BLOCH–KATO CONJECTURES FOR AUTOMORPHIC MOTIVES

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1. Introduction

Assume (for this paragraph only) the standard conjectures, and suppose that M is a pure irreducible Grothendieck motive over $\mathbf{Q}$ with coefficients in (say) a totally real field E. We make no assumption on the regularity or self-duality of M. According to conjectures of Hasse–Weil, Langlands, Clozel, and others, one expects that the motive M is automorphic, and corresponds to an algebraic cuspidal automorphic form π for $\mathrm{GL}(n)/\mathbf{Q}$ such that $L(\pi,s) = L(M,s)$. By a theorem of Jacquet and Shalika [JS81], the L-function

$$L(M \times M^\vee, s) = L(\pi \times \pi^\vee, s)$$

is meromorphic for $\mathrm{Re}(s) > 0$ and has a simple pole at $s = 1$. Let $\mathrm{ad}^0(M)$ be the pure motive of weight zero with coefficients in E such that $\mathrm{ad}^0(M) \oplus E = M \times M^\vee$. Then

$$L(\mathrm{ad}^0(M), s) = \frac{L(\pi \times \pi^\vee, s)}{\zeta(s)},$$

and $L(\mathrm{ad}^0(M), 1) \neq 0$ is finite. According to conjectures of Deligne and Bloch–Kato [BK90], for any pure de Rham representation V, there is an equality:

$$\dim H^1_f(G_{\mathbf{Q}}, V) - \dim H^0(G_{\mathbf{Q}}, V) = \mathrm{ord}_{s=1} L(V^*, s).$$

In particular, if we take $V = V^* = \mathrm{ad}^0(M)$, then we expect that $H^1_f(G_{\mathbf{Q}}, \mathrm{ad}^0(M))$ should vanish. This is a special case of the more general fact that $H^1_f(\mathbf{Q}, V)$ should be trivial for any p-adic representation V arising from a pure motive M of weight $w \geq 0$ which does not contain a copy of the trivial motive. One also conjectures that the value of $L(\mathrm{ad}^0(M), 1)$,

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after normalization by some suitable period should lie in $\mathbf{Q}^\times$. Moreover, after equating M with its étale realization for some prime p, the normalized L function should have the same valuation as the order of a corresponding Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0(M) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$.

No longer assuming any conjectures, suppose that $M = \{r_\lambda\}$ is now a weakly compatible system of n-dimensional irreducible Galois representations of $G_{\mathbf{Q}}$, and suppose moreover that M is automorphic, that is, it corresponds to a cuspidal form π for $\mathrm{GL}(n)/\mathbf{Q}$ in a manner compatible with the local Langlands correspondence. Then, even without the standard conjectures, it makes sense to ask whether, for a p-adic representation $r : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathcal{O})$ coming from M (for some finite extension $K/\mathbf{Q}_p$ with ring of integers $\mathcal{O}$), if the Selmer group

$$H^1_f(\mathbf{Q}, \mathrm{ad}^0(r) \otimes K/\mathcal{O})$$

is finite. Theorems of this kind were first proved for $n = 2$ by Flach [Fla92], and they are also closely related to modularity lifting theorems as proved by Wiles [Wil95, TW95], see (in particular) [DFG04]. More precisely, the order of this group is related to the order of a congruence ideal between modular forms. In this paper, we prove versions of these results for modular abelian surfaces and (conditionally) compatible families of n-dimensional representations whose existence was only recently proved to exist [HLTT]. The main theorem is the following.

Theorem 1.1. *Let $A/\mathbf{Q}$ be a semistable modular abelian surface with $\mathrm{End}(A) = \mathbf{Z}$. Let p be a prime such that:*

(1) p is sufficiently large with respect to some constant depending only on A.
(2) A is ordinary at p, and if α, β are the unit root eigenvalues of $D_{\mathrm{cris}}(V)$, then

$$(\alpha^2 - 1)(\beta^2 - 1)(\alpha - \beta)(\alpha^2\beta^2 - 1) \not\equiv 0 \mod p.$$

Then

$$H^1_f(\mathbf{Q}, \mathrm{asp}^0(r) \otimes \mathbf{Q}_p/\mathbf{Z}_p) = 0$$

where $\mathrm{asp}^0(r)$ is the 10-dimensional adjoint representation of $\mathrm{PGSp}(4)$. Moreover, the set of primes p satisfying these conditions has density one.

