X be a locally finite simplicial tree. Then the group $G = \mathrm{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, with a neighborhood basis of the identity consisting of automorphisms fixing larger and larger balls. A *lattice* in G is a discrete subgroup of cofinite (Haar) measure μ. The study of these "tree lattices" generalizes the theory of lattices in rank one Lie groups over nonarchimedean local fields, and provides a remarkably rich theory (see the now-standard reference [BL]). With the right normalization of the Haar measure μ, there is a combinatorial formula:

$$\mathrm{Vol}(\Gamma\backslash\backslash X) := \mu(\Gamma\backslash G) = \sum_{v \in V(A)} \frac{1}{|\Gamma_v|}$$

where the sum is taken over vertices v in a fundamental domain $A \subseteq X$ for the Γ–action, and $|\Gamma_v|$ is the order of the Γ–stabilizer of v. Recall that a lattice $\Gamma < G$ is *uniform* if $\Gamma\backslash G$ is compact, and is *nonuniform* otherwise.

One of the basic problems about a locally compact topological group G is to classify its lattices up to commensurability. Recall that two lattices

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$\Gamma_1, \Gamma_2 \leq G$ are *commensurable in* G if there exists $g \in G$ so that $g\Gamma_1 g^{-1} \cap \Gamma_2$ has finite index in both $g\Gamma_1 g^{-1}$ and Γ_2. Since covolume is multiplicative in index, two commensurable lattices have volumes which are commensurable real numbers, *i.e.*, which have a rational ratio. Other commensurability invariants are harder to come by unless G is linear.

When $G = \mathrm{Aut}(X)$, Bass–Kulkarni proved in [BK] that there is at most one commensurability class of uniform lattices in G. The case of nonuniform lattices, (which exist in abundance by a theorem of Carbone [Ca]), is much more complicated.

In this paper we will concentrate on the case of the biregular tree $X_{m,n}$ with degrees m and n. Bass–Lubotzky [BL] (for $m = n$) and Rosenberg [Ro] (for every $m \geq n \geq 3$) proved that for every real number $r > 0$, there exists a nonuniform lattice Γ in $G = \mathrm{Aut}(X_{m,n})$ with covolume r. This gives uncountably many commensurability classes of nonuniform lattices in G, one for each commensurability class of real numbers.

In Section 3 we find two new commensurability invariants. The *quotient growth type* (see §3.2) is essentially the "growth type" of the quotient graph $\Lambda = \Gamma\backslash X$, which is an equivalence class of functions measuring the growth of combinatorial balls in Λ. The *stabilizer growth type* (see §3.3) measures the growth of the order of the stabilizers of vertices in a fundamental domain as a function of their distance to a fixed basepoint. There is a refinement of this invariant for each prime p.

Our main result (see Sections 4 and 5) is the construction of lattices realizing every possible value of the invariants, which further indicates the richness of nonuniform lattices in G. Informally, we prove:

Theorem 1.1 (Main theorem, informal statement). *Suppose* $3 \leq m \leq n$*, and let* $G = \mathrm{Aut}(X_{m,n})$*. Let* $r > 0$ *be any real number, and let* $f, g \colon \mathbf{N} \to \mathbf{N}$ *be any functions which are possible quotient and stabilizer growth functions, respectively (see §3 for the precise conditions). Then there exists a nonuniform lattice* $\Gamma < G$ *with covolume* r*, quotient growth* f*, and (for* $n > 4$ *composite) stabilizer growth* g*.*

This result is stated precisely as Theorem 5.4 below. In particular we note the following.

Corollary 1.2. *Let $m, n \geq 3$ and let $G = \mathrm{Aut}(X_{m,n})$. Then for each real number $r > 0$, there exist uncountably many commensurability classes of nonuniform lattices in G, each having covolume r.*

As a concrete example, given any $r > 0$ and $1 < \alpha < 2$, we then construct lattices $\Gamma_{r,\alpha} < G$ each with covolume r, and with the growth type of the quotient tree proportional to α^k; the number α is a commensurability invariant. Similarly, we can build lattices whose quotients have "intermediate growth", i.e. whose R-balls have growth type e^{R^β} for any $0 < \beta < 1$. In contrast note that, for lattices in rank one Lie groups over nonarchimedean local fields, it follows from theorems of Lubotzky and Raghunathan (see [Lu]) that such lattices always have quotients with "linear growth type."

We would like to pose the following:

Problem 1.3. *Classify nonuniform lattices in* $\mathrm{Aut}(X_{m,n})$ *up to commensurability.*

As a special case, it would be interesting to give a commensurability classification for lattices all having a fixed quotient, for example a ray.

2 Preliminaries

In this section we present some needed background material.

2.1 Edge-indexed graphs and graphs of groups

In this subsection we briefly recall the basic tools in constructing tree lattices. See the book [BL] by Bass–Lubotzky for a more comprehensive introduction to this material.

