Week 2, Due Mon 10/9

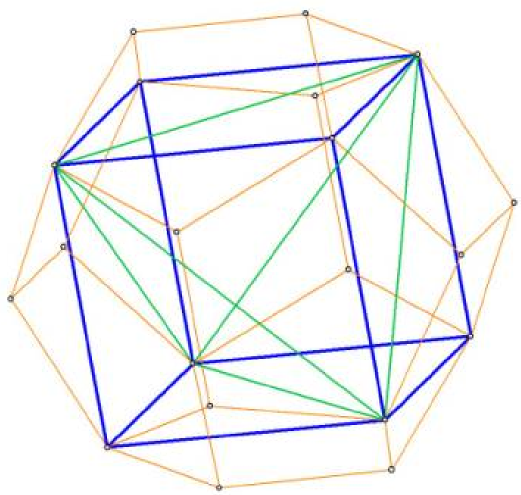
1. If σ is an element of S_n, then σ has a cycle decomposition into disjoint cycles of various lengths (let us include 1-cycles). Since disjoint cycles commute, the shape of the element is determined by the lengths of the various cycles, which we can assume are put in decreasing order. Any two elements with the same cycle shape are conjugate, so the conjugacy classes are determined by writing n ($= 52$ say) as a sum of decreasing integers.

   (a) Find (with proof) the conjugacy class in S_{52} with the largest number of elements.

   (b) Find (without proof) the conjugacy class in S_{52} which contains the element of largest order. (This question is somewhat computational so an explanation of your strategy plus the answer is sufficient.)

2. Fix $k \leq n$ be even. Prove that every element in S_n can be written as a product of k-cycles.

3. Let D be a regular dodecahedron. You may assume for this question that it is possible to inscribe a cube C on the vertices of D thus: **Remember the following distinction:** An object X in $\mathbf{R}^3$



   is **fixed pointwise** by g if every point on X is fixed by g, that is, if $gx = x$ for *every* $x \in X$. An object $X \in \mathbf{R}^3$ is **preserved** by g if every point on X maps to another (possibly different) point on X, i.e., for all $x \in X$ there exists $y \in X$ such that $gx = y$. As an example, the circle centered at the origin is preserved by any rotation through the origin, but it is not fixed pointwise unless the rotation is trivial.

   - If F is a face, call a line between two edges an *internal line* if it is a line between two vertices of F which are not adjacent. That is, a line between two vertices of the pentagon which is not an edge of the pentagon.
   - Observe that the cube C has 12 edges, and that each edge lies on exactly one of the twelve faces of D as an internal line.
   - Choose a face F of D, and let g be the symmetry of D of order 5 which is rotation by $2\pi/5$ through the line passing through the middle of F and the middle of the opposite face $-F$.

   (a) Label the vertices of a face F from 1 to 5. Suppose that $C = C_{(1,3)}$ intersects F in the internal edge from 1 to 3. [THERE'S NO QUESTION HERE, ADVANCE STRAIGHT TO GO!]

(b) Show that for any such g, the five cubes $C_{(1,3)}$, $C_{(2,4)}$, $C_{(3,5)}$, $C_{(1,4)}$, and $C_{(2,5)}$ obtained by applying the powers of g to each cube are distinct, because they intersect F in different internal lines (which are the lines between vertices indicated by the notation).

(c) Show that *any* symmetry of D takes C to one of these five cubes. Hint: show that if there is a sixth cube arising from C by a symmetry of D, there exists a pair of cubes which share two vertices $\mathbf{v}$ and $\mathbf{w}$ on F lying on an internal line of F which are connected by an edge of the cube. Given a cube centered at the origin with vertices $\mathbf{v}$ and $\mathbf{w}$ and $|\mathbf{v}| = |\mathbf{w}|$ connected by an edge, show that the eight vertices of the cube are as follows:

$$\pm\mathbf{v}, \pm\mathbf{w}, \pm\mathbf{u} \pm \left(\frac{\mathbf{v}-\mathbf{w}}{2}\right),$$

where $\mathbf{u}$ is the (unique up to a $\pm$ sign) vector with $3|\mathbf{u}|^2 = 2|\mathbf{v}|^2 = 2|\mathbf{w}|^2$ and $\mathbf{u}.\mathbf{v} = \mathbf{u}.\mathbf{w} = 0$, and hence deduce that these two cubes are the same cube.

(d) Let $\mathbf{v_i}$ indicate the vector corresponding to a vertex i of F. Deduce there are exactly two cubes which have $\mathbf{v_i}$ as a vertex, and that the only vertices these two cubes have in common is $\pm\mathbf{v}_i$.

(e) (*) Show that any rigid motion of D (an element of SO(3) preserving D) permutes the 5 cubes. Hint: show that if a symmetry σ preserves the two cubes passing through $\mathbf{v_i}$ then it preserves their intersection and deduce that

$$\sigma\mathbf{v}_i = \pm\mathbf{v}_i.$$

Deduce that this identity must hold for every i, and use this (and HW1) to show this implies that σ is the identity.

(f) Deduce that the symmetry group of the dodecahedron is a subgroup of S_5 of order 60.

4. Embed the cube inside $\mathbf{R}^3$ so that the centers of each face are at $A = (1,0,0)$, $B = (-1,0,0)$, $C = (0,1,0)$, $D = (0,-1,0)$, $E = (0,0,1)$, and $F = (0,0,-1)$. Considering the symmetry group of C as a subgroup of SO(3), write down the matrix of SO(3) corresponding to the following elements:

(a) $\sigma = (A,C,E)(B,D,F)$

(b) $\tau = (C,E,D,F)$

(c) $\sigma\tau = (A,C,E)(B,D,F)(C,E,D,F) = (A,C)(B,D)(E,F)$