For the families of Galois representations constructed in [HLTT], we prove the following conditional result.

Theorem 1.2. *Let π be a weight zero regular algebraic cuspidal representation for $\mathrm{GL}(n)/F$ for a CM field F and coefficients in E. Let λ be a prime of $\mathcal{O}_E$ dividing p, and let*

$$r = r_\lambda : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathcal{O})$$

be a p-adic representation associated to π with determinant $\epsilon^{n(n-1)/2}$. Assume that

(1) $\overline{r}$ has enormous image in the sense of [CGa] §9.2,
(2) The Serre level $N(\overline{r})$ is the level N of r.
(3) p is sufficiently large with respect to some constant depending only on π.

Assume all of Conjecture B of [CGa] except assumption (4). Then the Selmer group

$$H^1_f(F, \mathrm{ad}^0(r) \otimes K/\mathcal{O})$$

is trivial.

Note that Conjecture B of [CGa] consists of five parts: The first part concern local–global compatibility at $v|p$, which is still open. The second and third parts concern local–global compatibility at finite v. Here there is work in characteristic zero by Varma [Var14], although arguments of this nature should also apply to the Galois representations constructed by Scholze [Sch15], at least for modularity lifting purposes (since for modularity lifting it is usually sufficient to have local–global compatibility up to N-semi-simplification). The fifth part is essentially addressed in [CGa], and also (in a different and arguably superior manner) in [KT]. Hence the main remaining issue is local–global compatibility at $\ell = p$. Unlike the case of Theorem 1.1, we do not know whether this theorem applies for infinitely many p. One reason is that we do not even know that the representations r_λ are irreducible for sufficiently large p. Another is that we do not know whether the conductor of $N(\bar{r})$ is equal to the conductor of r for sufficiently large p, although this is predicted to hold by some generalization of Serre's conjecture. One example to which this does apply is to the Galois representations associated to Symmetric powers of elliptic curves E over F. (The theorem holds for F if it holds for some totally real extension F'/F, and any symmetric power of E is potentially modular in this case (as follows, for example, from the results of [BLGG11]).

If $\pi \simeq \pi^\vee \otimes \chi$ is RAESDC and $F = F^+$, then Theorem 1.2 may be deduced (in an unconditional form) from modularity results, modulo certain adequate hypothesis on the corresponding mod-p representation. Correspondingly, we deduce our theorems from the modularity lifting results of [CGa] and [CGb], of which we assume familiarity. One obstruction to directly applying these theorems is that the modularity results of *ibid.* require further unproven assumptions, namely, the vanishing of certain cohomology groups outside a prescribed range. The main observation here is that vanishing in these cases may be established for all sufficiently large p. For automorphic forms for $\mathrm{GL}(n)$, we require the extra assumption of local–global compatibility at $v|p$ for ordinary primes, which is not yet known.

2. Relation with special values of periods

The Bloch–Kato conjecture actually gives a more precise prediction of the exact order of the Selmer group in terms of the value of the L-function divided by a certain motivic period. One can think of this as two separate conjectures. The first is to show that the normalization of $L(1)$ by a suitable period is indeed rational. The second is to relate the corresponding p-adic valuation of this ratio to the order of a Selmer group. Our method naturally relates the order of a Selmer group to a certain tangent space. On the other hand, for most of the Galois representations we consider, it is not known whether there exists a corresponding motive, and so it is not clear exactly what it means to prove rationality. There are some formulations where one can establish certain forms of rationality (or even integrality) with respect to periods defined in terms of automorphic integrals (see, for example, [BA], [GHL16], and also [Urb98]). However, it is not clear to the authors how these results exactly relate to the (sometimes conjectural) motivic periods. An interesting test case is the following. Suppose that