A *graph of groups* $\mathbf{A} = (A, \mathcal{A})$ consists of a graph A with vertex set $V(A)$ and edge set $E(A)$, vertex groups $\mathcal{A}_v$ for each $v \in V(A)$, edge groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in E(A)$, and injections $\alpha_e \colon \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0(e)}$ from each edge group into the group at its terminal vertex. If $v_0 \in V(A)$ then the *fundamental group* $\Gamma = \pi_1(\mathbf{A}, v_0)$ acts without edge inversions on the *universal covering tree* $\widetilde{(\mathbf{A}, v_0)}$ with quotient A, so that for each lift $\tilde{e}$ of an edge e the inclusion α_e is isomorphic to the inclusion $\mathrm{Fix}_\Gamma(\tilde{e}) \hookrightarrow \mathrm{Fix}_\Gamma(\partial_0\tilde{e})$. Furthermore, every action (without inversions) arises in this manner from a quotient graph of groups $\Gamma\backslash\backslash X = (\Gamma\backslash X, \mathcal{A})$.

Let $i(e) = [\mathcal{A}_{\partial_0(e)} : \alpha_e\mathcal{A}_e]$, and let $I(\mathbf{A}) = (A, i)$ denote the corresponding *edge-indexed graph.* A *grouping* of an edge-indexed graph (A, i) is a graph of groups $\mathbf{A}$ with $I(\mathbf{A}) = (A, i)$. The grouping is *finite* if all the associated groups are finite and *faithful* if the action on the universal covering tree is faithful.

Note that the universal covering tree $(\widetilde{\mathbf{A}, v_0})$ depends only on the underlying edge indexed graph $I(\mathbf{A})$. In particular, if $\partial_0(e) = v$ then for each lift $\tilde{v}$ of v there are exactly $i(e)$ lifts of e with terminal vertex $\tilde{v}$.

In order to construct groupings of edge-indexed graphs, a useful intermediate step is an ordering. An *ordering* of an edge-indexed graph (A, i) is a function $N\colon V(A) \amalg E(A) \to \mathbf{Q}_+$ such that for each $e \in E(A)$ we have

$$N(e) = N(\overline{e}) = \frac{N\big(\partial_0(e)\big)}{i(e)}.$$

An edge indexed graph is *unimodular* if it admits an ordering. Any two orderings of (A, i) differ by a constant multiple, and thus an ordering is uniquely determined by its value at a single vertex.

An integral valued ordering gives a set of numbers that are combinatorially admissable as the orders of vertex and edge groups of a finite grouping of (A, i). More precisely, if $\mathbf{A} = (A, \mathcal{A})$ is a finite grouping of (A, i) then the orders of the vertex and edge groups define an ordering N with $N(v) = |\mathcal{A}_v|$.

Conversely, any integral ordering N has a finite cyclic grouping, constructed as follows. Let each vertex group [resp. edge group] be finite cyclic with order given by the value of N on that vertex [resp. edge], and let $\alpha_e\colon \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0(e)}$ be given by $[1] \mapsto [i(e)]$. This grouping is effective if and only if the values of N do not have a common factor.

2.2 Haar measure and covolume

Let G be a locally compact topological group with a left invariant Haar measure μ. A discrete subgroup $\Gamma \leq G$ is a *G–lattice* if the covolume $\mu(\Gamma\backslash G)$ is finite.

Suppose G acts on a set S with compact open stabilizers G_s for each $s \in S$. Then every discrete subgroup $\Gamma \leq G$ has finite stabilizers Γ_s, and we define the *S–covolume* of Γ to be the quantity

$$\mathrm{Vol}_S(\Gamma\backslash\backslash S) := \sum_{s\in\Gamma\backslash\backslash S} 1/\left|\Gamma_s\right|.$$

The following theorem relates Haar measure and S–covolume of discrete subgroups.

Theorem 2.1 ([BL], Chapter 1). *Let G be a locally compact topological group acting on a set S with compact open stabilizers and a finite quotient $G\backslash S$. Suppose further that G admits at least one lattice. Then there is a normalization of the Haar measure μ, depending only on the choice of G–set S, such that for each discrete subgroup $\Gamma \leq G$ we have*

$$\mathrm{Vol}_S(\Gamma\backslash\backslash S) = \mu(\Gamma\backslash G).$$

When G acts by automorphisms on a tree X, the usual convention in the literature is to compute covolumes of lattices with respect to the action on the set of all vertices of X. However, other natural choices for S often exist. For instance, one could let S be the set of edges of X, or S could be a union of G–orbits of vertices.

The preceding theorem shows that the particular choice of S is largely a matter of convenience, as long as the same choice is made for all computations.

3 Commensurability invariants

In this section we discuss and construct several commensurability invariants for discrete subgroups of $\mathrm{Aut}(X)$, for any locally finite tree X. The invariants include the covolume, the growth of the quotient graph, and the growth of the orders of vertex stabilizers in the quotient graph of groups. We remark that each of the invariants in this section can be easily extended to commensurability invariants of lattices in the automorphism group of any locally finite simplicial complex.

3.1 Covers of graphs of groups

Suppose X is a locally finite tree and H a discrete subgroup of $\mathrm{Aut}(X)$ acting without inversions on X. A subgroup $\Gamma \leq H$ corresponds to a cover $\Gamma\backslash\backslash X \to H\backslash\backslash X$ of the corresponding quotient graphs of groups, in a sense to be made precise below. In this subsection we examine some features of this correspondence, focusing on the case when Γ has finite index in H. Related results can be found in [BL] and [Ro].

