Week 6, Due Mon 11/06

1. Automorphisms of S_n.

   (a) Let $\psi : G \to G$ be an isomorphism. If $\{c\}$ is a conjugacy class of G, prove that the image $\psi(\{c\})$ of $\{c\}$ under ψ is the conjugacy class $\{\psi(c)\}$.

   (b) Deduce that $|\{c\}| = |\{\psi(c)\}|$.

   (c) Let $G = S_n$. Prove that $|\{(12)\}| = n(n-1)/2$.

   (d) If $n \neq 6$, and $\sigma \in S_n$ has order 2, prove that $|\{\sigma\}| = |\{(12)\}|$ if and only if σ is a 2-cycle.

   (e) Deduce that if $\psi : S_n \to S_n$ is an isomorphism, and $n \neq 6$, then ψ takes 2-cycles to 2-cycles.

   (f) Suppose that $\psi(12) = (ij)$, prove that, after possibly swapping i and j, that $\psi(13) = (ik)$ for some $k \notin \{i, j\}$.

   (g) Let $g \in S_n$ denote any element with $g(i) = 1$, $g(j) = 2$, and $g(k) = 3$. Let ϕ_g be the (inner) automorphism of S_n given by conjugation by g. After replacing ψ by $\phi_g \circ \psi$, deduce that one can assume that $\psi(12) = (12)$ and $\psi(13) = (13)$.

   (h) Assume that $\psi(1i) = (1i)$ for all $i < k$, with $k > 3$. Prove that $\psi(1k) = (1j)$ for some $j \geq k$. As in part (1g), show that after replacing ψ by $\phi_h \circ \psi$ for some h, one can assume in addition that $\psi(1k) = (1k)$.

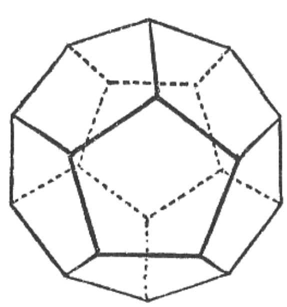
   (i) Deduce that ψ is the identity, and hence that any automorphism of S_n (for $n \neq 6$) is given by conjugation, i.e., $\mathrm{Out}(S_n) = 1$ for $n \neq 6$.

2. (This is due on next week's HW) Let H be a finite subgroup of G of index n. Let A be the set of left cosets G/H, and consider the left action of G on A. (See (4.2 (8)))

   (a) Let $n = |G/H|$, and consider the associated homomorphism $G \to S_{G/H} \simeq S_n$. Prove that the kernel of this map is a subgroup of H.

   (b) By considering the kernel of the map $G \to S_n$, deduce that G contains a normal subgroup N contained in H of index dividing $n!$ and divisible by n.

3. Let **Do** $\simeq A_5$ denote the symmetry group of the Dedecahedron: Fill out the missing entries in the table below for various sets X on which **Do** acts transitively. Since the action of **Do** is transitive for each X, all stabilizers S for any point $x \in X$ are conjugate to the stabilizers of any other point. Hence they are isomorphic as subgroups; simply list a group (from our known list of groups: symmetric, alternating, dihedral, cyclic, quaternion, etc.) isomorphic to any of the stabilizers.



Dodecahedron.

X	$\lvert X \rvert$	Faithful?	Stabilizer S of any x	Order of S
Dodecahedra	1	No	$S = \mathbf{Do} \simeq A_5$	60
Inscribed cubes				
Pairs of opposite faces				
Faces				
Vertices				
Edges				