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C})$$

is an irreducible odd representation. According to the Artin conjecture (known in this case, see [BT99, Buz03, KW09a, KW09b, Kis09]), one knows that ρ is modular of weight one. If one chooses a prime p, and supposes that ρ has a model over $\mathcal{O}$, the finiteness of the Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0(\rho) \otimes K/\mathcal{O})$ is a consequence of the finiteness of the p-class group of $\mathbf{Q}(\ker(\rho))$. (The former is a quotient of the latter.) The methods of this paper (following [CGa]) show that, at all primes $p > 2$ such that ρ is unramified, the Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0(\rho) \otimes K/\mathcal{O})$ is detected by congruences between the modular form f and other Katz modular forms of weight one which may not lift to characteristic zero. In particular, there exist such congruences if and only if $H^1_f(\mathbf{Q}, \mathrm{ad}^0(\rho) \otimes K/\mathcal{O})$ is non-zero. However,unlike in the case of higher weight modular forms, there does not seem to be an *a priori* way to relate this to a normalization of the adjoint L function $L(\mathrm{ad}^0(\rho), 1)$ (which in this case is an Artin L-function). The issue is that all such constructions (following Hida [Hid81]) proceed by understanding various parings on the Betti cohomology of arithmetic groups in characteristic zero, whereas weight one Katz modular forms only have an interpretation in terms of coherent cohomology. Even in cases where one does have access to Betti cohomology, say for regular algebraic cuspidal automorphic representations for $\mathrm{GL}(n)/F$ (even for $\mathrm{GL}(2)$ over imaginary quadratic fields F), it is not so clear whether the cohomological pairings one can define give the "correct" regulators or merely the regulators up to some finite multiple related to the torsion classes in cohomology. Since we have nothing to say about how to resolve these issues, we follow Wittgenstein's dictum ([Wit21] §7) and say no more about them.

3. Vanishing Theorems

The main idea of this paper is to note that the various vanishing theorems which are required inputs for the method of [CGa, CGb] may be established at least for p sufficiently large. This is not so useful for applications to modularity — if p is sufficiently large, then any completion of $\mathbf{T}$ at a maximal ideal $\mathfrak{m}$ of residue characteristic p will be formally smooth of dimension one, and so the only characteristic zero representation one can prove is modular is the representation one must assume is modular in the first place. However, with respect to Selmer groups, this statement does have content — it says that these representations will have no infinitesimal deformations.

3.1. **Betti Cohomology.** Let F be a CM field of degree $2d$. Let

$$l_0 := d\,(\mathrm{rank}(\mathrm{SL}_n(\mathbf{C})) - \mathrm{rank}(\mathrm{SU}_n(\mathbf{C}))) = d(n-1),$$
$$2q_0 + l_0 = d\,(\dim(\mathrm{SL}_n(\mathbf{C})) - \dim(\mathrm{SU}_n(\mathbf{C}))) = d(n^2-1),$$
$$q_0 = \frac{d(n^2-n)}{2}.$$

Fix a tempered cuspidal automorphic form π for $\mathrm{PGL}(n)/F$ of weight zero with coefficients in E. Let $Y = Y(K)$ be the corresponding arithmetic orbifold considered in §9 of [CGa]. For a prime v of $\mathcal{O}_E$, let

$$\overline{r}_v : G_F \to \mathrm{GL}_n(k)$$

be the corresponding semi-simple Galois representation, and let $\mathfrak{m}$ denote the corresponding maximal ideal of $\mathbf{T}$ acting on $H^*(Y, \mathbf{Z})$, where Y is assumed to have the same level as π.

Lemma 3.1. *For all sufficiently large v, and $\mathcal{O} = \mathcal{O}_{E,v}$, we have $H^i(Y, \mathcal{O}/\varpi^k)_{\mathfrak{m}} = 0$ unless $i \in [q_0, \ldots, q_0 + l_0]$.*

Proof. Assume otherwise. Since $H^*(Y, \mathbf{Z})$ is finitely generated, the groups $H^*(Y, \mathcal{O}) = H^*(Y, \mathbf{Z}) \otimes \mathcal{O}$ are torsion free and of finite rank over $\mathcal{O}$ for all i when $\mathcal{O}$ has sufficiently large residue characteristic. Moreover, there exist only finitely many systems of eigenvalues which occur in $H^*(Y, \mathbf{R})$. Assuming that the result if false (and there are infinitely many v), we deduce that there exists an eigenclass $[c]$ in $H^i(Y, \mathcal{O}_E)$ with $i \notin [q_0, \ldots, q_0 + l_0]$ such that the action of $\mathbf{T}$ on $[c]$ has support at $\mathfrak{m}$ for infinitely many primes v of $\mathcal{O}_E$. By the