Let $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{B} = (B, \mathcal{B})$ denote the graphs of groups $H\backslash\backslash X$ and $\Gamma\backslash\backslash X$, respectively, and let (A, i) and (B, j) be the underlying edge-indexed graphs. Then there is a natural quotient map $q\colon B \to A$. Now for each $b \in V(B)$ let $a = q(b)$ and choose a lift $\tilde{b}$ of b to X. Then the stabilizer $\mathrm{Fix}_\Gamma(\tilde{b})$ is naturally a subgroup of $\mathrm{Fix}_H(\tilde{b})$, so q induces a monomorphism $\mathcal{B}_b \hookrightarrow \mathcal{A}_a$ of vertex groups, which is well-defined up to conjugacy. Similarly, q induces monomorphisms of edge groups. Thus q induces a *covering of graphs of groups* in the sense of Bass ([Ba]).

Definition 3.1. The *degree* of the cover $q\colon \mathbf{B} \to \mathbf{A}$ is the quantity

$$\deg(q) := \sum_{b \in q^{-1}(a)} [\mathcal{A}_a : \mathcal{B}_b] \tag{1}$$

for any fixed vertex $a \in V(A)$.

Proposition 3.2. *The degree of q does not depend on the choice of $a \in V(A)$. Furthermore, if H and Γ each have finite covolume, then* $\deg(q) = [H : \Gamma]$.

For instance, if each inclusion $\mathcal{B}_b \hookrightarrow \mathcal{A}_a$ is an isomorphism, then each point of A lifts to exactly $\deg(q)$ points in B, and the graph map $q\colon B \to A$ is a covering of spaces whose topological degree is equal to $\deg(q)$. Alternately, if the graph map q is an isomorphism, then each point of A has exactly one lift to B, and we have $[\mathcal{A}_a : \mathcal{B}_b] = \deg(q)$ whenever $q(b) = a$.

Proof of Proposition 3.2. Since q is a cover of graphs of groups, it is easy to see that the induced map $q\colon (B, j) \to (A, i)$ is a covering of edge-indexed graphs in the sense of Bass–Lubotzky ([BL]). In other words, for each vertex

$b \in V(B)$ with $q(b) = a$ and each edge e with $\partial_0(e) = a$, we have

$$i(e) = \sum_{f \in q_b^{-1}(e)} j(f)$$

where q_b denotes the restriction of q to the edges in $\partial_0^{-1}(b)$.

Applying the definition of i and j, gives

$$\frac{|\mathcal{A}_a|}{|\mathcal{A}_e|} = \sum_{f \in q_b^{-1}(e)} \frac{|\mathcal{B}_b|}{|\mathcal{B}_f|}$$

or equivalently

$$[\mathcal{A}_a : \mathcal{B}_b] = \sum_{f \in q_b^{-1}(e)} [\mathcal{A}_e : \mathcal{B}_f]$$

Summing over all vertices b with $q(b) = a$ gives

$$\sum_{b \in q^{-1}(a)} [\mathcal{A}_a : \mathcal{B}_b] = \sum_{f \in q^{-1}(e)} [\mathcal{A}_e : \mathcal{B}_f]$$

Notice that the quantity on the left hand side is independent of the choice of $e \in \partial_0^{-1}(a)$, and the right hand side is unchanged if we replace e with $\overline{e}$. Hence both boundary vertices of an edge give the same degree. Since A is connected, it follows that $\deg(q)$ is independent of the choice of $a \in V(A)$.

Dividing both sides of (1) by $|\mathcal{A}_a|$ gives

$$\frac{\deg(q)}{|\mathcal{A}_a|} = \sum_{b \in q^{-1}(a)} \frac{1}{|\mathcal{B}_b|}$$

which immediately leads to the formula $\deg(q)\,\mathrm{Vol}(\mathbf{A}) = \mathrm{Vol}(\mathbf{B})$. (Here volumes are computed with respect to the set of all vertices.) Thus

$$\deg(q) = \frac{\mu(\Gamma\backslash G)}{\mu(H\backslash G)} = [H : \Gamma]$$

where the second equality is straightforward (see, *e.g.*, [BL]). $\square$

3.2 Quotient growth type

Let $f, g \colon \mathbf{N} \to \mathbf{N}$ be any two functions. We say that $f \preceq g$ if

$$f(k) \leq \alpha\, g(k + \beta)$$

for some $\alpha, \beta \in \mathbf{N}$. We say f and g are *equivalent*, denoted $f \simeq g$, if $f \preceq g$ and $g \preceq f$. Note that there is no multiplicative factor inside the argument of g.

Let A be a locally finite graph with basepoint $\star$, and let $g\colon \mathbf{N} \to \mathbf{N}$ be such that $g(k)$ the number of vertices in the combinatorial ball of radius k centered at $\star$. We define the *growth type* of $(A, \star)$ to be the equivalence class of the function g. It is easy to see that changing the basepoint does not change the growth type.

Now suppose $\Gamma \leq H$ are two discrete subgroups of $\mathrm{Aut}(X)$ with $[H : \Gamma] < \infty$. Let $q\colon \Gamma\backslash\backslash X \to H\backslash\backslash X$ be the corresponding finite degree covering map. The following result is an immediate corollary of Proposition 3.2.

Corollary 3.3. *For each vertex $a \in V(H\backslash X)$, the set $q^{-1}(a) \subseteq V(\Gamma\backslash X)$ contains at most* $\deg(q)$ *vertices.* $\square$

The following is an easy exercise using the previous corollary.

