Week 5, Due Mon 4/30

1. Let $S = \{p_1, p_2, \ldots, p_n\}$ be n distinct primes.

   (a) Let Σ denote the set consisting of non-trivial products of distinct elements of S. Prove that $|\Sigma| = 2^n - 1$.

   (b) If D_1 and D_2 denote elements of Σ, prove that $\mathbf{Q}(\sqrt{D_1}) \simeq \mathbf{Q}(\sqrt{D_2})$ if and only if $D_1 = D_2$.

   (c) Let $K = \mathbf{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$. Prove that K is the splitting field of
   $$(X^2 - p_1)(X^2 - p_2)\cdots(X^2 - p_n).$$

   (d) Prove that $\mathrm{Gal}(K/\mathbf{Q})$ is a subgroup of $(\mathbf{Z}/2\mathbf{Z})^n$.

   (e) Prove that K has at least $2^n - 1$ subfields of degree 2.

   (f) Prove that $\mathrm{Gal}(K/\mathbf{Q}) = (\mathbf{Z}/2\mathbf{Z})^n$, and deduce that $[K : \mathbf{Q}] = 2^n$.

2. [**Field Embeddings, I**] Let $E/\mathbf{Q}$ be a finite extension. Let $K/\mathbf{Q}$ be a Galois extension with Galois group $G = \mathrm{Gal}(K/\mathbf{Q})$. Let M be the set of subfields of K that are isomorphic to E.

   (a) Prove that M is empty, or there exists an inclusion from E to K.

   (b) Prove that G acts on M by sending $F \in M$ to $\phi(F)$.

   (c) If $F \in M$, prove that the stabilizer of F is the normalizer N_F of $\mathrm{Gal}(K/F)$.

   (d) Prove that G acts transitively on M.

   (e) Prove that $|M| = [G : N_F]$, for any $F \in M$.

   (f) Prove that $|M| = 1$ if and only if $E/\mathbf{Q}$ is Galois.

   (g) If $F \in M$, let (**typo corrected**) $H = K^{N_F}$. Prove that:

      i. H is contained in F.

      ii. F/H is Galois.

      iii. If $H' \subset F$ is any subfield of F such that F/H' is Galois, then H' contains H.

      iv. H does not depend on F.

   (h) Deduce that for any field $E/\mathbf{Q}$, there is a well defined minimal field $H/\mathbf{Q}$ in E such that E/H is Galois.

3. Let $F = \mathbf{C}(x_1, x_2, \ldots, x_n)$ be the field of fractions of the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$. Let s_i denote the elementary symmetric polynomials in the x_i, that is,
   $$\begin{aligned} s_1 &= x_1 + x_2 + \ldots + x_n \\ s_2 &= x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n \\ &\ddots \\ s_n &= x_1x_2\ldots x_n. \end{aligned}$$
   Let $E = \mathbf{C}(s_1, \ldots, s_n)$. Prove that, with respect to the natural inclusion $E \subset F$, that:

   (a) F/E is a finite Galois extension.

   (b) $\mathrm{Gal}(F/E) = S_n$.

4. Let $L/\mathbf{Q}$ be Galois with $\mathrm{Gal}(L/\mathbf{Q}) = Q = \{\pm i, \pm j, \pm k, \pm 1\}$, the quaternion group of order 8. Prove that any quadratic subfield of $K \subset L$ is a real quadratic field; that is, admits a ring homomorphism injection $K \to \mathbf{R}$.