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Almost Rational Torsion Points on Semistable Elliptic Curves

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1 Definitions and results

Let X be an algebraic curve of genus greater than 1. Let $J(X)$ be the Jacobian variety of X, and embed X in $J(X)$. The Manin-Mumford conjecture states that the set of torsion points $X_{\text{tors}} := X \cap J_{\text{tors}}$ is finite. This conjecture was first proved in 1983 by M. Raynaud [9]. It has long been known that the geometry of X imposes strong conditions on the action of Galois on X_{tors}. An approach to the Manin-Mumford conjecture using Galois representations attached to Jacobians was first suggested by S. Lang [6]. Recently, by exploiting the relationship between the action of Galois on modular Jacobians and the Eisenstein ideal (developed by B. Mazur [8]), M. Baker [1] and (independently) A. Tamagawa [18] explicitly determined the set of torsion points of $X_0(N)$ for N prime. Developing these ideas further, K. Ribet defined the notion of an *almost rational torsion point* and used this concept to derive the Manin-Mumford conjecture (see [12]) using some unpublished results of J.-P. Serre [14]. One idea suggested by these papers is that a possible approach to finding all torsion points on a curve X is to determine the set of almost rational torsion points on $J(X)$. Moreover, the concept of an almost rational torsion point makes sense for any Abelian variety or, more generally, for any commutative group scheme, and the set of such points may be interesting to study in their own right. In this paper we consider almost rational points on semistable elliptic curves over $\mathbb{Q}$, and we prove (in particular) that they are all defined over $\mathbb{Q}(\sqrt{-3})$.

Let K be a perfect field, and let A/K be a commutative algebraic group scheme.

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Definition 1. Let $P \in A(\overline{K})$. Then P is almost rational over K if the following holds. For all $\sigma, \tau \in \mathrm{Gal}(\overline{K}/K)$, the equality

$$\sigma P + \tau P = 2P$$

is satisfied only when $P = \sigma P = \tau P$. Equivalently, P is almost rational if any two nontrivial Galois conjugates of P have a sum different from $2P$. If P is also a torsion point of A, then we call P an *almost rational torsion point*.

Lemma 1.1. Let P be an almost rational point over K.

(1) If $L \supset K$ is a subfield of $\overline{K}$, then P is almost rational over L.

(2) If Q is a K-rational point of A, then Q is almost rational over K.

(3) If Q is a K-rational point of A, then $P + Q$ is almost rational over K.

(4) Let $\gamma \in \mathrm{Gal}(\overline{K}/K)$. Then γP is almost rational over K.

(5) Let $\sigma \in \mathrm{Gal}(\overline{K}/K)$. If $(\sigma - 1)^2 P = 0$, then $\sigma P = P$.

(6) If $2P$ is K-rational, then so is P. □

Proof. Statements (1), (2), and (3) are obvious. For statement (4), suppose that $\sigma\gamma P + \tau\gamma P = 2\gamma P$. Then

$$\gamma^{-1}\sigma\gamma P + \gamma^{-1}\tau\gamma P = 2P.$$

Since P is almost rational, $P = \gamma^{-1}\sigma\gamma P = \gamma^{-1}\tau\gamma P$, and thus $\gamma P = \sigma\gamma P = \tau\gamma P$. For statement (5), one has $\sigma^2 P - 2\sigma P + P = 0$. Applying σ^{-1} to the left-hand side of this equation, one finds that $\sigma P + \sigma^{-1} P = 2P$. Setting $\tau = \sigma^{-1}$ and using the fact that P is almost rational, we conclude that $P = \sigma P$. For statement (6), let σP be any Galois conjugate of P. Then

$$\sigma P + \sigma P = 2\sigma P = \sigma(2P) = 2P,$$

and so, by almost rationality, $P = \sigma P$ for all σ, and so P is K-rational. ■

Caution. We will see in Theorem 1.2 that the set of almost rational points does not necessarily form a group. In fact, the almost rationality of P does not imply the almost rationality of multiples of P.

Lemma 1.2. Let $A = \mathbb{G}_m/\mathbb{Q}$. Then the almost rational torsion points on A are exactly the points of order dividing 6. Let $H = \mu_n/K$, where K is any field such that $\mathrm{Gal}(K(\zeta_n)/K) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. Then the almost rational torsion points on H are the points of order dividing 6. □

Proof. Let P be a torsion point of order n on A. The Galois module generated by P is the cyclotomic module μ_n. The action of Galois on μ_n is via the mod n cyclotomic character $\chi_n : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to (\mathbb{Z}/n\mathbb{Z})^*$; that is, we have $\sigma P = \chi_n(\sigma)P$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To show that P is *not* almost rational, it suffices to find σ, τ such that $\chi_n(\sigma)P + \chi_n(\tau)P = 2P$ and $P \neq \sigma P$. In particular, since χ_n acts faithfully on μ_n, it suffices to find $a, b \in (\mathbb{Z}/n\mathbb{Z})^*$ such that $a \neq 1$ and $a + b = 2$. We attempt to do this using the Chinese remainder theorem. If $p^k \| n$ and $p \neq 3$, let $a \equiv 3 \mod p^k$ and $b \equiv -1 \mod p^k$. If $3^k \| n$, let $a \equiv 4 \mod 3^k$ and $b \equiv -2 \mod 3^k$. As long as $n \nmid 6$, we find that $a \neq 1$, and so P fails to be almost rational. If P is of order 1 or 2, then P is rational and so is almost rational by Lemma 1.1(2). If P is of order 3 (resp., 6), then the only nontrivial Galois conjugate of P is $\sigma P = 2P$ (resp., $\sigma P = 5P$). In either of these cases, $\sigma P + \sigma P \neq 2P$, and so P is almost rational. A similar argument proves the result for H. ■

The following lemma provides a connection between almost rational points and torsion points on curves, as well as providing a natural source of almost rational points.