Proposition 3.4. *Let X be a locally finite tree, and let Γ and Γ' be commensurable lattices of* $\mathrm{Aut}(X)$*. Then $\Gamma\backslash X$ and $\Gamma'\backslash X$ have the same growth type.* $\square$

3.3 Stabilizer growth type

Let $\mathbf{A} = (A, \mathcal{A})$ be a graph of finite groups such that A is a locally finite graph. Fix a basepoint $\star \in V(A)$. The *stabilizer growth type* of $(\mathbf{A}, \star)$ is the equivalence class of the function $g\colon \mathbf{N} \to \mathbf{N}$ such that $g(k)$ is the order of the largest vertex group associated to any vertex in the combinatorial ball of radius k centered at $\star$. Clearly the equivalence class of g does not depend on the choice of basepoint $\star$.

A finer invariant of $\mathbf{A}$ is the *p–stabilizer growth type* for any fixed prime p, defined as follows. If a finite group K has order $p_1^{n_1} \cdots p_\ell^{n_\ell}$ for distinct primes $p_1, \dots, p_\ell$, then the *p_i–order* of K, denoted $|K|_{p_i}$, is the natural number $p_i^{n_i}$. We define the *p–stabilizer growth type* of $(\mathbf{A}, \star)$, for p prime, as the equivalence class of the function $g\colon \mathbf{N} \to \mathbf{N}$ such that $g(k)$ is the maximal p–order of any vertex group associated to a vertex in the combinatorial ball of radius k centered at $\star$. The p–stabilizer growth type also does not depend on the choice of basepoint.

Suppose $\Gamma \le H$ are two discrete subgroups of $\mathrm{Aut}(X)$, for some locally finite tree X, such that $[H : \Gamma] < \infty$, and let $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{B} = (B, \mathcal{B})$ be the graphs of groups $H\backslash\backslash X$ and $\Gamma\backslash\backslash X$ respectively. Let $q\colon \mathbf{B} \to \mathbf{A}$ denote the associated covering map. The following lemma is an immediate corollary of Proposition 3.2.

Lemma 3.5. *For each $a \in V(A)$, each $b \in q^{-1}(a)$, and each prime p, the order [resp. p–order] of $\mathcal{B}_b$ divides the order [p–order] of $\mathcal{A}_a$. Furthermore, we have*

$$1 \le \frac{|\mathcal{A}_a|}{|\mathcal{B}_b|} \le \deg(q) \quad \textit{and} \quad 1 \le \frac{|\mathcal{A}_a|_p}{|\mathcal{B}_b|_p} \le \deg(q).$$

□

Corollary 3.6. *Let X be a locally finite tree, and let Γ and Γ' be commensurable lattices in $\mathrm{Aut}(X)$. Then $\Gamma\backslash\backslash X$ and $\Gamma'\backslash\backslash X$ have the same stabilizer growth type and also the same p–stabilizer growth type for each prime p.* □

Remark 3.7. Other related commensurability invariants exist using variations on the preceding definitions. For instance, combinatorial spheres are coarsely preserved by finite covers. So one could consider the growth of the minimal order, or minimal p–order, of a vertex group in the combinatorial sphere.

More generally, given a discrete $\Gamma \le \mathrm{Aut}(X)$ there is a spectrum of orders, and p–orders, of vertex groups in each combinatorial sphere of X, which is coarsely preserved under commensurability. It seems possible that a weighted variant of Patterson–Sullivan measure on ∂X could be associated to each commensurability class of discrete subgroup $\Gamma \le \mathrm{Aut}(X)$. The idea is to weight each vertex by the relative size of the associated vertex group compared to other vertex groups in the same sphere.

4 Constructions of lattices

As mentioned in the introduction, Bass–Lubotzky [BL] (for $m = n$) and Rosenberg [Ro] (for every $m \ge n \ge 3$) proved that for every real number

$r > 0$, there exists a nonuniform lattice Γ in $G = \mathrm{Aut}(X_{m,n})$ with covolume r. In this section we give a different construction which will allow us the flexibility of exhibiting the lattices promised in Theorem 5.2.

The construction will occur in three steps: the basic building block will be given in §4.1; these will be used in §4.2 to give lattices with arbitrary covolume bigger than a certain constant (depending on m, n); and the constant will be improved to zero in §4.3.

4.1 Star trees and their associated lattices

Let $X_{m,n}$ denote the biregular tree with degrees m and n, for $3 \leq n \leq m$. In this subsection we construct a family of discrete subgroups of $G = \mathrm{Aut}(X_{m,n})$ which will be the basic building blocks for our construction of the lattices. Each subgroup Γ will constructed as the fundamental group of a graph of groups $\mathbf{A} = (A, \mathcal{A})$ with A a tree.

A *star of degree* m is a tree with one vertex of degree m, called the *center* of the star, and m vertices of degree one. A *star tree of degree* m is a bipartite tree with vertex sets V_0 and V_1, such that every vertex of V_0 has degree m and every vertex of V_1 has degree either one or two. Any star tree of degree m can be uniquely expressed as a union of stars of degree m provided $m \geq 3$.

Let A be a star tree of degree m with basepoint $v_0 \in V_0$. Put the following edge indexing on A. For each edge e such that $\partial_0(e) \in V_0$, set $i(e) = 1$. For each edge e such that $\partial_0(e)$ is a vertex of V_1 with degree one, set $i(e) = n$. If $v \in V_1$ has degree two, let e and e' be the two edges with terminal vertex v, chosen so that e is closer than e' to v_0 . Let $i(e) = n - 1$ and $i(e') = 1$. Note that the edge-indexed graph (A, i) has universal cover $X_{m,n}$.

