Putnam Questions, Week 1

1. Prove that \( \frac{1}{1 + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \ldots + \frac{1}{\sqrt{99} + \sqrt{100}} = 9. \)

2. Prove that there exists an integer \( n \) such that the first four digits of \( 2^n \) are 2, 0, 0, 9.

3. Given a set of \( n + 1 \) integers between 1 and \( 2n \), prove that one number must divide another. Prove that this is not necessarily true for \( n \) integers between 1 and \( 2n \).

4. Let \( f(x) \) be a polynomial, and suppose that \( f(x) + f'(x) > 0 \) for all \( x \). Prove that \( f(x) > 0 \) for all \( x \).

5. For which real numbers \( c \) is \( (e^x + e^{-x})/2 \leq e^{cx^2} \) for all real \( x \)?

6. Evaluate the infinite product \( \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} \).