Week 7, Due Mon 11/13
$\star$ = not for submission

You should think and try to solve the starred questions, but several of them are quite messy and some are difficult, so only submit the ones without stars.

1. 4.1 Question 7, Question 8.

2. 4.2 Question 9 (see also Question 8 from previous HW)

3. Suppose that G acts transitively and faithfully on a finite set X, and that G is abelian. Prove that $|G| = |X|$. Show that the equality need not hold if G is not abelian.

4. Let G be a finite group and let H be any subgroup.

   (a) Prove that the left action of G on the coset space G/H has kernel $N := \bigcap_{g \in G} gHg^{-1}$.

   (b) Prove that $N := \bigcap_{g \in G} gHg^{-1}$. is the largest normal subgroup of G contained in H.

5. **The Quaternions.** Let $\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$ be a 4-dimensional vector space over $\mathbf{R}$. Define a non-commutative associative multiplication structure on $\mathbf{H}$ by the formulae

$$ij = -ji = k, jk = -kj = i, ki = -ik = j; \quad i^2 = j^2 = k^2 = -1.$$

   (a) ($\star$) Show that there is a map ϕ from $\mathbf{H}$ to 2×2 matrices $M_2(\mathbf{C})$ over $\mathbf{C}$ by sending

$$i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

   so that:

      i. ϕ is injective as a map of vector spaces over $\mathbf{R}$

      ii. ϕ respects multiplication; if q_1 and q_2 are two quaternions then $\phi(q_1 q_2) = \phi(q_1)\phi(q_2)$. This should reduce easily enough to the case when q_i and q_j are elements of the set $\phi(1), \phi(i), \phi(j), \phi(k)$. The map ϕ is not a group homomorphism since 0 is not an invertible quaternion, but we shall see below in part (5c) hat non-zero quaternions form a group, so ϕ restricted to $\mathbf{H}^\times$ is actually a homomorphism from $\mathbf{H}^\times$ to $\mathrm{GL}_2(\mathbf{C})$.

   (b) Define the conjugate of a quaternion $q = a + bi + cj + dk$ by $\overline{q} := a - bi - cj - dk$. Prove that $N(q) := q\overline{q} = a^2 + b^2 + c^2 + d^2$.

   (c) Prove that non-zero quaternions $\mathbf{H}^\times$ form a group under multiplication.

   (d) Let $Q = \langle i, j \rangle$ be the subgroup of $\mathbf{H}^\times$ generated by i and j. Prove that Q is a group of order 8. (Q is known as the "quaternion group".)

   (e) Prove that every subgroup of Q is normal.

   (f) Let $N = \pm 1 \subset Q$. Prove that $Q/N \simeq (\mathbf{Z}/2\mathbf{Z})^2$ and that Q/N is not isomorphic to a subgroup of Q.

   (g) ($\star$) Let Γ be the subgroup of $\mathbf{H}^\times$ generated by the elements of Q together with $\dfrac{1+i+j+k}{2}$. Prove that Γ is a group of order 24.

   (h) Prove that Γ is *not* isomorphic to S_4, and Q is *not* isomorphic to D_8. In fact, $\Gamma = \mathrm{SL}_2(\mathbf{F}_3)$.

   (i) ($\star$) Construct a surjective homomorphism from Γ to A_4.

(j) Prove that the subgroup $\mathbf{H}^1$ of quaternions q with $N(q) = 1$ is a subgroup of $\mathbf{H}^\times$. Deduce that the 3-sphere $S^3 \subset \mathbf{R}^4$ defined by $a^2 + b^2 + c^2 + d^2 = 1$ has a natural structure of a group. Note that S^1 also has a natural group structure given by rotations in $\mathrm{SO}(2)$. It turns out that S^n has a natural (= continuous) group structure only for $n = 1$ and $n = 3$.

(k) ($\star$) Say that a quaternion is *pure* if it is of the form $bi + cj + dk$, i.e. $a = 0$. We may identify pure quaternions with $\mathbf{R}^3$. Show that if u is a pure quaternion then quq^{-1} is still a pure quaternion for any $q \in \mathbf{H}^\times$.

(l) ($\star$) Prove that the action of q on $\mathbf{R}^3$ by $q.u = quq^{-1}$ is via element of $\mathrm{SO}(3)$, and deduce that there is a homomorphism $\mathbf{H}^\times \to \mathrm{SO}(3)$.

(m) ($\star$) Prove that the restriction of this homomorphism to $\mathbf{H}^1 \to \mathrm{SO}(3)$ is surjective and has kernel of order 2.