Since A is a tree, any edge indexing on A is unimodular. In particular, if we let N be the ordering of (A, i) normalized so that $N(v_0) = 1$, it is easy to see that N is integral.

The *canonical grouping* of (A, i) is the cyclic grouping $\mathbf{A}$ associated to N, which is necessarily faithful, since each lift of v_0 has a trivial stabilizer. The canonical grouping is a graph of finite groups $\mathbf{A}$ with universal cover $X_{m,n}$.

Convention 4.1 (Star tree covolumes). For convenience, we compute all covolumes of star tree lattices with respect to the set of vertices V_0, using

Theorem 2.1. (In case $m = n$, we first pass to the index two subgroup $G_0 < G$ stabilizing the lift of V_0.) For instance, if A consists of just a single star, then its canonical grouping $\mathbf{A}$ has V_0–covolume 1. On the other hand, the covolume with respect to the full set of vertices $V = V_0 \coprod V_1$ is $(n+m)/n$. Thus the reader can multiply any V_0–covolume by the factor $(n+m)/n$ to obtain the corresponding V–covolume.

For each $v \in V_0$, the *level* of v, denoted $\ell(v)$, is the number of vertices of V_1 on the unique edge-path from v to the basepoint v_0. If $\mathbf{A}$ is the canonical grouping $\mathbf{A}$ of a star tree A, then for each $v \in V_0$, we have $N(v) = (n-1)^{\ell(v)}$. Therefore the covolume of $\mathbf{A}$ is given by

$$\sum_{v \in V_0} \frac{1}{(n-1)^{\ell(v)}},$$

establishing the following theorem.

Theorem 4.2. *Suppose $3 \leq n \leq m$, and let A be any star tree of degree m. Then there is a discrete subgroup $\Gamma(A) \leq \mathrm{Aut}(X_{m,n})$, such that $\Gamma(A)\backslash X_{m,n} = A$, and $\Gamma(A)$ has covolume*

$$\sum_{v \in V_0(A)} \frac{1}{(n-1)^{\ell(v)}}$$

Example 4.3. A *star ray* is a minimal infinite star tree. It consists of an infinite sequence of stars connected end to end. A star ray R of degree 4 is illustrated in Figure 1(a). The ordering of the canonical grouping of (R, i) is shown in Figure 1(b) in the case $m = 4$ and $n = 3$. This grouping has covolume

$$\sum_{\ell=0}^{\infty} \frac{1}{(n-1)^{\ell}} = \frac{n-1}{n-2} = 2.$$

4.2 Arbitrary volumes bounded away from zero

In this subsection we construct uncountably many lattices in G with covolume κ, for each $\kappa > \kappa_0 := (n-1)/(n-2)$.

If the canonical grouping of a star tree A is to be a nonuniform lattice, it is necessary that A contains at least one ray. Thus the covolume of any

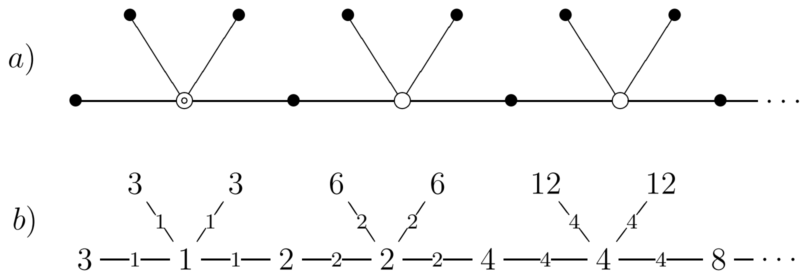

Figure 1: (a) A star ray R of degree 4. The basepoint is indicated by a double circle. (b) The ordering of the canonical grouping of (R, i) in the case $m = 4$ and $n = 3$.

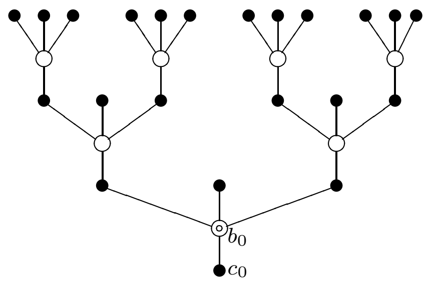

Figure 2: The tree B_3 in the case $m = 4$ and $n = 3$.

nonuniform lattice produced in this manner is bounded below by κ_0, which is the covolume associated to a star ray R (see Example 4.3).

For each natural number p, let B_p be a finite star tree of degree m with basepoint b_0 such that $V_0(B_p)$ contains exactly $(n-1)^j$ vertices of level j for $j = 0, \ldots, p-1$ and no vertices of any higher level. Let $c_0 \in V_1(B_p)$ be a vertex of degree one adjacent to b_0. The tree B_3 is illustrated in Figure 2 in the case $m = 4$, and $n = 3$.

