Name: ______________________

Id #:

Math 25700 Midterm

Autumn Quarter 2023

Monday, October 23, 2023

Name:

Instructions:
Show *all* your work (unless otherwise noted). Make sure that your final answer is clearly indicated. This test has six problems. Good luck!

Prob.	Possible points	Score
1	10	
2	20	
3	20	
4	20	
5	30	
6	10	
TOTAL	110	

PART I: Computational Questions

(No working is required for these problems, but some partial credit is available)

Question 1. *(10 points) Let* $\sigma = (1,2,3)(4,5,6,7) \in S_8$. *Find an element* $\tau \in S_8$ *such that*

$$\tau\sigma\tau^{-1} = (3,1,4,5)(9,2,6).$$

We may also write $\sigma = (1,2,3)(4,5,6,7)(8)(9)$. We have

$$\tau\sigma\tau^{-1} = (\tau(1),\tau(2),\tau(3))(\tau(4),\tau(5),\tau(6),\tau(7))(\tau(8))(\tau(9)).$$

Writing the element $(9,2,6)(3,1,4,5)(7)(8)$ directly underneath σ, this leads to (one of many) choices of τ as follows:

n	1	2	3	4	5	6	7	8	9
$\tau(n)$	9	2	6	3	1	4	5	7	8

with $\tau = (1,9,8,7,5)(3,6,4)$.

Question 2. *(20 points)*

1. *Find all the conjugacy classes inside S_6 which contain an element of order 2.*

There are three conjugacy classes, given by the partitions

$$6 = 2 + 2 + 2$$
$$6 = 2 + 2 + 1 + 1$$
$$6 = 2 + 1 + 1 + 1 + 1$$

2. *Determine the number of elements of S_6 of order exactly 2.*

We simply compute the orders of the conjugacy classes given in the last answer.

(a) The conjugacy class of $(**)$ has $\binom{6}{2} = 15$ elements.

(b) The conjugacy class of $(**)(**)$ has $\frac{1}{2!}\binom{6}{2}\binom{4}{2} = 45$ elements.

(c) The conjugacy class of $(**)(**)(**)$ has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15$ elements.

Hence there are

$$15 + 45 + 15 = 75$$

elements of order two.

PART II: Theoretical Questions

Question 3. *(20 points) Let x, y be elements of a finite group G. Prove or disprove: the order of xy is always equal to the order of yx.*

Note that $xy = y^{-1}(yx)y$ is conjugate to yx, and conjugate elements have the same order. (Proof: if $h^n = e$, then $(ghg^{-1})^n = gh^n g^{-1} = e$. Conversely, if $(ghg^{-1})^n = e$, then $e = (ghg^{-1})^n = gh^n g^{-1}$, and then $h^n = g^{-1}g = e$.)

Question 4. *(20 points) Prove that there exist finite groups G of arbitrarily large order such that every element $g \in G$ is conjugate to its inverse g^{-1}.*

Let $G = (\mathbf{Z}/2\mathbf{Z})^n$. Then every element in G has order 1 or 2, so $g^2 = e$, and then $g^{-1} = g$. But every element is conjugate to itself.

Alternatively: Let $G = S_n$. Then the conjugacy class of $g \in S_n$ is given by its cycle shape. But the inverse of a cycle is the cycle in the reverse order, so g^{-1} has the same cycle shape, so g^{-1} is always conjugate to g in S_n.

Question 5. *(10+10+10 points) Let G be a group, and let x be a fixed element of G.*

1. *Let $C_G(x)$ denote the subset of elements in G such that $hxh^{-1} = x$. Prove that $C_G(x)$ is a subgroup of G.*

   It suffices to prove that $C_G(x)$ is closed under multiplication, under inverses, and is non-empty.

   (a) Certainly $exe^{-1} = x$, so $e \in C_G(x)$.

   (b) If $a, b \in C_G(x)$, then

   $$abx(ab)^{-1} = abxb^{-1}a^{-1} = a(bxb^{-1})a^{-1} = axa^{-1} = x,$$

   where the last two equalities follow from $b \in C_G(x)$ and $a \in C_G(x)$.

   (c) If $a \in C_G(x)$, then $x = axa^{-1}$, so

   $$a^{-1}xa = a^{-1}(axa^{-1})a = x.$$

   Hence $C_G(x)$ is a subgroup.

2. *Let $\{x\}$ denote the conjugacy class of G, that is, the set of elements in G which are conjugate to x, and consider $y \in \{x\}$. Let S denote the subset of elements in G such that $gxg^{-1} = y$. Prove that S is a left coset of $C_G(x)$ in G.*

Since $y \in \{x\}$, there exists at least one $a \in S$, so $axa^{-1} = y$. We have:

$$
\begin{aligned}
b \in S &\Leftrightarrow bxb^{-1} = y \\
&\Leftrightarrow bxb^{-1} = axa^{-1} \\
&\Leftrightarrow a^{-1}bxb^{-1}a = x \\
&\Leftrightarrow a^{-1}bx(a^{-1}b)^{-1} = x \\
&\Leftrightarrow a^{-1}b \in C_G(x) \\
&\Leftrightarrow b \in aC_G(x),
\end{aligned}
$$

so $S = aC_G(x)$.

3. Deduce that

$$|\{x\}| \cdot |C_G(x)| = |G|.$$

Certainly $|G| = |C_G(x)| \cdot |G/C_G(x)|$, so it suffices to give a bijection between $y \in \{x\}$ and left cosets of $C_G(x)$. Let $y \mapsto S$ where S is the set of elements such that $gxg^{-1} = y$. From the last part, S is a left coset of $C_G(x)$. This map is injective; if y and y' map to the same coset S containing g then $y' = gxg^{-1} = y$. Conversely, if $g \in G$ is any element, and $y = gxg^{-1}$, then y maps to a left coset containing g which therefore equals $gC_G(x)$ (because this is the only left coset containing g), and so this map is surjective as well.

Question 6. *(5+5 points) Find (with proof) the smallest n such that the dihedral group D_{24} of order 24 is isomorphic to a subgroup of S_n. That is, for some $n = m$, prove that $D_{24} \simeq H \subset S_m$, and prove that D_{24} is not isomorphic to any subgroup of S_{m-1}.*

D_{24} has an element of order 12, so it cannot be isomorphic to a subgroup of S_6, since S_6 has no such element.

If $r = (1,2,3,4)(5,6,7)$, and $s = (1,4)(2,3)(5,7)$, then $srs^{-1} = r^{-1}$ and these elements generate D_{24}.

Alternatively: Inside the dodecagon, you can inscribe four equilateral triangles $1,2,3,4$ and 3 square $5,6,7$. Now D_{24} permutes these four triangles and three squares, and this gives a map $D_{24} \to S_7$ which one can check is injective. (For a rotation to fix the squares it must have order dividing 3, and to fix the triangles it must have order dividing 4; on the other hand, no reflection fixes more than one square.)