Lemma 1.3. Let X/K be a curve of genus $g \geq 2$, and let $Q \in X(K)$ be a K-rational point on X. Let J/K be the Jacobian of X, and let $i_Q : X \hookrightarrow J$ be the Albanese map $P \mapsto [P - Q]$ defined over K. Then, for any point $P \in X(\overline{K})$, either P is a hyperelliptic branch point of X, or $i_Q(P)$ is almost rational over K. □

Proof. Since i_Q is a closed immersion, we identify points of X with their images. Assume that $\sigma P + \tau P = 2P$. Then, since $J(\overline{K}) \simeq \mathrm{Pic}^0(X(\overline{K}))$, the divisor $D = (\sigma P) + (\tau P) - 2(P)$ is principal and so equals (f) for some function f. Either f is constant, in which case $P = \sigma P = \tau P$, or f is of degree 2, in which case P is a hyperelliptic branch point of f. The result follows. ■

One of the main motivations for studying almost rational torsion points is the following result of Ribet [12].

Theorem 1.1. Let A/K be an Abelian variety. Then the set of almost rational torsion points on A is finite. □

Remark. Ribet's proof depends on some of Serre's unpublished "big-image" theorems (see [14]) and is not effective. Note that the Manin-Mumford conjecture follows from Theorem 1.1 when X is embedded in its Jacobian via an Albanese map. This paper provides an effective version of Theorem 1.1 when $A/\mathbb{Q}$ is a semistable elliptic curve, and in this case we give a complete classification of the possible almost rational torsion points that can arise.

In contrast, the following result (pointed out by B. Poonen) shows that there is an abundance of almost rational points that are *not* torsion points.

Lemma 1.4. For any $P \in A(\overline{\mathbb{Q}})$, there exists an $n > 0$ such that nP is almost rational. □

Proof. Since P is algebraic, it is defined over some Galois field $K/\mathbb{Q}$, and it has only finitely many Galois conjugates. Choose n such that every torsion point of the form $\sigma P - \tau P$ is killed by n. Then $Q = nP$ is almost rational. Indeed, by construction, the Galois orbit S of Q injects into $A(K) \otimes \mathbb{R}$. If Q' is some extremal point of the convex hull of S (with respect to the canonical height), then the identity $\sigma Q' + \tau Q' = 2Q'$ can hold only if $Q' = \sigma Q' = \tau Q'$. Thus Q' is almost rational. Since Q' is a Galois conjugate of Q, it follows by Lemma 1.1(4) that Q is also almost rational. ■

The main result of this paper is the following theorem.

Theorem 1.2. Let P be an almost rational torsion point over $\mathbb{Q}$ on a semistable elliptic curve $E/\mathbb{Q}$. Then either P is rational, or it can be written as a sum $Q + R + S$, where

(1) Q generates a (cyclotomic) μ_3 subgroup of $E[3]$;
(2) R is a rational point of 3-power order, $R \in E(\mathbb{Q})[9]$;
(3) S is a $\mathbb{Q}(\zeta_3)$-rational point of 2-power order, $S \in E(\mathbb{Q}(\sqrt{-3}))[16]$.

Moreover, all sums of this form are almost rational. □

Example. Let E be the elliptic curve

$$y^2 + yx + y = x^3 + 354x + 4684$$

of conductor $N = 1302 = 2 \cdot 3 \cdot 7 \cdot 31$. Then (as Galois modules) $E[3] \simeq \mu_3 \oplus \mathbb{Z}/3\mathbb{Z}$, and the point $S = (-3 - 6\sqrt{-3}, 5 + 12\sqrt{-3})$ is a torsion point of order 2. If Q is a 3-torsion point that generates μ_3, then $P = Q + S$ is an almost rational torsion point on E. Note that $3P = S$ is *not* an almost rational torsion point.

The key idea of the proof of Theorem 1.2 is to limit the ramification of $E[\ell]$ for the largest prime divisor ℓ of $|P|$. Ribet's level lowering result [10] can then be used to show that $E[\ell]$ is reducible. There are several reasons why we restrict our attention to elliptic curves instead of higher-dimensional Abelian varieties. In particular, we use many strong results about representations arising from elliptic curves (such as modularity) which have no convenient equivalent in higher dimensions. Secondly, we rely on the explicit determination, by Mazur [8], of the possible rational torsion subgroups in $E(\mathbb{Q})$. Some results, however, do apply in more generality, such as Lemma 1.5.

Definition 2. Let P be an almost rational torsion point. Then the Galois module M associated to P is the module generated by P and all its Galois conjugates:

$$M = \sum \mathbb{Z} \cdot \sigma P.$$

Since P is a torsion point, M is finite as a Galois module.

Example. If P is rational of order n, then (as Galois modules) $M \simeq \mathbb{Z}/n\mathbb{Z}$.

Remark. Throughout this paper, the word "finite" is used in two different senses. "M is finite as a Galois module" means that, as an Abelian group, M is finite. "M is finite as a group scheme" means that there exists some finite flat group scheme $\mathfrak{M}/\operatorname{Spec}\mathbb{Z}$ such that $\mathfrak{M}(\mathbb{Q}) \simeq M$. Unless explicitly stated, we reserve finite to mean finite as a group scheme. "M is finite at a prime p" means that there exists some finite flat group scheme $\mathfrak{M}_p/\operatorname{Spec}\mathbb{Z}_p$ such that $\mathfrak{M}_p(\mathbb{Q}_p) \simeq M$ as $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ modules. M is finite if and only if it is finite at all primes p. If the cardinality of M is coprime to p, then M is finite at p if and only if M is unramified at p.

Lemma 1.5. Let A/K be a semistable Abelian variety, and let $P \in A(\overline{K})$ be an almost rational point on A of order n. Then the Galois module M generated by P is finite at all primes not dividing n. In other words, M is *unramified* outside primes dividing n. □

Proof. Let $\mathfrak{p}$ be a prime coprime to n. Let $I_\mathfrak{p} \subseteq \mathrm{Gal}(\overline{K}/K)$ be a choice of inertia group above $\mathfrak{p}$. Then, since A is semistable, the action of $I_\mathfrak{p}$ on the Tate module $T_\ell(A)$ for $\ell \mid n$ (or for any $(\ell, \mathfrak{p}) = 1$) satisfies $(\sigma - 1)^2 = 0$ for all $\sigma \in I_\mathfrak{p}$ (see [4, exposé IX, Proposition 3.5]). In particular, writing P in terms of its ℓ-primary components, we find that $(\sigma - 1)^2 P = 0$. Since P is almost rational, by Lemma 1.1(5), $\sigma P = P$. Applying the same argument to all the conjugates of P (which are still almost rational by Lemma 1.1(4)), we find that M is unramified at $\mathfrak{p}$, and we are done. ■

Remark. For semistable elliptic curves, the fact that inertia at p satisfies $(\sigma - 1)^2 \mid T_\ell(E) = 0$ for $(\ell, p) = 1$ can be proved directly without appeal to results of A. Grothendieck. In particular, either E has good reduction at p, in which case $T_\ell(E)$ is unramified at p (by the Néron-Ogg-Shafarevich criterion), or E has multiplicative reduction at p, in which case the result follows from the explicit description of Tate curves given, for example, in [17, Chapter V].