Consider the star tree obtained by gluing a copy of B_p to the star ray R along the vertex c_0 so that b_0 has level j with respect to the basepoint of R. We refer to this process as *gluing B_p to R at level j*. Such a gluing increases the covolume by precisely

$$\frac{p}{(n-1)^j}.$$

It is now easy to construct lattices with any given covolume $\kappa > \kappa_0$, as follows. Choose a sequence of natural numbers (e_j) such that

$$\rho := \kappa - \kappa_0 = \frac{e_1}{n-1} + \frac{e_2}{(n-1)^2} + \cdots + \frac{e_j}{(n-1)^j} + \cdots$$

and let A be the tree obtained by gluing for each $j > 0$ a copy of B_{e_j} to R at level j. The canonical grouping of A is then a lattice Γ with covolume κ.

Choosing the sequence (e_j) is similar to writing ρ in base $n-1$, except we do not require that each e_j be smaller than $(n-1)$. Consequently, there is a large amount of flexibility in choosing the e_j, even if we restrict ourselves to bounded sequences. In particular, by varying the sequence (e_j), it is easy to obtain uncountably many lattices with a given covolume. We will take advantage of this flexibility below

4.3 Arbitrarily small volumes

In this subsection we modify the lattices constructed above to produce lattices with arbitrarily small covolumes.

Recall that a star tree A of degree m gives rise to an edge-indexed graph (A, i) with universal cover $X_{m,n}$. Thus far, we have only considered the covolume of the canonical grouping of (A, i). Consequently, we have not produced lattices with covolume arbitrarily close to zero. In order to achieve smaller covolumes, we must increase the orders of the edge and vertex groups.

The main source of tension in increasing the orders of the local groups is that the grouping must remain faithful. If a grouping $(\mathbf{A}, \mathcal{A})$ is not faithful, then the kernel of the action on the universal cover is a group K which can be embedded as a nontrivial normal subgroup in every edge group and vertex group in a manner which is equivariant with respect to the edge maps (see [Ba, I.1.23]).

Loosely speaking, a nonuniform lattice arising as the canonical grouping of a star tree is based on a sequence of inclusions

$$G_0 \xrightarrow{\iota_0} G_1 \xrightarrow{\iota_1} \cdots \xrightarrow{\iota_{j-1}} G_j \xrightarrow{\iota_j} \cdots,$$

where $|G_j| = (n-1)^j$. Our strategy now is to find a group H and a sequence of maps $\phi_j \colon H \to \mathrm{Aut}(G_j)$ equivariant with respect to the ι_j. The ι_j then induce a sequence of inclusions

$$G_0 \rtimes_{\phi_0} H \longrightarrow G_1 \rtimes_{\phi_1} H \longrightarrow \cdots \longrightarrow G_j \rtimes_{\phi_j} H \longrightarrow \cdots$$

Since G_0 is the trivial group, the corresponding grouping is faithful if and only if H does not have a subgroup H' which is normal in every semidirect product, or equivalently if each nontrivial element $h \in H$ acts nontrivially on some G_i.

Note that the volume of this new lattice equals $\rho / |H|$, where ρ is the volume of the original lattice, since the new ordering differs from the old only by the multiplicative factor $|H|$. To get arbitrarily small covolumes, we want to be able to choose the group H arbitrarily large.

It remains to produce specific groups H and G_j and actions $H \to \mathrm{Aut}(G_j)$ satisfying the preceding properties. To this end, fix a natural number k, and define

$$G_j = \begin{cases} \mathbf{Z}/(n-1)^j\mathbf{Z}, & \text{if } j \leq k; \\ \mathbf{Z}/(n-1)^k\mathbf{Z} \times \mathbf{Z}/(n-1)^{j-k}\mathbf{Z}, & \text{if } j > k. \end{cases}$$

Let $\iota_j \colon G_j \to G_{j+1}$ be defined as follows. If $j < k$, then ι_j is multiplication by $n-1$, and if $j \geq k$, it is the identity on the first factor and multiplication by $n-1$ on the second factor.

We now let H be the multiplicative group of the ring $\mathbf{Z}/(n-1)^k\mathbf{Z}$. There is a natural monomorphism $H \hookrightarrow \mathrm{Aut}(G_k)$ given by multiplication, which pulls back to maps $H \to \mathrm{Aut}(G_j)$ for each $j \leq k$. For $j \geq k$, let H act by ϕ_k on the left factor and trivially on the right factor. It is easy to see that this defines a sequence of maps $\phi_j \colon H \to \mathrm{Aut}(G_j)$ equivariant with respect to the ι_j. Furthermore, by construction, every element of H acts nontrivially on G_k. Since the order of H is at least $(n-1)^{(k-1)}$ we acheive arbitrarily small covolumes as $k \to \infty$.

For each star tree A of degree m, the preceding discussion can be applied to produce finite groupings of (A, i) with arbitrarily small covolumes as follows. For each vertex $v \in V_0$, set $\mathcal{A}_v = G_{\ell(v)} \rtimes H$. If e is an edge with $\partial_0(e) = v$, set $\mathcal{A}_e = \mathcal{A}_v$. If $w \in V_1$ has degree one and e is the edge with $\partial_0(e) = w$, set $\mathcal{A}_w = \mathcal{A}_e \times \mathbf{Z}/n\mathbf{Z}$ and let the edge map be the inclusion of the first factor. If $w \in V_1$ has degree two, then $\partial_0^{-1}(w)$ consists of two edges

e_1 and e_2 such that $\mathcal{A}_{e_1} = G_i \rtimes H$ and $\mathcal{A}_{e_2} = G_{i+1} \rtimes H$. In this case, set $\mathcal{A}_w = \mathcal{A}_{e_2}$, and let the map $\mathcal{A}_{e_1} \hookrightarrow \mathcal{A}_w$ be the map $G_i \rtimes H \hookrightarrow G_{i+1} \rtimes H$ induced by ι_i as above.