Chinese remainder theorem, the Hecke eigenvalues of $[c]$ coincide with those of π. Eigenclasses in cohomology may be realized by isobaric automorphic representations; suppose that $[c]$ corresponds to such an automorphic form Π. Because of the degree where $[c]$ occurs, we deduce [BW80] that Π is not tempered. Yet by strong multiplicity one [JS81], there is an isomorphism $\Pi \simeq \pi$. $\square$

Theorem 3.2. *Suppose that* $H^i(Y, \mathcal{O}/\varpi^n)_{\mathfrak{m}} = 0$ *unless* $i \in [q_0, \ldots, q_0 + l_0]$. *Let* Q *be a collection of Taylor-Wiles primes* x *such that* $\overline{r}(\mathrm{Frob}_x)$ *has distinct eigenvalues. Then*

$$H^i(Y_1(Q), \mathcal{O}/\varpi^n)_{\mathfrak{m}_\alpha} = 0$$

for all $i \notin [q_0, \ldots, q_0 + l_0]$, *where* α *is any collection of Eigenvalues of* $\overline{r}(\mathrm{Frob}_x)$ *for* x *dividing* Q. *In particular, the conclusions of this theorem apply for all sufficiently large* p.

Proof. By Poincaré duality, it suffices to prove the result for $i < q_0$. Let i be the smallest integer for which the inequality is violated. Then, by the Hochschild–Serre spectral sequence, we deduce that

$$H^i(Y_0(Q), k)_{\mathfrak{m}_\alpha} \neq 0.$$

As in §9.2 of [CGa], we deduce that $H^i(Y_0(Q), k)_{\mathfrak{m}_\alpha} \simeq H^i(Y, k)_{\mathfrak{m}}$. The result then follows by Theorem 3.1. $\square$

The assumption that $\overline{r}_v$ has enormous image is exactly the assumption that there exists arbitrary many sets Q of auxiliary Taylor–Wiles primes satisfying the hypothesis that $\overline{r}(\mathrm{Frob}_x)$ has distinct eigenvalues. (For a different treatment of Taylor–Wiles primes using the enormous image hypothesis, see [KT]).

3.2. **Coherent Cohomology.** Let $\mathcal{O}$ denote the ring of integers in some finite extension of $\mathbf{Q}_p$. Let X denote a toroidal compactification of a Seigel 3-fold Y of level prime to p over $\operatorname{Spec} \mathcal{O}$, and let Z denote the minimal compactification. Let $\pi : \mathcal{A} \to Y$ denote the universal abelian variety, let $\mathscr{E} = \pi_* \Omega^1_{\mathcal{A}/X}$, let $\omega = \det \mathscr{E}$, and, by abuse of notation, also let ω denote the canonical extension of ω to X or the corresponding ample line bundle on Z. Fix a cuspidal automorphic form π for $\mathrm{GSp}(4)/\mathbf{Q}$ corresponding to a modular abelian surface A which we assume has no endomorphisms over $\mathbf{Q}$, and hence to a cuspidal Siegel modular form of scalar weight 2. Let

$$\overline{r}_p : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_p)$$

be the corresponding semi-simple representation for each prime p. Let $\mathfrak{m}$ denote the corresponding maximal ideal of $\mathbf{T}$.

Lemma 3.3. *For all sufficiently large* p, $H^2(X, \omega^2_{\mathcal{O}/\varpi^n})_{\mathfrak{m}} = 0$.

Proof. This is a consequence of the proof of Theorem 7.11 of [CGb]. We give a brief sketch here of the idea: as in the proof of Lemma 3.1, we otherwise deduce that there exists a characteristic zero form in $H^2(X, \omega^2_{\mathbf{C}})$ giving rise to infinitely many of these classes. The representation $\overline{r}_p$ will be irreducible for all sufficiently large p (because $\mathrm{End}_{\mathbf{Q}}(A) = \mathbf{Z}$ — see also the proof of Lemma 4.1). If follows that the transfer of this form to $\mathrm{GL}(4)$ must be cuspidal, and moreover (by multiplicity one) coincide with the transfer of the representation coming from the holomorphic Siegel modular form. But such a representation only contributes to cohomology in degrees 0 and 1. (For details relating the coherent cohomology of Siegel threefolds and their relation to automorphic forms we refer the reader back to [CGb].) $\square$

