Problems

As usual, we let **R**, **Q**, **C** denote the real, rational, and complex numbers respectively, and let **Z** denote the integers. Some of these problems are taken directly from the book, and in such cases a reference to the appropriate section is given.

You may assume:

- All rings R are commutative.
- All rings R have multiplicative identity.
- All ring homomorphisms $R \to S$ send 1_R to 1_S.

1. (7.1.11) Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.
2. Prove that for any commutative ring R, there is a canonical (unique) ring homomorphism from $\mathbf{Z}$ to R.
3. (7.3.26) Define the characteristic of a ring R to be the smallest positive integer n such that $1 + 1 + \ldots + 1$ (n times) is equal to 0 in R. If no such n exists say that R has characteristic zero.
   (a) If R is a domain, show that the characteristic is either 0 or prime.
   (b) If $\mathbf{Z} \to R$ is the canonical ring homomorphism, show that the kernel of this homomorphism is the ideal (n), where n is the characteristic of R.
   (c) Deduce that if p is prime, then R has characteristic p if and only if there is a ring homomorphism $\mathbf{Z}/p\mathbf{Z} \to R$.
4. (7.3.26) Let R be a commutative ring of characteristic p for some prime p. Prove that the map $\phi : R \to R$ given by $\phi(r) = r^p$ is a homomorphism.
5. If $r \in R$ is nilpotent, prove that $1 + r$ is a unit in R.
6. (7.3.33) Let $p(x) \in R[x]$, and suppose that $p(x) = a_0 + a_1 x + \ldots + a_n x^n$. Prove that $p(x)$ is nilpotent in $R[x]$ if and only if $a_i \in R$ is nilpotent for every i.
7. (7.3.32) Let $\phi : R \to S$ be a homomorphism of rings. Prove that if $r \in R$ is nilpotent, then $\phi(r) \in S$ is nilpotent.
8. Suppose that R is a commutative ring. Suppose that the ideals (a) and (b) are equal.
   (a) (7.4.8) If R is a domain, prove that there exists a unit $u \in R$ such that $a = bu$.
   (b) * Is the result still true if R is not assumed to be a domain?
9. We call $e \in R$ **idempotent** if $e^2 = e$, and $e \neq 0$. Show that $eR := \{er \mid r \in R\}$ can be given the structure of a ring (with multiplicative identity e) such that the map $\psi : R \to eR$ with $\psi(r) = er$ is a homomorphism.
10. For any commutative ring R, prove that there is a unique homomorphism $\phi : \mathbf{Z} \to R$. If R is a domain, prove that either $\ker(\phi) = (0)$ or $\ker(\phi) = (p)$ for some prime p.
11. If $f(x) \in \mathbf{Q}[x]$ is irreducible, then is $f(x^2)$ always irreducible?

12. Find (with proof) all prime ideals in the ring of dual numbers $\mathbf{R}[\epsilon]/\epsilon^2$.

13. (7.4.13) Let $\phi : R \to S$ be a homomorphism of commutative rings. Suppose that $P \subset S$ is a prime ideal.

    (a) Prove that
    $$\phi^{-1}(P) := \{r \in R \mid \phi(r) \in P\}$$
    is a prime ideal of R.

    (b) If P is a *maximal* ideal of S, show that $\phi^{-1}(P)$ is not always a maximal ideal or R.

    (c) if ϕ is surjective, and P is a *maximal* ideal of S, show that $\phi^{-1}(P)$ is a maximal ideal or R.

    (d) If Q is a prime ideal of R, is $\phi(Q)$ always a prime ideal of S?

14. (8.1.8) Let $\omega = \frac{1+\sqrt{-7}}{2}$, and let $R = \mathbf{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbf{Z}\}$. Prove that R is a ring with the obvious operations of multiplication and addition. Define a norm on R by the formula:
    $$N(a + b\omega) = a^2 + ab + 2b^2.$$

    (a) Show that if $x, y \in R$ then $N(xy) = N(x)N(y)$, and that $N(x) = 0$ if and only if $x = 0$.

    (b) Prove that R is a Euclidean Domain with respect to this norm.

    (c) Write the $(\omega - 7, 11)$ as a principal ideal of R.

