Week 8, Due Mon 11/27

1. Suppose that $\mathbf{Z}/m\mathbf{Z}$ is a subgroup of S_n for some n and $m > 2$. Prove that D_{2m} is also a subgroup of S_n.

2. Let G be a group, and let $N \subseteq G$ be the subgroup generated by the elements $xyx^{-1}y^{-1}$ for all pairs x, $y \in G$. Prove that N is a normal subgroup, and that G/N is abelian.

3. **Projective Linear Groups.** Let $\mathrm{GL}_2(\mathbf{R})$ be the group of invertible matrices of $\mathbf{R}$, and let $\mathrm{SL}_2(\mathbf{R}) \subset \mathrm{GL}_2(\mathbf{R})$ denote the subgroup of matrices of determinant one.

   (a) Let $\mathcal{L}$ denote the set of lines through the origin, where $x \in \mathcal{L}$ can be thought of as $\mathbf{vR}$ for some non-zero vector $\mathbf{v}$ (not unique!). Prove that
   $$g.[\mathbf{vR}] = [g.\mathbf{vR}]$$
   gives a well-defined action of $\mathrm{GL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{L}$.

   (b) Prove that this action is transitive for both $\mathrm{GL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{R})$, and that the kernel consists precisely of the scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ in either $\mathrm{SL}_2(\mathbf{R})$ or $\mathrm{GL}_2(\mathbf{R})$.

   (c) Prove that one can identify $\mathcal{L}$ with $X = \mathbf{R} \cup \infty$ by defining the "slope" $s(\mathbf{vR})$ of the line $\mathbf{vR}$ to be $x = p/q$ when $\mathbf{v} = [p, q]$ and ∞ if $q = 0$. Show that the action of $\mathrm{GL}_2(\mathbf{R})$ on X is given by
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \begin{cases} \dfrac{ax+b}{cx+d}, & x \neq -d/c, \\ \infty, & x = -d/c, \\ \dfrac{a}{c}, & x = \infty. \end{cases}$$

4. **Projective Linear Groups over Finite Fields.** Let p be prime, and let $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Note that one can add and multiply elements of $\mathbf{F}_p$. Let $\mathrm{GL}_2(\mathbf{F}_p)$ be the group of invertible matrices over $\mathbf{F}_p$, and let $\mathrm{SL}_2(\mathbf{F}_p) \subset \mathrm{GL}_2(\mathbf{F}_p)$ denote the subgroup of matrices of determinant one.

   (a) There are $p^2 - 1$ non-zero vectors $\mathbf{v} \in \mathbf{F}_p^2$. Let a "line" be $\mathbf{vF}_p$, the scalar multiples of $\mathbf{v}$. Prove that the set $\mathcal{L}$ of lines has cardinality $|\mathcal{L}| = p + 1$.

   (b) Prove that $\mathrm{SL}_2(\mathbf{F}_p)$ and $\mathrm{GL}_2(\mathbf{F}_p)$ act naturally on $\mathcal{L}$ by $g.[\mathbf{vF}_p] = [g.\mathbf{vF}_p]$.

   (c) Prove that this action is transitive for both $\mathrm{GL}_2(\mathbf{F}_p)$ and $\mathrm{SL}_2(\mathbf{F}_p)$.

   (d) Prove that the kernel of the action consists precisely of the scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ in either $\mathrm{SL}_2(\mathbf{F}_p)$ or $\mathrm{GL}_2(\mathbf{F}_p)$.

   (e) Let $\mathrm{PGL}_2(\mathbf{F}_p)$ and $\mathrm{PSL}_2(\mathbf{F}_p)$ denote the quotient of G and H by the subgroup of scalar matrices. Prove that $|\mathrm{PGL}_2(\mathbf{F}_p)| = (p^2 - 1)p$ and $|\mathrm{PSL}_2(\mathbf{F}_p)| = 6$ if $p = 2$ and $\frac{1}{2}(p^2 - 1)p$ otherwise.

   (f) Prove that $\mathrm{PGL}_2(\mathbf{F}_2) = \mathrm{PSL}_2(\mathbf{F}_2) = S_3$.

   (g) Prove that $\mathrm{PGL}_2(\mathbf{F}_3) = S_4$ and $\mathrm{PSL}_2(\mathbf{F}_3) = A_4$.

   (h) Prove that $\mathrm{PSL}_2(\mathbf{F}_5) = A_5$ and $\mathrm{PGL}_2(\mathbf{F}_5) = S_5$. (Hint: using that A_6 is simple, prove that any index 6 subgroup of A_6 or S_6 is A_5 or S_5 respectively).