Theorem 3.4. *Suppose that $H^2(X, \omega^2_{\mathcal{O}/\varpi^n})_{\mathfrak{m}} = 0$. Suppose, moreover, that $\overline{r}_p$ is absolutely irreducible. Then, for $i \ge 2$,*

$$H^i(X_1(Q), \omega^2_{\mathcal{O}/\varpi^n})_{\mathfrak{m}_\alpha} = H^i(X_0(Q), \omega^2_{\mathcal{O}/\varpi^n})_{\mathfrak{m}_\alpha} = 0,$$

where Q is any collection of Taylor–Wiles primes and α is any collection of Eigenvalues of $\rho(\mathrm{Frob}_x)$ for x dividing Q. In particular, the conclusions of this theorem apply for all sufficiently large p.

Proof. By dévissage, we can reduce to the case where the coefficients are a finite field k, and the case $Q = x$. We start by proving a different statement, namely that the maps pr_x of [CGa] §7 induce isomorphisms:

$$\mathrm{pr}_x : H^i(X, \omega^m)_{\mathfrak{m}} \to H^i(X_0(x), \omega^m)_{\mathfrak{m}_\alpha},$$

$$\mathrm{pr}_x : H^i(X, \omega^m(-\infty))_{\mathfrak{m}} \to H^i(X_0(x), \omega^m(-\infty))_{\mathfrak{m}_\alpha},$$

for all i and all m. This implies the second equality. To deduce vanishing for $X_1(x)$, we first use Serre duality to reduce the problem to $H^i(X_1(x), \omega(-\infty))_{\mathfrak{m}}$ for $i \le 1$. Yet, by the Hochschild–Serre spectral sequence for $X_1(x) \to X_0(x)$, this reduces to the statement claimed above. Hence we may concentrate on the claim that pr_x induces an isomorphism. Note that, for H^0 and H^1, the required claim is Lemma 7.4 and 7.5 of [CGb] — our argument here is an elaboration of that argument. Let Z denote the minimal compactification of the Siegel 3-fold. There is a projection $\pi : X \to Z$, and the higher direct images of $\pi_*\omega^m$ vanish. The varieties Z and $Z_0(x)$ admit an Ekedahl–Oort stratification

$$Z = Z^0 \supset Z^1 \supset Z^2 \supset Z^3, \qquad Z_0(x) = Z^0_0(x) \supset Z^1_0(x) \supset Z^2_0(x) \supset Z^3_0(x)$$

corresponding to the (closures of the) rank 1 locus, supersingular locus, and superspecial locus respectively. Hence we are reduced to the claim that, for any $i \ge 0$ and the Hecke equivariant sheaf $V = \pi_*\mathcal{O}_X$ and any of these strata, we have isomorphisms

$$\mathrm{pr}_x : H^i(Z^j, V \otimes \omega^m)_{\mathfrak{m}} \to H^i(Z^j_0(x), V \otimes \omega^m)_{\mathfrak{m}}.$$

We prove this result first by downwards induction on i, then upwards induction on j. Consider the generalized Hasse invariant $A_j \in H^0(Z^j, \omega^N)$ constructed by Boxer [Box15]. Replacing A_j by a power of A_j if necessary, we may assume that ω^{n+M} has no higher cohomology on any of the strata (from Serre vanishing and the ampleness of ω). Multiplication by A_j induces a map

$$\begin{tikzcd} H^i(Z^j, V \otimes \omega^m)_{\mathfrak{m}} \arrow[r, "A_j"] \arrow[d, "\mathrm{pr}_x"] & H^i(Z^j, V \otimes \omega^{m+N})_{\mathfrak{m}} \arrow[d, "\mathrm{pr}_x"] \\ H^i(Z^j_0(x), V \otimes \omega^m)_{\mathfrak{m}} \arrow[r, "A_j"] & H^i(Z^j_0(x), V \otimes \omega^{m+N})_{\mathfrak{m}} \end{tikzcd}$$

