Week 7, Due Mon 5/12

1. Let L be the splitting field of the polynomial $x^4 - x - 1$ over $\mathbf{Q}$, and denote the roots by α_1, α_2, α_3, and α_4. You may assume that $G = \mathrm{Gal}(L/\mathbf{Q}) = S_4$.

   (a) Determine the number n of subfields E of L. (Thus two fields $E \subset L$ and $F \subset L$ count as one if and only if $E = F$ inside L.)

   (b) Determine the number m of subfields E of L up to isomorphism. (Thus two fields $E \subset L$ and $F \subset L$ count as one if and only if there is an isomorphism $E \simeq F$.)

   (c) For each of the n subfields E in part (1a), write down a primitive element $\theta \in L$; that is, an element $\theta \in L$ such that $E = \mathbf{Q}(\theta) \subset L$.

   (d) For each of the n subfields E and elements θ of part (1c), write down the irreducible polynomial of θ in $\mathbf{Q}[x]$.