2 Finiteness of $\widetilde{\rho}$

Let p be prime. Let ρ_p denote the Galois representation associated to the Tate module $T_p(E)$. Let $\widetilde{\rho}_p$ denote the mod p representation arising from the action of Galois on $E[p]$.

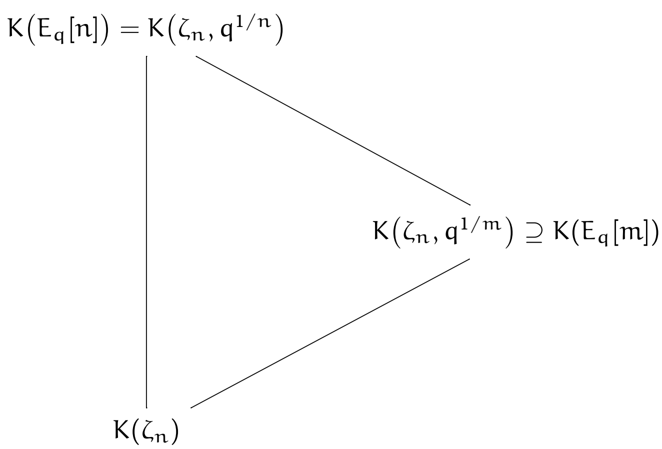
One sees that $\widetilde{\rho}_p$ is the reduction of $\rho_p \mod p$.

Fix a semistable elliptic curve $E/\mathbb{Q}$, fix an almost rational torsion point P of order n, and fix its associated Galois module M. Lemma 1.5 shows how one can control the ramification of M at primes away from n. In this section, we show how it is also sometimes possible to control the ramification at primes p dividing n.

Theorem 2.1. Let $p \mid n$ be a prime such that $E[p]$ is irreducible. Then $\widetilde{\rho}_p$ is finite at p. □

Proof. If E has good reduction at p, then we are done, so we may assume that E has multiplicative reduction at p. We use the criterion, due to Serre [15, Proposition 5, p. 191], that $\widetilde{\rho}_p$ is finite at p (*peu ramifié*) if and only if $v_p(\Delta) \equiv 0 \mod p$.

Let $n = p^k m$ with $(m, p) = 1$. Write $P = P_p + P'$, where P_p is of order p^k, and P' is of order m. Since $M \cap E[p] \neq 0$ and $E[p]$ is irreducible, $E[p] \subseteq M$. Let E_q denote the Tate curve isomorphic to E over K, where $[K : \mathbb{Q}_p]$ is an unramified extension of degree less than or equal to 2. Since $K(\zeta_n)(E_q[n]) = K(\zeta_n, q^{1/n})$, we have the following diagram of fields:



We show that the extension $K(\zeta_n, q^{1/n})/K(\zeta_n, q^{1/m})$ is trivial. Assume otherwise. Then, since $K(E_q[n])/K$ is Galois, there exists a nontrivial Galois automorphism $\sigma \in \mathrm{Gal}(K(E_q[n])/K)$ fixing $K(\zeta_n, q^{1/m})$. If σ exists, then we may take it to be of order p and of the form $\sigma : q^{m/n} \longrightarrow q^{m/n}\zeta_p$. We choose a basis $\{q^{m/n}, \zeta_{p^k}\}$ for $E_q[p^k]$ such that σ acts via the matrix

$$\begin{pmatrix} 1 & p^{k-1} \\ 0 & 1 \end{pmatrix} \mod p^k.$$

Since σ fixes $E_q[m]$, $(\sigma - 1)^2 P = 0$. Considering σ as an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via the inclusion

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/K) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

we conclude from the almost rationality of P that $\sigma P = P$. Thus, with respect to our chosen basis for $E_q[p^k]$, P_p is of the form

$$\begin{pmatrix} a \\ bp \end{pmatrix}.$$

From Lemma 1.1(4), we may apply the same argument above to γP for every $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and so all Galois conjugates of P_p are also of this form. It follows that the mod p Galois representation $\widetilde{\rho}_p$ is upper triangular, contradicting the irreducibility of $E[p]$. Thus $K(\zeta_n, q^{1/n})/K(\zeta_n, q^{1/m})$ is the trivial extension.

It follows that $q^{1/n} \in L := K(\zeta_n, q^{1/m})$, and so in particular $q^{1/m}$ is a p^kth power in L, and thus its normalized valuation with respect to this field satisfies $\nu_L(q) \equiv 0 \mod p^k$. Since $(m, p) = 1$, the wild ramification of $K(\zeta_n, q^{1/m})$ is limited to $K(\zeta_{p^k})$, for which $p^{k-1} \| e(K(\zeta_{p^k})/\mathbb{Q}_p)$. Since $\nu_L(q) = e(L/\mathbb{Q}_p) \cdot \nu_{\mathbb{Q}_p}(q)$, the valuation of q with respect to $\mathbb{Q}_p$ must be divisible by p. Thus

$$\mathrm{ord}_p(\Delta_q) = \mathrm{ord}_p\left(q \prod_{i=1}^{\infty}(1-q^n)^{24}\right) = \mathrm{ord}_p(q) \equiv 0 \mod p,$$

and therefore $E[p]$ is finite at p. ■

Theorem 2.2. Let ℓ be the largest prime factor of $n = |P|$. Then $E[\ell]$ is reducible. □

Proof. Assume otherwise. From Theorem 2.1, since $E[\ell]$ is irreducible, $\widetilde{\rho}_\ell$ is finite at ℓ. Let $p \neq \ell$ be prime. We show that $E[\ell]$ is unramified at p or that $\ell = 3$ and $p = 2$. In fact, with no assumptions on $E[\ell]$, we show more generally that $M[\ell^\infty]$ is unramified at p. The irreducibility assumption on $E[\ell]$ then ensures that $E[\ell] \subseteq M[\ell^\infty]$ and thus that $E[\ell]$ is also unramified at p.