By the argument above, this construction produces for each k a finite grouping $\mathbf{A}_k$ of (A, i). The grouping $\mathbf{A}_k$ is faithful provided that the tree A contains at least one vertex at level k. Thus if A is infinite, $\mathbf{A}_k$ is faithful for every k.

5 Realizing all values of the invariants

In this section we state the precise version of Theorem 1.1 (see Theorem 5.4 below) and use the constructions given in §4 to exhibit its proof.

5.1 Arbitrary growth type and covolume

Throughout this section, all quotient growths should be computed with respect to the metric on $X_{m,n}$ in which each edge has length $1/2$. In this metric, the function ℓ measures the distance from a vertex to the basepoint.

A function $f\colon \mathbf{N} \to \mathbf{N}$ is an *acceptible quotient growth function* if it satisfies

$$f(0) = 1, \quad 1 \leq f(j+1) \leq 2f(j), \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{f(j)}{2^j} < \infty.$$

(The third condition is essential only in the case when $n = 3$.)

Lemma 5.1. *Let f be any acceptible quotient growth function. Then there is a star tree T_f of degree m whose growth is equivalent to f. Furthermore, we may assume that T_f has a vertex $c_0 \in V_1(T_f)$ of degree one adjacent to the basepoint.*

Proof. Construct T_f with exactly $f(j)$ stars at level j. The inequality $1 \leq f(j+1) \leq 2f(j)$ quarantees that this can always be done, since each level contains at most twice the number of stars as the previous level and $2 \leq m - 1$. $\square$

Theorem 5.2. *Let f be any acceptible quotient growth function, and κ any positive real number. Then there is a lattice Γ in $\mathrm{Aut}(X_{m,n})$ with covolume κ and quotient growth type f. In particular, there are uncountably many commensurability classes of lattices with covolume κ, and also uncountably many commensurability classes with growth type f.*

Proof. We will construct our lattices as canonical groupings of star trees. Without loss of generality, we may assume that $\kappa > \kappa_0$, since the techniques of §4.3 can then be used to obtain arbitrary positive covolumes.

Let T_f be a star tree of degree m with growth equivalent to f, as given by Lemma 5.1. The canonical grouping of such a tree is a lattice with finite covolume ν.

Fix a natural number k sufficiently large that $\nu/(n-1)^k < \kappa - \kappa_0$, and choose a bounded sequence of natural numbers $(e_j)_{j\neq k}$ so that

$$\kappa - \kappa_0 = \frac{e_1}{n-1} + \cdots + \frac{e_{k-1}}{(n-1)^{k-1}} + \frac{\nu}{(n-1)^k} + \frac{e_{k+1}}{(n+1)^{k+1}} + \cdots$$

Form a star tree $A(f,\kappa)$ by gluing T_f to the star ray R at level k and for each $j \neq k$ gluing a copy of B_{e_j} at level j. Since the B_{e_j} have uniformly bounded depth, they do not affect the growth type of A, which is therefore equivalent to f. Furthermore, the canonical grouping of A has covolume κ by construction.

The last assertion of the theorem follows from the fact that for each $\alpha \in (1,2)$ there is an acceptible quotient growth function equivalent to $j \mapsto \alpha^j$ and two such functions are inequivalent whenever $\alpha \neq \alpha'$. □

5.2 Arbitrary stabilizer growths

In the previous sections we constructed lattices in $\mathrm{Aut}(X_{m,n})$ as groupings of the canonical edge indexing of a star tree. All nonuniform lattices arising from such a canonical edge indexing necessarily have the same stabilizer growth type, namely the type of the exponential function given by $h(k) = (n-1)^k$. In this subsection we construct lattices with different stabilizer growths by varying the edge indexing. The techniques of this subsection require that n be composite.

As in the previous subsection, we continue to compute all growth functions using the metric in which each edge has length 1/2.

We say that a sequence $s = (s_k)$ is *n–admissible* if each s_k is an integer dividing n and if $2 < s_k \leq n$. For each k, let $r_k := n/s_k$. Given an n–admissible sequence s and an infinite star tree A of order m with basepoint v_0, there is an edge indexing $I(A, s)$ with universal cover $X_{m,n}$ defined as follows. If $\partial_0(e) \in V_0(A)$, set $i(e) := 1$. If $\partial_0(e) \in V_1(A)$ has degree one, set $i(e) := n$. If $w \in V_1$ has degree two, then $\partial_0^{-1}(w) = \{e, e'\}$, where e is closer to the basepoint v_0 than e'. Then $\ell\big(\partial_1(e)\big) = k - 1$ and $\ell\big(\partial_1(e')\big) = k$ for some k. In this case, set

$$i(e) := n - r_k = r_k(s_k - 1) \quad \text{and} \quad i(e') := r_k$$

Observe that the canonical edge indexing is the special case when $s_k = n$ and $r_k = 1$ for all k.

