Week 6, Due Mon 11/12

1. 4.1 (7,8), 4.2 (8,9).

2. Suppose that G acts transitively and faithfully on a finite set X, and that G is abelian. Prove that $|G| = |X|$. Show that the equality need not hold if G is not abelian.

3. **The Quaternions.** Let $\mathbf{H} = \mathbf{R} \oplus \mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k$ be a 4-dimensional vector space over $\mathbf{R}$. Define a non-commutative associative multiplication structure on $\mathbf{H}$ by the formulae

$$ij = -ji = k, jk = -kj = i, ki = -ik = j; \quad i^2 = j^2 = k^2 = -1.$$

   (a) Show that there is a map from $\mathbf{H}$ to 2×2 matrices $M_2(\mathbf{C})$ over $\mathbf{C}$ by sending

$$i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

   (b) Define the conjugate of a quaternion $q = a + bi + cj + dk$ by $\overline{q} := a - bi - cj - dk$. Prove that $N(q) := q\overline{q} = a^2 + b^2 + c^2 + d^2$.

   (c) Prove that non-zero quaternions $\mathbf{H}^\times$ form a group under multiplication.

   (d) Let $Q = \langle i, j \rangle$ be the subgroup of $\mathbf{H}^\times$ generated by i and j. Prove that Q is a group of order 8. (Q is known as the "quaternion group".)

   (e) Let Γ be the subgroup of $\mathbf{H}^\times$ generated by the elements of Q together with $\dfrac{1+i+j+k}{2}$. Prove that Γ is a group of order 24.

   (f) Prove that Γ is *not* isomorphic to S_4, and Q is *not* isomorphic to D_8. In fact, $\Gamma = \mathrm{SL}_2(\mathbf{F}_3)$.

   (g) Construct a surjective homomorphism from Γ to A_4.

   (h) Prove that the subgroup $\mathbf{H}^1$ of quaternions q with $N(q) = 1$ is a subgroup of $\mathbf{H}^\times$. Deduce that the 3-sphere $S^3 \subset \mathbf{R}^4$ defined by $a^2+b^2+c^2+d^2 = 1$ has a natural structure of a group. Note that S^1 also has a natural group structure given by rotations in $SO(2)$. It turns out that S^n has a natural (= continuous) group structure only for $n = 1$ and $n = 3$.

4. **Projective Linear Groups over Finite Fields.** Let p be prime, and let $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Note that one can add and multiply elements of $\mathbf{F}_p$. Let $\mathrm{GL}_2(\mathbf{F}_p)$ be the group of invertible matrices over $\mathbf{F}_p$, and let $\mathrm{SL}_2(\mathbf{F}_p) \subset \mathrm{GL}_2(\mathbf{F}_p)$ denote the subgroup of matrices of determinant one.

   (a) There are $p^2 - 1$ non-zero vectors $v \in \mathbf{F}_p^2$. Let a "line" $\ell = [v] \subset \mathbf{F}_p^2$ denote the scalar multiples λv of a non-zero vector v. Prove that the set X of lines has cardinality $|X| = p+1$.

   (b) Prove that $\mathrm{SL}_2(\mathbf{F}_p)$ and $\mathrm{GL}_2(\mathbf{F}_p)$ act naturally on X by $g.[v] = [g.v]$.

   (c) Prove that this action is transitive for both $\mathrm{GL}_2(\mathbf{F}_p)$ and $\mathrm{SL}_2(\mathbf{F}_p)$.

   (d) Prove that the kernel of the action consists precisely of the scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ in either $\mathrm{SL}_2(\mathbf{F}_p)$ or $\mathrm{GL}_2(\mathbf{F}_p)$.

   (e) Let $\mathrm{PGL}_2(\mathbf{F}_p)$ and $\mathrm{PSL}_2(\mathbf{F}_p)$ denote the quotient of G and H by the subgroup of scalar matrices. Prove that $|\mathrm{PGL}_2(\mathbf{F}_p)| = (p^2-1)p$ and $|\mathrm{PSL}_2(\mathbf{F}_p)| = 6$ if $p = 2$ and $\frac{1}{2}(p^2-1)p$ otherwise.

   (f) Prove that $\mathrm{PGL}_2(\mathbf{F}_2) = \mathrm{PSL}_2(\mathbf{F}_2) = S_3$.

   (g) Prove that $\mathrm{PGL}_2(\mathbf{F}_3) = S_4$ and $\mathrm{PSL}_2(\mathbf{F}_3) = A_4$.

   (h) Prove that $\mathrm{PSL}_2(\mathbf{F}_5) = A_5$ and $\mathrm{PGL}_2(\mathbf{F}_5) = S_5$. (Hint: using that A_6 is simple, prove that any index 6 subgroup of A_6 or S_6 is A_5 or S_5 respectively).