For primes $p \nmid n$, the result follows from Lemma 1.5. Hence it suffices to consider $p \neq \ell$ such that $p \mid n$. Write P in terms of its q-primary components

$$P = P_2 + P_3 + \cdots + P_p + \cdots + P_\ell,$$

where P_q is a torsion point of order $q^k \| n$. Let I_p denote a choice of the inertia group at p. Let $\sigma \in I_p$ be an element of the inertia group. Then, by the proof of Lemma 1.5 (or by the subsequent remark), $(\sigma - 1)^2 P_q = 0$ for $q \neq p$. Consider the following diagram of fields:

$$\begin{array}{ccc} \mathbb{Q}(M[\ell^\infty]) & \text{———} & \mathbb{Q}(M) \\ | & & | \\ \mathbb{Q} & \text{———} & \mathbb{Q}(M[p^\infty]) \end{array}$$

For $\sigma \in I_p$ fixing $M[p^\infty]$, $\sigma P_p = P_p$, and so $(\sigma - 1)^2 P_p = 0$. For such σ, $(\sigma - 1)^2 P = 0$, and thus, by Lemma 1.1(5), $\sigma P = P$. Applying this to the Galois conjugates of P, one concludes that the extension $\mathbb{Q}(M)/\mathbb{Q}(M[p^\infty])$ is unramified at all primes above p. Comparing ramification indices at p in the diagram above,

$$e_p\big(\mathbb{Q}(M)/\mathbb{Q}(M[\ell^\infty]\big) \cdot e_p\big(\mathbb{Q}(M[\ell^\infty])/\mathbb{Q}\big) = e_p\big(\mathbb{Q}(M[p^\infty])/\mathbb{Q}\big).$$

If P_p is of order p^k, then $M[p^\infty] \subseteq E[p^k]$; thus $e_p(\mathbb{Q}(M[p^\infty])/\mathbb{Q})$ divides the order of $GL_2(\mathbb{Z}/p^k\mathbb{Z}) = (p^2 - 1)(p - 1)p^{4k-3}$, and so, in particular, $e := e_p(\mathbb{Q}(M[\ell^\infty]/\mathbb{Q})$ also divides this number. Yet the action of I_p on $T_\ell(E)$ is unipotent, and so e is either 1 or some power of ℓ. Since ℓ is the *largest* prime factor of n, $p < \ell$, and thus

$$\big(\ell, (p^2 - 1)(p - 1)p^{4k-3}\big) = (\ell, p + 1),$$

which equals 1 unless $(\ell, p) = (3, 2)$. Thus if $\ell \neq 3$, $M[\ell^\infty]$ is unramified outside ℓ, and, for all ℓ, $M[\ell^\infty]$ is unramified outside 2 and ℓ. If $E[\ell]$ is irreducible, then the Serre conductor (see [15]) $N(\widetilde{\rho}_\ell)$ is equal to 1 or 2. (Since E is semistable, the exponent of 2 in the Serre conductor must be at most 1.) By A. Wiles [22] and R. Taylor and Wiles [20], $\widetilde{\rho}_\ell$ is modular. Since $\widetilde{\rho}_\ell$ is finite at ℓ, for $\ell > 2$, Ribet's level lowering result in [10] implies that $\widetilde{\rho}_\ell$ arises from a weight 2 modular form of level 1 or 2. No such form exists. This is a contradiction, and the theorem follows. For $\ell = 2$ and $E[2]$ irreducible, one finds that $N(\widetilde{\rho}_2) = 1$, contradicting a theorem of J. Tate [19]. ■

Corollary 2.1. Let ℓ be any prime divisor of n. Then $M[\ell^\infty]$ is at most ramified at ℓ and primes $p|n$ such that $\ell \mid (p^2 - 1)$. Moreover, if $p \neq \ell$ and $E[p]$ is reducible, then $M[\ell^\infty]$ can only be ramified at p if $\ell \mid (p - 1)$. □

Proof. The first part of the corollary is proved during the proof of Theorem 2.2. If $E[p]$ is reducible, then $[\mathbb{Q}(E[p]) : \mathbb{Q}]$ divides $p(p - 1)$, and so arguing as in Theorem 2.2 we conclude that ℓ divides $(p - 1)$. ■

Corollary 2.2. Let P be an almost rational torsion point of order n. Then if ℓ is the largest prime dividing n, $\ell \leq 7$. □

Proof. If E is reducible at p, then, since E is semistable, it follows from Serre [13, Proposition 21, p. 306] (and subsequent remarks) that either E or some isogenous curve E' has a rational point of order p. The result then follows from the following theorem of Mazur [8, Theorem 8, p. 35]. ■

Theorem 2.3 [8]. Let Φ be the torsion subgroup of the Mordell-Weil group of an elliptic curve over $\mathbb{Q}$. Then Φ is isomorphic to one of the following 15 groups: $\mathbb{Z}/m\mathbb{Z}$ for $m \leq 10$ or 12; $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2k\,\mathbb{Z}$ for $k \leq 4$. □

3 The possible cases

Corollary 2.2 severely limits the possible prime divisors of n. In this section, we eliminate all possibilities not allowed by Theorem 1.2. Let S denote the set of primes dividing n. Corollary 2.2 restricts S to 15 nontrivial possibilities, which divide into three cases.

3.1 Case I: $S = \{p\}$, $n = p^k$

When $n = p^k$, our strategy is as follows. First we show that M is an extension of a trivial module by a cyclotomic module. If $p \geq 3$, this sequence splits, and it suffices to find almost rational torsion points on the cyclotomic module μ_{p^r}, which we calculated in Lemma 1.2. For $p = 2$, some complications arise, but the essential ideas remain the same. Some of our arguments can be shortened using results of Tamagawa (e.g., [18, Theorem 3.2]); however, we proceed directly in order to be self-contained.

We begin by recalling an elementary result from the theory of cyclotomic fields.

Lemma 3.1. Let $h(K)$ denote the class number of the field K. Suppose that p is inert in K. Then, for $k \geq 1$,

$$p \mid h(K(\zeta_p)) \Longleftrightarrow p \mid h\big(K(\zeta_{p^k})\big).$$

In other words, p divides $h(K(\zeta_{p^k}))$ for all k if and only if p is an irregular prime with respect to K. □

For a proof, see, for example, L. Washington [21, Theorem 10.4].