If $i > 0$, the groups in the right vertical column vanish. The kernel comprises exactly of terms coming from the cohomology of Z^{j+1}. Hence we may reduce the statement to the case where $i = 0$. In this case, the result is true for $Z = Z^0$ and Z^1 by Lemma 4.12 of [CGb]. (This is where the hypothesis that $x \in Q$ is a Taylor–Wiles prime is used.) Note that there is a natural trace map tr_x in the opposite direction induced from the finite map $Z^j_0(x) \to Z^j$, and the composition $\mathrm{tr}_x \circ \mathrm{pr}_x$ on global sections is an invertible scalar $d_x = [\mathrm{GSp}_4(\mathbf{Z}_x) : \Pi(x)]$.

(This doesn't depend on any Taylor–Wiles hypothesis or any assumption on $\mathfrak{m}$.) Hence it suffices to show that pr_x is surjective.

We now argue by upwards induction on j. We first consider the case when m is sufficiently large, and thus, by Serre vanishing, we have a commutative diagram:

$$\begin{tikzcd}
H^0(Z^{j-1}, V \otimes \omega^m)_{\mathfrak{m}} \arrow[r] \arrow[d, equal, "\mathrm{pr}_x"] & H^0(Z^j, V \otimes \omega^m)_{\mathfrak{m}} \arrow[d, "\mathrm{pr}_x"] \\
H^0(Z_0^{j-1}(x), V \otimes \omega^m)_{\mathfrak{m}_\alpha} \arrow[r, two heads] & H^0(Z_0^j(x), V \otimes \omega^m)_{\mathfrak{m}_\alpha}
\end{tikzcd}$$

Since the diagram commutes, we deduce that pr_x is surjective, as required. We now use downward induction on m. We have a commutative diagram:

$$\begin{tikzcd}
H^0(Z^j, V \otimes \omega^m)_{\mathfrak{m}} \arrow[r, hook, "A_j"'] \arrow[d, hook] & H^0(Z^j, V \otimes \omega^{m+M})_{\mathfrak{m}} \arrow[d, equal, "\mathrm{pr}_x"] \\
H^0(Z_0^j(x), V \otimes \omega^m)_{\mathfrak{m}_\alpha} \arrow[r, hook, "A_j"'] & H^0(Z_0^j(x), V \otimes \omega^{m+M})_{\mathfrak{m}_\alpha}
\end{tikzcd}$$

To prove surjectivity of the first injection, it suffices (by a diagram chase) to prove that the map of cokernels is an injection. But the map on cokernels is also given in terms of pr_x on H^0 of Z^{j+1} and $Z_0^{j+1}(x)$, and since pr_x is injective, the result holds for all m, and we are done by induction. □

4. Proofs

If one has an isomorphism $R \simeq \mathbf{T}$ for all sufficiently large p satisfying the required hypothesis, then since one also will have an isomorphism $\mathbf{T} \simeq \mathcal{O}$, this would immediately imply that the tangent space to R along the projection to $\mathcal{O}$ is trivial, and hence the corresponding adjoint Selmer groups are trivial. Hence Theorem 1.2 is now an immediate consequence of Theorem 5.16 of [CGa] (as modified by the fact that we have local smoothness, see Theorem 6.4 of [CGa]), noting that Theorem 3.2 provides a substitute for the vanishing assumption required in Conjecture B of *ibid.* Equally, Theorem 1.1 follows as in Theorem 1.1 of [CGb], where the vanishing result of Theorem 3.4 replaces the vanishing results of Lan–Suh [LS11] for other low weight local systems used in [CGb]. The modularity argument above requires a large image hypothesis which we assume in Theorem 1.2 and which we are required to prove (for sufficiently large p) for Theorem 3.2. Furthermore, we must justify the claim in Theorem 3.2 that the assumptions hold for a set of primes p of density one. Hence it remains to prove the following:

Lemma 4.1. *Let A be a semistable abelian surface of conductor N with $\mathrm{End}(A) = \mathbf{Z}$. Then:*

(1) For sufficiently large p, the residual representation $\overline{r}_p : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_p)$ is surjective with minimal conductor N.

