Week 5, Due Mon 11/05

1. Automorphisms of S_n.

   (a) Let $\psi : G \to G$ be an isomorphism. If $\langle c \rangle$ is a conjugacy class of G, prove that the image $\psi(\langle c \rangle)$ of $\langle c \rangle$ under ψ is the conjugacy class $\langle \psi(c) \rangle$.

   (b) Deduce that $|\langle c \rangle| = |\langle \psi(c) \rangle|$.

   (c) Let $G = S_n$. Prove that $|\langle (12) \rangle| = n(n-1)/2$.

   (d) If $n \neq 6$, and $\sigma \in S_n$ has order 2, prove that $|\langle \sigma \rangle| = |\langle (12) \rangle|$ if and only if σ is a 2-cycle.

   (e) Deduce that if $\psi : S_n \to S_n$ is an isomorphism, and $n \neq 6$, then ψ takes 2-cycles to 2-cycles.

   (f) Suppose that $\psi(12) = (ij)$, prove that, after possibly swapping i and j, that $\psi(13) = (ik)$ for some $k \notin \{i, j\}$.

   (g) Deduce that, after replacing ψ by $\psi_g \psi$ where ψ_g is carefully chosen inner automorphism (given by conjugation by g for some g), one has $\psi(12) = (12)$ and $\psi(13) = (13)$.

   (h) Assume that $\psi(1i) = (1i)$ for all $i < k$, with $k > 3$. Prove that $\psi(1k) = (1j)$ for some $j \geq k$. Deduce that, after replacing ψ again by $\psi_g \psi$ for some g, that $\psi(1i) = (1i)$ for all $i \leq k$.

   (i) Deduce that ψ is the identity, and hence that the original ψ was a product of inner automorphisms, and thus $\mathrm{Out}(S_n) = \mathrm{Aut}(S_n)/\mathrm{Inn}(S_n) = 1$ for $n \neq 6$.

2. A_n is simple for $n \geq 5$. Let H be a normal subgroup of A_n.

   (a) Prove that A_n is generated by 3-cycles.

   (b) Prove that for any two three cycles (a, b, c) and (x, y, z) (warning; the sets $\{a, b, c\}$ and $\{x, y, z\}$ may not be disjoint), there exists an element σ of A_n such that

   $$\sigma(a, b, c)\sigma^{-1} = (x, y, z) \text{ or } (x, z, y)$$

   (c) Deduce that if H contains a 3-cycle, then H contains all 3-cycles, and hence $H = A_n$.

   (d) Suppose that $\sigma = (a_1, a_2, a_3, a_4, \ldots)(\ldots)$ contains a cycle of length ≥ 4. Prove that σ is conjugate (in A_n) to $\tau = (a_2, a_3, a_1, a_4, \ldots)(\ldots)$, where all the other entries of σ remain unchanged.

   (e) Show that $\sigma\tau^{-1} = (a_1, a_4, a_2)$, and deduce that either $H = A_n$ or all the cycles in the cycle decomposition of $\sigma \in H$ have length ≤ 3.

   (f) Suppose that $\sigma = (a, b, c)(d, e, f)\ldots$ contains at least two 3-cycles. Prove that σ is conjugate (in A_n) to $\tau = (a, b, d)(e, c, f)\ldots$, where all the other entries of σ remain unchanged.

   (g) Show that $\tau\sigma = (a, d, c, b, f)\ldots$, and, by part (e), deduce that either $H = A_n$ or all the cycles in the cycle decomposition of $\sigma \in H$ have at most one 3-cycle and are otherwise composed of 2-cycles.

   (h) If the cycle decomposition of σ is a 3-cycle times a product of 2-cycles, show that σ^2 is 3-cycle. Deduce that either $H = A_n$ or that all the cycles in the cycle decomposition of $\sigma \in H$ are products of 2-cycles.

   (i) Suppose that $\sigma = (a, b)(c, d)(e, f)\ldots$. Prove that σ is conjugate to $\tau = (a, c)(e, b)(d, f)\ldots$.

   (j) Show that $\tau\sigma = (a, e, d)(c, f, b)\ldots$, and by part (g), deduce that either $H = A_n$ or that all the cycles in the cycle decomposition of $\sigma \in H$ consist of at most two 2-cycles.

   (k) Deduce that if H is proper (i.e. not the entire group), then every non-trivial element of H has cycle decomposition $(a, b)(c, d)$.

   (l) If $n \geq 5$, prove that $\sigma = (a, b)(c, d)$ is conjugate to $\tau = (a, e)(c, d)$, and deduce from the fact that $\tau\sigma = (a, b, e)$ that the only normal subgroup of A_n for $n \geq 5$ is either A_n or is trivial.