Lemma 3.2. If $p \leq 7$, $p \nmid h(\mathbb{Q}(\zeta_{p^k}))$. Moreover, $2 \nmid h(\mathbb{Q}(\zeta_{3\cdot 2^k}))$ and $2 \nmid h(\mathbb{Q}(\zeta_{5\cdot 2^k}))$. For each of these fields K, if L/K is a Galois extension such that $\mathrm{Gal}(L/K)$ is a p-group and L/K is unramified, then $L = K$. □

Proof. The first statement is a consequence of Lemma 3.1, of the fact that the fields $\mathbb{Q}(\zeta_n)$ have class number 1 for $n = 1, 3, 4, 5, 7, 12, 20$, and of the fact that 2 is inert in $\mathbb{Q}(\zeta_p)$ for $p = 3, 5$. The second statement follows from the identification of the class group with the

Galois group of the Hilbert class field and from the fact that all p-groups are solvable. ■

Assume that $|P| = n = p^k$ for some k. Recall that M is the module generated by P and its conjugates.

Lemma 3.3. We have $\mathbb{Q}(M) = \mathbb{Q}(M[p^k]) \subseteq \mathbb{Q}(\zeta_{p^k})$. □

Proof. By Lemma 1.5, we find that M is unramified outside p. By Theorem 2.2, E is reducible at p. Since elliptic curves with supersingular reduction at p are automatically irreducible at p, E has either multiplicative reduction or good ordinary reduction at p. In either case, the action of inertia at p on $E[p^k]$ is given by

$$(\rho_p \mod p^k)|_{I_p} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix},$$

where χ is the cyclotomic character, and $*$ is possibly trivial. It follows from the almost rationality of P that M is unramified over $\mathbb{Q}(\zeta_{p^k})$, since

$$\chi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^k}))} \equiv 1 \mod p^k,$$

and so all inertial elements fixing $\mathbb{Q}(\zeta_{p^k})$ satisfy $(\sigma - 1)^2 P = 0$.

Since E is reducible at p, $\mathbb{Q}(E[p], \zeta_p)/\mathbb{Q}(\zeta_p)$ is a Galois extension of degree dividing p. $\mathrm{Gal}(\mathbb{Q}(E[p^k])/\mathbb{Q}(E[p]))$ is automatically a p-group, and thus $\mathrm{Gal}(\mathbb{Q}(E[p^k])/\mathbb{Q}(\zeta_p))$ is also a p-group. Since $M \subseteq E[p^k]$, it follows that the extension $\mathbb{Q}(M, \zeta_{p^k})/\mathbb{Q}(\zeta_{p^k})$ is an unramified Galois extension whose Galois group is a p-group. Thus, by Lemma 3.2, $\mathbb{Q}(M) \subseteq \mathbb{Q}(\zeta_{p^k})$, as claimed. ■

Suppose that E has multiplicative reduction at p. Then, locally at p, E is given by a Tate curve E_q. Let I_p denote the absolute inertia group of $\mathbb{Q}_p$. For each n, we have an exact sequence of I_p-modules:

$$0 \longrightarrow \mu_{p^n} \longrightarrow E_q[p^n] \xrightarrow{\psi} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0.$$

Let $M' = M \cap \mu_{p^k}$ and $M'' = \psi(M) \subseteq \mathbb{Z}/p^k\mathbb{Z}$. Then by the snake lemma we have an exact sequence of I_p-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Since M is defined over $\mathbb{Q}(\zeta_{p^k})$, this is in fact an exact sequence of G-modules, where $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^k})/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^k})/\mathbb{Q}_p) \hookrightarrow I_p$ and where the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on M factors through G. Moreover, we observe that (as Galois modules) M'' is constant and M' is cyclotomic.

Now suppose that E has ordinary good reduction at p. On the level of inertia, multiplicative reduction and ordinary good reduction are highly analogous. In particular, from [16, Section VII, Proposition 2.1] and [13, Proposition 11], $E[p^n]$ sits in an exact sequence of I_p-modules:

$$0 \longrightarrow \mu_{p^n} \longrightarrow E[p^n] \xrightarrow{\psi} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0,$$

where ψ is the reduction mod p on the elliptic curve. This is precisely the I_p-module sequence arising when E has multiplicative reduction, and so we may treat these cases simultaneously.

Assume for the moment that $p \geq 3$. Taking G-invariants of the above sequence, we obtain an exact sequence of Galois cohomology:

$$0 \longrightarrow H^0(G, M) \longrightarrow H^0(G, M'') \longrightarrow H^1(G, M').$$

Since G is cyclic and $p \geq 3$, by Sah's lemma (see [7, Chapter 8, Lemma 8.1]), $H^1(G, M') = 0$. Since M'' is a constant module, $H^0(G, M'') = M''$. Thus we obtain a map

$$M'' \simeq H^0(G, M) \hookrightarrow M$$

which induces a splitting (as Galois modules) of our exact sequence. Thus $M \simeq \mu_{p^r} \oplus \mathbb{Z}/p^s\mathbb{Z}$ for some integers r, s. From Lemma 1.2, it follows either that P is rational or that $p = 3$, $r = 1$, and P is a rational point of 3-power order plus a point that generates a μ_3 subgroup of $E[3]$. From Theorem 2.3, this rational point is of order dividing 9.

Now assume that $p = 2$. Consider the following sequence of G-modules:

$$0 \longrightarrow M' \longrightarrow M \xrightarrow{\psi} M'' \longrightarrow 0.$$

Let H be the minimal quotient of G which acts on M'. Then $H = \mathrm{Gal}(\mathbb{Q}(\zeta_{2^r})/\mathbb{Q})$, where $r = \log_2 |M'|$. We show that the action of G on M factors through H. Let $\sigma \in G$ become trivial in H (so σ acts trivially on M'). Since $\psi(\sigma P - P) = \sigma\psi(P) - \psi(P) = 0$, one sees that $\sigma P - P \in M'$. Thus $(\sigma - 1)^2 P = (\sigma - 1)(\sigma P - P) = 0$. By almost rationality, $\sigma P = P$. Since all the conjugates of P are also almost rational, we see that σ fixes M, and the claim follows.