(2) For a set of primes p of density one, we have

$$\alpha\beta(\alpha^2 - 1)(\beta^2 - 1)(\alpha - \beta)(\alpha^2\beta^2 - 1) \not\equiv 0 \mod p.$$

Proof. Since $\mathrm{End}(A) = \mathbf{Z}$, the residual image is surjective for all sufficiently large p by [Ser00a], Corollaire au Théorème 3. In order to ensure that the conductor of $\overline{r}_p$ at a prime ℓ dividing N matches that of A, it suffices to take p co-prime to the (finite) order of the component group $\Phi_{\mathcal{A}}$ of the Néron model of $A^\vee$ at ℓ. This proves the first claim.

For the second claim, let the characteristic polynomial of Frobenius (for p not dividing the discriminant on the étale cohomology $V_\ell = H^1(A, \overline{\mathbf{Q}}_\ell)$ at any prime $\ell \neq p$) be

$$X^4 + a_p X^3 + (2p + b_p) X^2 + p a_p X + p^2.$$

Let the roots of this polynomial be α, β, $\alpha^{-1}p$ and $\beta^{-1}p$ respectively; by the Riemann hypothesis for curves (Weil bound) they are Weil numbers of absolute value $\sqrt{p}$. Note that $2p + b_p$ is the trace of Frob_p on $\wedge^2 V_\ell$ for all but finitely many ℓ, and a_p^2 is the trace of Frob_p on $V_\ell \otimes V_\ell$. We use the following lemma, which is essentially an observation of Ogus (2.7.1 of [DMOS82]).

Lemma 4.2. *There is no fixed linear relation between* 1, p, b_p, a_p, *and* a_p^2 *which can hold for a set* $p \in S$ *of positive density.*

Proof. From such an equality, we can build two finite dimensional representations A_ℓ and B_ℓ built out of copies of $\wedge^2 V_\ell$, $\mathbf{Q}_p$, $\mathbf{Q}_p(1)$, V_ℓ, and $V_\ell \otimes V_\ell$ respectively which have equal trace on Frob_p for infinitely many p. There must be at least one quadratic field with a positive density of inert primes in S, twisting by this representation we arrive at a representation W_ℓ with a set S of positive density such that Frob_p has trace zero for $p \in S$. For sufficiently large ℓ, our assumptions on A implies ([Ser00b]) that the image of $G_{\mathbf{Q}}$ on V_ℓ is $\mathrm{GSp}_4(\mathbf{Z}_\ell)$ if $\mathrm{End}(A) = \mathbf{Z}$. By Cebotarev, it follows that the appropriate identity must also hold on an subset of these groups of positive measure. Yet the distribution of the appropriate eigenvalues for $\mathrm{GSp}_4(\mathbf{Z}_\ell)$ does not have any atomic measure. In particular, writing the eigenvalues in either case as x, y, δ/x, and δ/y, we would obtain an relation between the polynomials

$$1, \quad \delta, \quad xy + \delta x/y + \delta y/x + \delta^2/xy, \quad x + y + \delta/x + \delta/y, \quad (x + y + \delta/x + \delta/y)^2$$

that holds on an open set (and consequently holds everywhere). There are no such relations by inspection. □

Returning to the proof of Lemma 4.1, we deduce from the Weil bounds that $|a_p|^2 \leq 16p$ and $|b_p| \leq 4p$. Hence, if we have any linear expression in a_p, a_p^2, b_p and 1 which is congruent to zero modulo p for a set of positive density, then it must also equal a constant multiple of p for a set of positive density, and we would obtain a contradiction by Lemma 4.2. We show that this holds in each of the possible cases when our congruence above holds. (We take advantage of the symmetry in α and β and consider a reduced number of cases.)

1. Suppose that neither β and $\beta^{-1}p$ are units. Then $b_p \equiv 0 \mod p$.
2. Suppose that $\alpha\beta \equiv \varepsilon \mod p$ for some fixed $\varepsilon \in \{\pm 1\}$. Then $b_p \equiv \varepsilon \mod p$.
3. Suppose that $\alpha = \varepsilon \mod p$ for some fixed $\varepsilon \in \{\pm 1\}$. Then $b_p - \varepsilon a_p + 1 \equiv 0 \mod p$.
4. Suppose that $\alpha - \beta \equiv 0 \mod p$. Then $4b_p - a_p^2 \equiv 0 \mod p$.

□

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