Week 4, Due Mon 10/23

1. Let H and K be normal subgroups of G such that $H \cap K$ is trivial. Prove that $xy = yx$ for all $x \in H$ and $y \in K$. (3.1, (42)).

2. Show that S_4 does not have a normal subgroup of order 3 or order 8.

3. If H is a subgroup of G, define the **normalizer** of H to be:
$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}.$$
   (a) Prove that $N_G(H) = G$ if and only if H is normal.
   (b) Prove that $N_G(H)$ contains H.
   (c) Prove that H is a *normal* subgroup of $N_G(H)$.
   (d) Compute $N_G(H)$ for the following pairs (G, H):
      i. $(S_4, \langle(1234)\rangle)$,
      ii. $(S_5, \langle(12345)\rangle)$,

4. Prove that the subgroup N generated by elements of the form $x^{-1}y^{-1}xy$ for all x, $y \in G$ is normal. (3.1 (41)).

5. Prove that if $G/\mathrm{Z}(G)$ is cyclic, then G is abelian. (For a hint, see 3.1 (36)).

6. Let G be a finite group, and let $H \subset G$ be a subgroup of index two — i.e. $|G|/|H| = 2$. Prove that H is normal.

7. Let G be a finite group, and let $H \subset G$ be a subgroup of index three — i.e. $|G|/|H| = 3$. Show that H is not necessarily normal.

8. **Automorphism Groups**. (see 4.4) Define an automorphism of a group G to be an isomorphism $\phi : G \to G$ from G to itself.
   (a) Prove that the identity map is an automorphism.
   (b) Prove that the composition of two automorphisms is an automorphism.
   (c) Prove that the set of automorphisms forms a group under composition.
   (d) If $g \in G$ is a fixed element, prove that the map $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$ is an isomorphism.
   (e) Prove that the map $\psi : G \to \mathrm{Aut}(G)$ given by $\psi(g) = \phi_g$ (sending the element g to the automorphism ϕ_g) is a homomorphism of groups.
   (f) Prove that the kernel of the map $\psi : G \to \mathrm{Aut}(G)$ is the center
$$\mathrm{Z}(G) := \{g \in G \mid gx = xg, \ \forall x \in G\}.$$
   (g) Define the inner automorphism group $\mathrm{Inn}(G)$ of G to be the subgroup of $\mathrm{Aut}(G)$ given by the image of G under ψ. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
   (h) Show that if G is abelian, then $\mathrm{Inn}(G)$ is trivial.
   (i) Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Prove that
      i. $\mathrm{Aut}(\mathbf{Z}/3\mathbf{Z}) = \mathrm{Out}(\mathbf{Z}/3\mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$,
      ii. $\mathrm{Out}(S_3) = \{1\}$.
      iii. $\mathrm{Aut}(K) \simeq \mathrm{Out}(K) \simeq S_3$, where $K = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ is the Klein 4-group.

9. Let p be an odd prime number. Prove that there are no surjective homomorphisms from S_n to $\mathbf{Z}/p\mathbf{Z}$ for any prime p. (Hint: consider the image of the two-cycles.)