Let $Q \in M$. The map $H \to M'$ given by $\sigma \mapsto \sigma Q - Q$ defines an element of

$$H^1(H, M') = H^1\left(\mathrm{Gal}\left(\mathbb{Q}(\zeta_{2^r})/\mathbb{Q}\right), \mu_{2^r}\right) \simeq H^1\left(\left(\mathbb{Z}/2^r\mathbb{Z}\right)^*, \mathbb{Z}/2^r\mathbb{Z}\right).$$

Sah's lemma shows only that this group is killed by 2. In fact, an elementary argument shows that it is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, where the nontrivial cocycle class is given by

$$f(-1 \bmod 2^r) = 2^{r-1} \mod 2^r, \qquad f(-3 \bmod 2^r) = 0 \mod 2^r.$$

(Here 0 and 2^{r-1} correspond to the elements 1 and -1 in μ_{2^r}.) The cocycle picked out by Q is the image of Q under the composition

$$M \xrightarrow{\psi} M'' \xrightarrow{\delta} H^1(H, M').$$

This cocycle is given by $\delta_{\psi(Q)} : \sigma \mapsto \sigma Q - Q$. Since $H^1 \simeq Z/2\mathbb{Z}$, we may write $\delta_{\psi(Q)}$ as 0 or f plus some coboundary. In particular, $\sigma Q - Q = \sigma Q' - Q' + T_\sigma$ for all σ and some fixed (depending only on Q) $Q' \in M'$, and where T_σ is either 0 or $f(\sigma)$. In particular, T_σ is Galois invariant (since ± 1 is Galois invariant in μ_{2^r}) and killed by 2. Thus for any $Q \in M$ we may write $Q = Q' + Q''$ with $Q' \in M'$ such that

$$\sigma Q = Q + \sigma Q' - Q' + T_\sigma = \sigma Q' + Q'' + T_\sigma.$$

Moreover, since T_σ is a cocycle, $T_{\tau\sigma} = \tau T_\sigma + T_\tau = T_\sigma + T_\tau$.

Choose $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\chi(\sigma) \equiv 1 + 4k \mod 2^r, \qquad \chi(\tau) \equiv 1 - 4k \mod 2^r.$$

Note that $(1+4k)(1-4k) \equiv 1 \mod 8$, and so $\tau\sigma^{-1} = \gamma^2$ for some $\gamma \in H$. Thus $T_\tau = T_{\tau\sigma^{-1}} + T_\sigma = 2T_\gamma + T_\sigma = T_\sigma$. One finds that $\sigma P + \tau P - 2P = T_\sigma - T_\tau = 0$. By almost rationality, $P = \sigma P$. Thus the action of H on P factors through

$$\mathrm{Gal}\left(\mathbb{Q}(\zeta_{2^r})/\mathbb{Q}\right)/\{\sigma \mid \chi(\sigma) \equiv 1 \mod 4\} \simeq \begin{cases} \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}), & r \geq 2, \\ 1, & r \leq 1, \end{cases}$$

and so P (and hence M) is defined over $\mathbb{Q}(i)$. In particular, since $H = \mathrm{Gal}(\mathbb{Q}(\zeta_{2^r})/\mathbb{Q})$ acts faithfully on $M' \subseteq M$, we conclude that $r \leq 2$. Since $r \leq 2$ and $P' \in M'$, it follows that $2P'$ is either 1 or -1 in μ_{2^r} and so in particular is rational. For any $\sigma \in H$,

$$\sigma(2P) = 2\sigma(P' + P'') = 2\sigma P' + 2P'' + 2T_\sigma = \sigma(2P') + 2P'' = 2P,$$

and so 2P is rational. It follows from Lemma 1.1(6) that P is rational.

3.2 Case II: $S = \{2, p\}$, $n = p^m 2^k$, $p = 5, 3$

From Theorem 2.2, $E[p]$ is reducible. We show that $E[2]$ is also reducible. Assume otherwise. From Lemma 1.5, $E[2]$ is unramified outside 2 and p, and thus $N(\widetilde{\rho}_2)$ is equal to 1 or p. Since $E[2]$ is irreducible, Tate's theorem in [19] eliminates the first possibility. Thus the representation $\widetilde{\rho}_2$ is genuinely ramified at p and, from Theorem 2.1, is finite at 2. The arguments of [10] do not apply at the prime 2. However, another result of Ribet [11] allows us (since E is modular) to conclude that $\widetilde{\rho}_2$ arises from a modular form of weight 2 and level at most p. Since $p \leq 5$, the space of such forms is trivial, and so $E[2]$ is reducible.

Since $E[2]$ is reducible, Corollary 2.1 shows that $M[p^m]$ is unramified outside p. Arguing as in Lemma 3.3, we infer that $\mathbb{Q}(M[p^m]) \subseteq \mathbb{Q}(\zeta_{p^m})$. For the proof to go through, it suffices to recall that any $\sigma \in I_p$ satisfying $(\sigma - 1)^2 P_p = 0$ (indeed, any $\sigma \in I_p$) automatically satisfy $(\sigma - 1)^2 P_2 = 0$.

Lemma 3.4. $\mathbb{Q}(M[2^k], \zeta_p)/\mathbb{Q}(\zeta_p)$ is unramified at all primes above p. □

Proof. Consider the following diagram of fields:

$$\begin{array}{ccc} \mathbb{Q}(\zeta_{p^m}) & \text{———} & \mathbb{Q}(\zeta_{p^m}, M) = \mathbb{Q}(\zeta_{p^m}, M[2^k]) \\ | & & | \\ \mathbb{Q}(\zeta_p) & \text{—————} & \mathbb{Q}(\zeta_p, M[2^k]) \end{array}$$

