Week 7, Due Mon 11/19

1. Suppose that $\mathbf{Z}/m\mathbf{Z}$ is a subgroup of S_n for some n. Prove that D_{2m} is also a subgroup of S_n.

2. (See (4.2 (8))) Let H be a finite subgroup of G of index n. Let A be the set of left cosets G/H, and consider the left action of G on A.

   (a) Let $n = |A|$, and consider the associated homomorphism $G \to S_A \simeq S_n$. Prove that the kernel of this map is a subgroup of H.

   (b) By considering the kernel of the map $G \to S_n$, deduce that G contains a normal subgroup N contained in H of index dividing $n!$ and divisible by n.

3. Let G be a group, and let $N \subseteq G$ be the subgroup generated by the elements $xyx^{-1}y^{-1}$ for all pairs $x, y \in G$. Prove that N is a normal subgroup, and that G/N is abelian.

4. Compute the order of the following groups as well as a set of generators:

   (a) The centralizer of (12345) in A_7.

   (b) The centralizer of $((12), (123))$ in $S_5 \times S_5$.

   (c) The normalizer of $H = \langle (12), (34), (56), (78) \rangle$ in S_8.

5. The "exotic" automorphism of S_6. You may use the fact proved in class that, for $n \geq 5$, the only normal subgroups of S_n are the trivial group, A_n, and S_n.

   (a) Let $G = S_5$. Prove that there are 24 elements of G of order 5, and 6 subgroups of G of order 5. Prove they are all conjugate.

   (b) Let $P = \langle (12345) \rangle \subset G = S_5$. Prove that the normalizer $N = N_G(P)$ of P has order 20. (Hint: use the Orbit–Stabilizer Theorem to determine the order of N.)

   (c) Show that the left action of G on G/N gives rise to a homomorphism $\phi : G \to S_6$.

   (d) Prove that ϕ is injective, and deduce that $G \simeq \mathrm{im}(\phi)$.

   (e) Prove that $\mathrm{im}(\phi)$ inside S_6 is not one of the "natural" copies of S_5 in S_6 given by subgroups which stabilize a point. (Hint: use the fact that the action in part 5c is transitive.) It is an "exotic" copy of S_5 inside S_6.

   (f) Since $[S_6 : \mathrm{im}(\phi)] = 6$, show that the left action of S_6 on $S_6/\mathrm{im}(\phi)$ gives rise to a homomorphism $\psi : S_6 \to S_6$.

   (g) Prove that ψ is injective and deduce that ψ is automorphism.

   (h) Prove that the the image of the exotic $S_5 \simeq \mathrm{im}(\phi) \subset S_6$ maps under ψ to a copy of $S_5 \subset S_6$ which stabilizes a point.

   (i) Deduce that ψ is *not* given by conjugation by some element of S_6.

   (j) Prove that $\mathrm{Out}(S_6) = \mathbf{Z}/2\mathbf{Z}$ is generated by ψ. Hint: by using results from a previous homework question, show that any automorphism of S_6 which is not inner must send the conjugacy class $(**)$ to $(**)(**)(**)$ and send $(**)(**)(**)$ to $(**)$. Deduce that the product of any two elements in $\mathrm{Aut}(S_6)$ not in $\mathrm{Inn}(S_6)$ are inner, and hence trivial in $\mathrm{Out}(S_6)$.

   (k) Prove that the group $\mathrm{Aut}(S_6)$ has order $2 \times 6! = 1440$.