    (d) Write $(\omega - 23, 127)$ as a principal ideal of R.

15. Let a and b be positive integers, and suppose that $(a, b) = (d)$, with $d > 0$ in $\mathbf{Z}$. Prove that $(x^a - 1, x^b - 1) = (x^d - 1)$ in $\mathbf{Q}[x]$.

16. Exercise 8.3.8

17. (9.2.3) Let $p(x)$ be a polynomial in $K[x]$, for a field K. Prove that $K[x]/p(x)$ is a field if and only if $p(x)$ is irreducible.

18. (9.1.8) Let F be a field, and let $R = F[x, x^2y, x^3y^2, x^4y^3, \ldots, x^n y^{n-1}, \ldots]$ be a subring of $F[x, y]$.

    (a) Prove that the field of fractions of R and $F[x, y]$ are the same.

    (b) Prove that R contains an ideal that is not finitely generated.

19. Let R be a commutative ring. We call $r \in R$ **nilpotent** if $x^n = 0$ in R for some integer n. Let I denote the set of nilpotent elements in R. Prove that if $P \subset R$ is a prime ideal of R, then $I \subset P$.

20. (7.5.3) Let K be a field. Prove that either:

    (a) K has characteristic $p > 0$, and K contains $\mathbf{Z}/p\mathbf{Z}$ as a subfield.

    (b) K has characteristic 0, and K contains $\mathbf{Q}$ as a subfield.

21. Let K be a field, and let $K((T))$ denote formal sums:
    $$\left\{ \sum_{-\infty}^{\infty} a_n T^n, \;\middle|\; a_n \in K,\ a_n = 0 \text{ for } n \ll 0 \right\}.$$
    Prove that $K((T))$ is a field.

22. Let $R \subset \mathbf{Q}[x]$ be the subring consisting of all polynomials with vanishing x coefficient. Prove that R is not a P.I.D.

23. Let $R \subset \mathbf{Q}[x]$ be the subring consisting of all polynomials with vanishing x coefficient. Let $S = \mathbf{Q}[y,z]/(y^2 - z^3)$. Prove that the map $S \to R$ given by sending y to x^3 and z to x^2 is an isomorphism.

24. If I and J are two ideals of R, recall that $I + J$ is the ideal generated by $i + j$ for all $i \in I$ and $j \in J$, and $I \cap J$ is the ideal of elements that lie in I and in J. Suppose that $I + J = R$. Prove that

$$R/(I \cap J) \simeq R/I \oplus R/J.$$

Hint: first show that the map $R \to R/I \oplus R/J$ sending r to $(r \mod I, r \mod J)$ has kernel $I \cap J$. Then show that this map is surjective.

25. Let R be a commutative ring, and let R^* denote the set of units in R. Prove that R^* is an abelian group with respect to the multiplcation on R.

26. Let R be a commutative ring with a *unique* maximal ideal M. Prove that $r \in R$ is a unit if and only if $r \notin M$.

27. Let $R = \mathbf{Z}[2i] = \{a + 2bi \mid a, b \in \mathbf{Z}\}$, where $i^2 = -1$. Show that R is a domain, but *not* a P.I.D.

28. (7.4.33) Let R be the ring of all continuous functions from the closed interval $[0,1]$ to $\mathbf{R}$. Let $I_c := \{f \in R \mid f(c) = 0\}$.

    (a) Prove that I_c is an ideal.

    (b) Prove that the map: $R \to c$ defined by sending f to $f(c)$ has kernel I_c, and deduce that I_c is a prime ideal and also a maximal ideal.

    (c) Prove that if $f \in R$ does not vanish on $[0,1]$ then f is a unit.

    (d) Prove that if $f, g \in R$ are two functions such that there does *not* exist a point $c \in \mathbf{R}$ such that $f(c) = g(c) = 0$, then the ideal (f, g) contains a unit. (Hint: squares are always non-negative).

    (e) Deduce that any maximal ideal of R is of the form I_c for some $c \in R$,

    (f) Prove that if b and c are two distinct points of $[0,1]$ then $I_b \neq I_c$.

    (g) Prove that $I_c \neq (x - c)R$.

    (h) Prove that I_c is not finitely generated.