Any element of $\mathrm{Gal}(\mathbb{Q}(M, \zeta_{p^m})/\mathbb{Q}(\zeta_{p^m}))$ (trivially) fixes $\mathbb{Q}(\zeta_{p^m})$ and thus $M[p^m]$ (since $\mathbb{Q}(M[p^m]) \subseteq \mathbb{Q}(\zeta_{p^m})$). Since E is semistable, any $\sigma \in I_p$ satisfy $(\sigma - 1)^2 P_2 = 0$. In particular, any inertia at primes above p in $\mathbb{Q}(M, \zeta_{p^m})/\mathbb{Q}(\zeta_{p^m})$ satisfies $(\sigma - 1)^2 P_2 = 0$ and $\sigma P_p = 0$, and so $(\sigma - 1)^2 P = 0$. By almost rationality, $\sigma P = P$, and thus the extension $\mathbb{Q}(\zeta_{p^m}, M)/\mathbb{Q}(\zeta_{p^m})$ is unramified at all primes above p. One concludes that the ramification index $e_p(\mathbb{Q}(\zeta_{p^m}, M)/\mathbb{Q}(\zeta_p))$ is equal to $e_p(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}(\zeta_p)) = p^{m-1}$. Considering the other part of the diagram, however, one notes (again using semistability) that the ramification of $\mathbb{Q}(M[2^k])/\mathbb{Q}$, and thus of $\mathbb{Q}(M[2^k], \zeta_p)/\mathbb{Q}(\zeta_p)$ at p, is of 2-power order. Since this order must divide p^{m-1}, we conclude that $\mathbb{Q}(\zeta_p, M[2^k])/\mathbb{Q}(\zeta_p)$ is unramified at all primes above p. ■

From Lemma 1.5, M is unramified outside 2 and p. As in the proof of Lemma 3.3, any inertial elements $\sigma \in I_2$ fixing $\mathbb{Q}(\zeta_{2^k})$ satisfy $(\sigma - 1)^2 P_2 = 0$, and, by semistability, $(\sigma - 1)^2 P_p = 0$. Thus $\mathbb{Q}(M, \zeta_{2^k})$ is unramified at all primes above 2 in $\mathbb{Q}(\zeta_{2^k})$. Combining this with Lemma 3.4, we infer that $\mathbb{Q}(M[2^k], \zeta_p, \zeta_{2^k})/\mathbb{Q}(\zeta_p, \zeta_{2^k})$ is unramified everywhere. Since $\mathbb{Q}(E[2^k])/\mathbb{Q}$ is a 2-extension (as $E[2]$ is reducible), we infer from Lemma 3.2 that

$$\mathbb{Q}(M[2^k]) \subseteq \mathbb{Q}(\zeta_{2^k}, \zeta_p).$$

Thus we have shown that $\mathbb{Q}(M) \subseteq \mathbb{Q}(\zeta_n)$. In particular, the Galois group $\mathrm{Gal}(\mathbb{Q}(M, \zeta_p)/\mathbb{Q}(\zeta_p))$ decomposes as a product of two groups, each of which acts trivially on either $M[p^m]$ or $M[2^k]$.

Lemma 3.5. P_p and P_2 are themselves almost rational over $\mathbb{Q}(\zeta_p)$. □

Proof. Let $\sigma, \tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_p))$ satisfy $\sigma P_p + \tau P_p = 2P_p$. Then there exist $\sigma', \tau' \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_p))$ such that $\sigma = \sigma'$ and $\tau' = \tau$ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}(\zeta_p))$, and $\sigma' = \tau' = 1$ in $\mathrm{Gal}(\mathbb{Q}(\zeta_{2^k}, \zeta_p), \mathbb{Q}(\zeta_p))$. This is because

$$\mathrm{Gal}\big(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_p)\big) \simeq \mathrm{Gal}\big(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}(\zeta_p)\big) \oplus \mathrm{Gal}\big(\mathbb{Q}(\zeta_{2^k}, \zeta_p)/\mathbb{Q}(\zeta_p)\big).$$

Since $\mathbb{Q}(\zeta_p, P_2) \subseteq \mathbb{Q}(\zeta_p, \zeta_{2^k})$ and $\mathbb{Q}(\zeta_p, P_p) \subseteq \mathbb{Q}(\zeta_{p^m})$, it follows that $\sigma' P_2 = \tau' P_2 = P_2$ and $\sigma' P_p + \tau' P_p = 2P_p$, and so $\sigma' P + \tau' P = 2P$. By almost rationality, $P = \sigma' P = \tau' P$, and so $P_p = \sigma' P_p = \tau' P_p$, and P_p is almost rational over $\mathbb{Q}(\zeta_p)$. An identical argument works for P_2. ■

We now limit possible P_p as in Case I. Exactly as in Case I, as $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{p_m})/\mathbb{Q})$-modules, $M[p^m] \simeq \mu_{p^r} \oplus \mathbb{Z}/p^s\mathbb{Z}$ for some $r, s \leq m$. (This result required only the fact that $\mathbb{Q}(M[p^m]) \subseteq \mathbb{Q}(\zeta_{p^m})$.)

Choose $\sigma, \tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}(\zeta_p))$ such that

$$\chi(\sigma) \equiv 1 + p \mod p^k, \qquad \chi(\tau) \equiv 1 - p \mod p^k.$$

We see that $\sigma P_p + \tau P_p - 2P_p = 0$. Since P_p is almost rational over $\mathbb{Q}(\zeta_p)$, this implies that $\sigma P_p = P_p$, and thus either P_p is $\mathbb{Q}$-rational or $r = 1$ and $M[p^m] \simeq \mu_p \oplus \mathbb{Z}/p^s\mathbb{Z}$. If P_p is $\mathbb{Q}$-rational, then $P_2 = P - P_p$ is almost rational over $\mathbb{Q}$, and so P_2 is also $\mathbb{Q}$-rational, from Case I. Hence (possibly after subtracting a rational point of 3-power order, allowed by Lemma 1.1(3)), we may assume that P_p generates a μ_p subgroup of $E[p]$.

The argument for $n = 2^k$ applies equally well over the field $\mathbb{Q}(\zeta_p)$ as over $\mathbb{Q}$, and we may similarly conclude that P_2 is defined over $\mathbb{Q}(\zeta_p)$. If $p = 3$, then we are in the $(P = Q + R + S)$-case of Theorem 1.2. Note that such a point is almost rational, since the only nontrivial Galois conjugate of P is $\sigma P = -Q + R + \sigma S$, and so $2\sigma P - 2P = -Q \neq 0$. The only thing left to check is that S has order less than 32. If S had order divisible by 32, then E and the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-3}))$-module generated by S would give rise to a $\mathbb{Q}(\sqrt{-3})$-rational point of $X_0(32)$. Since $X_0(32)$ is explicitly given by the elliptic curve $y^2 = x^3 - x$, we may calculate its $\mathbb{Q}(\sqrt{-3})$-rational points (equivalently, the $\mathbb{Q}$-rational points of $X_0(32)$ and its twist $y^2 = x^3 - 9x$ by $\mathbb{Q}(\sqrt{-3})$). One finds that both curves have rank zero and that the torsion points do not correspond to semistable elliptic curves over $\mathbb{Q}$. The existence of the 3-rational point over $\mathbb{Q}(\sqrt{-3})$ suggests that this analysis could be refined to further limit the order of S.

