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Irrationality of Certain p-adic Periods for Small p

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1 Introduction

Apéry's proof [13] of the irrationality of $\zeta(3)$ is now over 25 years old, and it is perhaps surprising that his methods have not yielded any significant new results (although further progress has been made on the irrationality of zeta values [1, 14]). Shortly after the initial proof, Beukers produced two elegant reinterpretations of Apéry's arguments; the first using iterated integrals and Legendre polynomials [2], and the second using modular forms [3]. It is this second argument that we will apply to study the irrationality of certain p-adic periods, in particular, the p-adic analogues of $\zeta(3)$ and Catalan's constant for small p. To relate certain classical periods to modular forms, Beukers considers various integrals of holomorphic modular forms that themselves satisfy certain functional equations (analogous to the functional equation for the nonholomorphic Eisenstein series of weight two). That these integrals satisfy functional equations is a consequence of the theory of Eichler integrals. The periods arise as coefficients of the associated period polynomials. In our setting, these auxiliary functional equations are replaced by the notion of overconvergent p-adic modular forms [6, 7, 10]. In this guise, our p-adic periods will occur as coefficients of overconvergent Eisenstein series of negative integral weight. These p-adic periods are equal to special values of Kubota–Leopoldt p-adic L-functions. Thus $\zeta(3)$ is replaced by $\zeta_p(3) = L_p(3, \mathrm{id})$ and Catalan's constant

$$G = L(2, \chi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \tag{1.1}$$

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is replaced by $L_p(2,\chi)$, where χ is the character of conductor 4. We will prove that $\zeta_p(3)$ is irrational for $p = 2$ and 3, and that $L_p(2,\chi)$ is irrational for $p = 2$.

2 Elementary remarks and definitions

2.1 Irrationality in $\mathbb{Q}_p$

Let p be prime. Let $\|\cdot\|_p$ denote the p-adic absolute value normalized by $\|p\|_p = 1/p$, and let $|\cdot|$ denote the Archimedean absolute value. Given $r = a/b \in \mathbb{Q}$, how well can r be approximated p-adically by other (distinct) rational integers?

Lemma 2.1. Suppose that

$$\left\|\frac{a}{b} - \frac{c}{d}\right\|_p \leq \frac{1}{p^n}. \tag{2.1}$$

Then $\max\{|c|,|d|\} \geq p^n/(|a|+|b|)$. □

Proof. The inequality above implies that $ad - bc \equiv 0 \bmod p^n$. In particular, it must be the case that $|ad - bc| \geq p^n$, and the lemma readily follows. ■

An element $\eta \in \mathbb{Q}_p$ is irrational if it does not lie in $\mathbb{Q}$. From Lemma 2.1, we may derive a simple criterion for irrationality.

Lemma 2.2 (criterion for p-adic irrationality)**.** Let p_n/q_n be rational numbers with q_n unbounded and suppose there exists a $\delta > 0$ such that

$$0 < \left\|\eta - \frac{p_n}{q_n}\right\|_p \leq \frac{1}{\left(\max\{|p_n|,|q_n|\}\right)^{1+\delta}} \tag{2.2}$$

for sufficiently large n. Then η is irrational. □

2.2 Overconvergent Eisenstein series and p-adic zeta functions

Let B_n denote the nth Bernoulli number, defined by the identity

$$\frac{x}{2} + \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}. \tag{2.3}$$

It is a classical result of Euler and Riemann that for nonnegative integers k,

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}, \tag{2.4}$$

where ζ is the Riemann zeta function. Let $\zeta_p^*(s) = (1-p^{-s})\zeta(s)$. It is a consequence of the Kummer congruences that the values $\zeta_p^*(s)$ at negative odd integers p-adically interpolate. In our context, these numbers arise as constant terms of Eisenstein series. Let $q = e^{2\pi i \tau}$ and suppose $2k \geq 2$ is an even integer. Let $\sigma_{2k-1}^*(n) = \sum_{d|n}^{d\nmid p} d^{2k-1}$. Then the Eisenstein series

$$E_{2k}^*(\tau) = \frac{\zeta_p^*(1-2k)}{2} + \sum_{n=1}^{\infty} q^n \sigma_{2k-1}^*(n) \tag{2.5}$$

is a holomorphic modular form of weight $2k$ for the group $\Gamma_0(p)$. The form E_{2k}^* is related to the more familiar Eisenstein series of level one

$$E_{2k}(\tau) = \frac{\zeta_p(1-2k)}{2} + \sum_{n=1}^{\infty} q^n \sigma_{2k-1}(n) \tag{2.6}$$

