Putnam Questions, Week 5

1. \(\sqrt{44} = 6\) and \(\sqrt{4444} = 66\). Generalize.

2. Prove that \((2 + \sqrt{5})^{1/3} + (2 - \sqrt{5})^{1/3}\) is rational.

3. Let \(\alpha = \frac{3 + \sqrt{13}}{2}\). What is the last digit of \([\alpha^{2009}]\)?

4. Suppose that \(\alpha, \beta, \text{ and } \gamma\) are real numbers such that
   \[
   \begin{align*}
   \alpha + \beta + \gamma &= 2, \\
   \alpha^2 + \beta^2 + \gamma^2 &= 14, \\
   \alpha^3 + \beta^3 + \gamma^3 &= 17.
   \end{align*}
   
   Find \(\alpha\beta\gamma\).

5. How many ways are there to make 1 with quarters, dimes, and nickels? How about if one also allows 1 cent coins?

6. Find a formula for \(a_n\), where \(a_0 = a_1 = 2\), and \(a_{n+1} = 4a_n - 4a_{n-1}\).

7. Find \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n\).

8. Suppose that \(x_0 = 18\), \(x_{n+1} = \frac{10x_n}{3} - x_{n-1}\), and that the sequence \(\{x_n\}\) converges to some real number. Find \(x_1\).

9. Let \(\{x_n\}_{n \geq 0}\) be a sequence of nonzero real numbers such that \(x_n^2 - x_{n-1}x_{n+1} = 1\) for all \(n \geq 1\). Prove there exists a real number \(a\) such that \(x_{n+1} = ax_n - x_{n-1}\) for all \(n \geq 1\).

10. Let 1, 2, 3, \ldots, 2006, 2007, 2009, 2012, 2016, \ldots be a sequence defined by \(x_k = k\) for \(k = 1\) to 2006 and \(x_{k+1} = x_k + x_{k-2005}\) for \(k \geq 2006\). Show that the sequence has 2005 consecutive terms each divisible by 2006.