It remains to eliminate the possibility that $p = 5$. If E is a semistable elliptic curve with $\mu_5 \hookrightarrow E[5]$, then there exists a 5-isogenous curve E' with a rational point of order 5. Moreover, $E[2] \simeq E'[2]$, and so E' also has a rational point of order 2. Hence E' has a rational point of order 10. Such curves are parameterized by the genus zero curve $X_1(10)$.

We first consider the case when $E[2] \subseteq M[2]$. Since $E[2]$ is reducible, it is defined over some field of degree $d \leq 2$. Since $\mathbb{Q}(M) \subseteq \mathbb{Q}(\zeta_5)$, $\mathbb{Q}(E[2])$ must be either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. Since $E[2] \simeq E'[2]$, the same must be true of $E'[2]$. D. Kubert [5] gives an explicit parameterization of the genus zero curve $X_1(10)$ in terms of some uniformizing parameter f. The discriminant of the cubic in the Weierstrass equation for E' is a (rational) square times $(2f-1)(4f^2-2f-1)$. Hence if $\mathbb{Q}(E'[2]) \subseteq \mathbb{Q}(\sqrt{5})$, then one of the equations

$$y^2 = (2f-1)\big(4f^2-2f-1\big), \qquad 5y^2 = (2f-1)\big(4f^2-2f-1\big)$$

must have a rational solution. The first curve is $E_1 = 20A2$ in Cremona's tables (see [3]); the second curve is its twist $E_2 = 100A1$. We find that $E_1(\mathbb{Q}) \simeq \mathbb{Z}/6\mathbb{Z}$ and $E_2(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z}$, where each torsion point corresponds to a curve such that $\Delta_{E'} = 0$ and so does not correspond to an actual elliptic curve. Hence $E[2] \not\subseteq M[2]$.

Now suppose that $E[2]$ does not contain $M[2]$. If P is of order 10, then $5P \in M[2]$ is defined over either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. If $5P$ is not defined over $\mathbb{Q}$, then, since $E[2]$ is reducible, it must contain some nontrivial rational point, and then all of $E[2]$ is defined over $\mathbb{Q}(\sqrt{5})$. This situation was considered above. If $5P \in E(\mathbb{Q})$, then $\sigma P + \sigma^2 P = 2P$, where $\chi_5(\sigma) \equiv 3 \mod 5$ and $\chi_5(\sigma^2) \equiv 4 \mod 5$. This contradicts the almost rationality of P. Hence P is of order divisible by 20.

Let $Q = mP$ be of order 4. Then, since the Galois module generated by Q does not contain $E[2]$, it must be cyclic. By assumption, $E[5]$ also contains a cyclic Galois submodule of order 5. Together they generate a Galois invariant cyclic subgroup of order 20, which is impossible since $X_0(20)$ has no noncuspidal rational points. We conclude that M cannot contain a μ_5 subgroup.

3.3 Case III: Remaining S

For all other possible n, we may rule out the existence of P by the following simple arguments. Denote by S the set of primes dividing n.

- $S = \{7,5,3,2\}, \{7,5,3\}, \{7,5,2\}, \{7,5\}$. From Theorem 2.2, $E[7]$ is reducible. Since $(7^2-1, 5) = 1$, by Corollary 2.1, $M[5]$ is unramified outside 5. If $E[5]$ was irreducible, then it would be finite at 5 and unramified outside 5. Arguing as in the proof of Theorem 2.2, we infer that $E[5]$ must be reducible. Since E is semistable, there exists an isogenous curve E' with a rational point of order 35. This contradicts Theorem 2.3.
- $S = \{7,3,2\}, \{7,3\}$. From Theorem 2.2, $E[7]$ is reducible. If $E[3]$ is reducible, then

there exists an isogenous curve E' with a rational point of order 21, which contradicts Theorem 2.3. Thus $E[3]$ is irreducible and so finite at 3 by Theorem 2.2. One sees that $N(\widetilde{\rho}_3)$ is either 1, 2, 7, or 14. By level lowering (see [10]), the only allowable possibility is that $N(\widetilde{\rho}_3) = 14$. In this case, $\widetilde{\rho}_3$ must arise as the Galois representation attached to some modular form in $S_2(\Gamma_0(14))$ (and trivial character). This space is 1-dimensional and corresponds to the elliptic curve $X_0(14)$ of conductor 14. Yet from Cremona's tables in [3], all curves of conductor 14 are reducible at 3, which implies that $\widetilde{\rho}_3$ must also be reducible. This is a contradiction.

- $S = \{5, 3, 2\}, \{5, 3\}$. From Theorem 2.2, $E[5]$ is reducible. If $E[3]$ is reducible, then there exists an isogenous curve E' with a rational point of order 15, which contradicts Theorem 2.3. Thus $E[3]$ is irreducible and so finite at 3 by Theorem 2.1. One sees that $N(\widetilde{\rho}_3)$ is either 1, 2, 5, or 10. All cases are impossible, by Ribet's theorem in [10].

- $S = \{7, 2\}$. From Theorem 2.2, $E[7]$ is reducible. If $E[2]$ is reducible, then there exists an isogenous curve E' with a rational point of order 14, which contradicts Theorem 2.3. Thus $E[2]$ is irreducible and so finite at 2 by Theorem 2.1. One sees that $N(\widetilde{\rho}_2)$ is either 1 or 7. The first possibility is excluded by Tate [19]. In the second case, since $\widetilde{\rho}_2$ is genuinely ramified at 7, we may use Ribet [11] to conclude that $\widetilde{\rho}_2$ arises from some modular form in $S_2(\Gamma_0(14))$. Again, from Cremona's tables in [3], all elliptic curves of conductor 14 are reducible at 2, and thus $\widetilde{\rho}_2$ is also, a contradiction.

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