Let N be the ordering associated to I such that $N(v_0) = 1$. If $v, v' \in V_0(A)$ with $\ell(v) = k - 1$ and $\ell(v') = k$, then we have

$$N(v') = N(v)\,(s_k - 1) = (s_1 - 1) \cdots (s_k - 1)$$

In particular, N is integral and $N(v)$ depends only on the level of $v \in V_0(A)$. The value $N(w)$ for any $w \in V_1$ is within a factor of n of $N(v)$ for some $v \in V_0$ at a combinatorial distance 1 from w. Thus we may ignore vertices in V_1 when considering the stabilizer growth type of N. It is now clear that N has stabilizer growth type equivalent to the function h given by

$$h(k) = (s_1 - 1) \cdots (s_k - 1)$$

In particular, if $t > 2$ is any nontrivial factor of n (*i.e.*, not equal to either 1 or n), then for each real number $\lambda \in [t-1, n-1]$ there is a sequence $s = (s_k)$ with $s_k \in \{t, n\}$ such that the ordering N associated to s has stabilizer growth equivalent to the function $k \mapsto \lambda^k$. Furthermore, distinct choices of λ give rise to inequivalent growth types. Thus we have established the following result.

Proposition 5.3. *Suppose that $3 \leq m \leq n$ and that $n > 4$ is composite. Then $G = \mathrm{Aut}(X_{m,n})$ admits uncountably many nonuniform lattices with distinct stabilizer growth types.*

Notice that the stabilizer growth type of any ordering N associated to an n–admissible sequence s depends only on the choice of s. The growth type is independent of the choice of ordering N of I, and is also independent of the choice of infinite star tree A.

Our goal for the remainder of this subsection is to modify the constructions in the previous subsections to produce lattices modeled on an arbitrary n–admissible sequence s that also have arbitrary acceptible quotient growth and arbitrary covolume. Note that these constructions involved modifying only the tree A and the grouping of I. Thus they do not affect the stabilizer growth of the corresponding lattices.

Theorem 5.4. *Suppose that $3 \leq m \leq n$ and that $n > 4$ is composite. Let f be any acceptible quotient growth function, $s = (s_k)$ any n–admissible sequence, and κ any positive real number. Then $G = \mathrm{Aut}(X_{m,n})$ contains nonuniform lattices with quotient growth type f, covolume κ, and stabilizer growth type given by the function*

$$h(0) = 1; \quad h(j) = (s_1 - 1) \cdots (s_j - 1)$$

Proof. Let R be the star ray of degree m with basepoint v_0, let N be the ordering associated to the edge indexing $I(R, s)$ such that $N(v_0) = 1$, and let

$$\kappa_0 := \sum_{j=0}^{\infty} \frac{1}{h(j)}$$

which is the covolume of the ordering N.

We first consider the special case when $\kappa > \kappa_0$, as in §4.2. Let T_f be a star tree with growth f, which has a vertex $c_0 \in V_1(T_f)$ of degree one adjacent to the basepoint. Gluing T_f onto the star ray R at any level k, has the effect of increasing the covolume of the ordering N by

$$\nu_k := \sum_{j=0}^{\infty} \frac{f(j)}{h(j+k)}$$

Since s is n–admissible, it follows that $s_j - 1 \geq 2$, so that $h(j) \geq 2^j$. Therefore

$$\nu_k \leq 2^{-k} \sum_{j=0}^{\infty} \frac{f(j)}{2^j}$$

The preceding sum converges since f is an acceptible growth function. As in the proof of Theorem 5.2, we fix a natural number k sufficiently large that $\nu_k < \kappa - \kappa_0$, and glue T_f to R at level k.

The next step is to modify the construction of the bounded trees B_p from §4.2 to take into account the variable stabilizer growth provided by the sequence s. For each $p, q \in \mathbf{N}$ let $B_{p,q}$ be a finite depth star tree of degree m with exactly $h(q+j)/h(q)$ vertices of level j for $j = 0, \ldots, p-1$ and no vertices of any higher level. Thus the process of gluing $B_{p,q}$ to R at level q contributes $h(q+j)/h(q)$ vertices of level $q+j$ for each $j = 0, \ldots, p-i$. So $B_{p,q}$ increases the covolume of N by

$$\sum_{j=0}^{p-1} \frac{1}{h(q)} = \frac{p}{h(q)}$$

We now construct a star tree A with growth type f and covolume κ as follows. Choose a bounded sequence $(e_j)_{j \neq k}$ satisfying

$$\kappa - \kappa_0 = \frac{e_1}{h(1)} + \cdots + \frac{e_{k-1}}{h(k-1)} + \nu_k + \frac{e_{k+1}}{h(k+1)} + \cdots$$

Then the star tree A is formed by gluing T_f onto the star ray R at level k, and for each $j \neq k$ gluing a copy of $B_{e_j,j}$ at level j. The resulting tree A, with the induced ordering N, has quotient growth type f, covolume κ, and stabilizer growth type h, as desired.

Finally we must consider the case when $\kappa \leq \kappa_0$. But this is easily dealt with by a construction similar to that in §4.3, replacing $(n-1)^j$ with $h(j)$ throughout. Therefore, given any star tree A and any n–admissible sequence s, the associated edge indexing $I(A, s)$ admits faithful finite groupings with arbitrarily small covolumes. Since quotient growth and stabilizer growth types depend only on the underlying edge indexed graph, this process completes the proof of the theorem. $\square$

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