by the relation $E_{2k}^*(\tau) = E_{2k}(\tau) - p^{2k-1}E_{2k}(p\tau)$. If $2k$ and $2k'$ are integers congruent modulo $p-1$ and close p-adically, then the q-expansions of E_{2k}^* and $E_{2k'}^*$ are highly congruent. Thus by considering certain limits of q-expansions one may define p-adic Eisenstein series that are p-adic modular forms in the sense of Serre [12]. However, one can also make a more precise analytic statement. The usual context in which to view p-adic families of p-adic eigenforms with finite slope (nonzero T_p-eigenvalue) is Coleman's theory of overconvergent modular forms [6, 7] (see also [10]). A fine introduction can be found in [9]. We must be content with only a brief discussion. Holomorphic modular forms are given by sections of certain line bundles over modular curves, and their q-expansions are obtained by evaluating these sections at the cusp at infinity. In weight zero, modular forms are simply modular functions, and their q-expansion is their Taylor series expansion with respect to an appropriate local parameter at the infinite cusp. A p-adic modular form of integral weight is a section of the same sheaf, except considered only over a certain region of the modular curve defined by points whose corresponding elliptic curve is ordinary (more precisely the connected component of this space containing the cusp at infinity). This region is not open in the sense of the Zariski topology, but rather in the sense of rigid analytic geometry (the analytic geometry of the p-adic world). In terms of q-expansions, p-adic modular forms are characterized as the p-adic limits of usual holomorphic modular forms (here two q-expansions are considered "close" if the coefficients of q^n are simultaneously highly congruent p-adically for all n). Finally, an overconvergent modular form of integral weight is a p-adic modular form which extends as a section to some rigid analytic region of the modular curve strictly containing the (connected component containing infinity of) the ordinary locus. It is precisely this overconvergence which will be the p-adic analogue of Beukers' construction whereby a certain analytic expression $A(z) + \theta B(z)$ converges further for some special value of θ than for any other. The q-expansions of overconvergent modular forms are harder to characterize. A difficult theorem of Coleman implies that the q-expansions of all overconvergent eigenforms of finite slope (with $a_p \neq 0$) are exactly those p-adic modular q-expansions that are limits of classical holomorphic *eigenforms*. As an example, the q-expansion E_2, which is not a classical modular form, is known to be a finite slope p-adic modular eigenform [12]. However, a result of Coleman, Gouvêa, and Jochnowitz [8] implies that E_2 is *not* overconvergent. This reflects the (nonobvious) fact that although E_2 can be approximated by classical modular forms, it cannot be approximated by classical eigenforms. In practice, the usual method of writing down explicit overconvergent modular forms involves working with a particular modular curve and an algebraic description of the ordinary locus. For example, the curves $X_0(p)$ for $p \in \{2, 3, 5, 7, 13\}$ have genus zero, and so are parameterized by some explicit meromorphic modular function z (unique after suitable normalization). For these p, the ordinary locus containing the cusp at infinity consists of the $\mathbb{C}_p$ points of the rigid analytic space of $X_0(p)$ such that $\|z\|_p \leq 1$ (for z suitably normalized, of course). In this setting, p-adic modular forms are exactly given by power series in z converging in the disc $\|z\|_p \leq 1$, whilst overconvergent forms are exactly those power series whose radius of convergence is strictly greater than 1. Since z can be written as a power series in q, this gives at least one concrete description of overconvergent p-adic modular functions of tame level 1 and small p.

In general, the weight of a modular form can vary over a rigid analytic space $\mathcal{W}$ (weight space) of characters (see [5, page 27]). Let $q = p$ if p is odd and $q = 4$ if $p = 2$. A $\mathbb{C}_p$-point of $\mathcal{W}$ corresponds to a map

$$\Lambda = \mathbb{Z}_p\big[\big[(\mathbb{Z}/q\mathbb{Z})^\times \times \big(1 + q\mathbb{Z}_p\big)\big]\big] \simeq \varprojlim \mathbb{Z}_p\big[\big[\big(\mathbb{Z}/p^n\mathbb{Z}\big)^\times\big]\big] \longrightarrow \mathbb{C}_p. \tag{2.7}$$

The classical even integral weights correspond to the characters that send a with $a \in \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times$ to a^{2k}. Let $\mathcal{W}^+$ be the part of weight space consisting of characters κ such that $\kappa(-1) = 1$. For $n \geq 1$, let

$$\sigma^*_\kappa(n) := \sum_{d|n}^{(d,p)=1} \kappa(d) d^{-1}. \tag{2.8}$$

Theorem 2.3. There exists a function $\zeta_p(\kappa)$ on $\mathcal{W}^+$ which is rigid analytic outside $\kappa = 1$ and has a simple pole at $\kappa = 1$, such that if

$$E_\kappa = \frac{\zeta_p(\kappa)}{2} + \sum_{n=1}^{\infty} \sigma^*_\kappa(n) q^n, \tag{2.9}$$

then E_κ varies rigid analytically over weight space (away from $\kappa = 1$) and for each point $\kappa \in \mathcal{W}^+$ specializes to an overconvergent eigenform. Moreover, if $2k \geq 2$ is an even positive integer and κ is the character sending a to a^{2k}, then $E_\kappa = E^*_{2k}$. □

This theorem in this generality is essentially proved in [7]. By continuity, we may recover the value of $\zeta_p(\kappa)$ at characters $a \mapsto a^{-2n}$ for positive integers n as follows.

Lemma 2.4. Let $\zeta_p(1+2n)/2$ be the constant term of E_κ at the character $a \mapsto a^{-2n}$. Then

$$\zeta_p(1+2n) = \lim_{k \to n} \zeta(1-2k), \tag{2.10}$$

where the limit runs over strictly increasing integers k approaching $-n$ in $\mathbb{Z}_p$ with the added restriction that $2k \equiv -2n \bmod p-1$. □

The added restriction comes from the fact that the characters $a \mapsto a^{2k}$ and $a \mapsto a^{2k'}$ are close if $2k \equiv 2k' \bmod (p-1)p^n$ for large n, as follows from Euler's version of Fermat's little theorem. The imposition that k approaches infinity means we can neglect the Euler factor term, which for large values of k tends p-adically to 1.

Remark 2.5. As with the classical zeta values, not much is known about the arithmetic nature of $\zeta_p(1+2n)$. Indeed it could be argued that the situation is worse, as little is known about the following conjecture, even for $n = 1$.

Conjecture 2.6. For all integers $n > 0$ and primes p, the values $\zeta_p(1+2n) \neq 0$. □

It was the (failed) attempt to prove this conjecture for $n = 1$ that led to this paper, the idea being that if one can prove that $\zeta_p(3)$ is irrational, then one has also shown it is nonzero!

Let $2k$ be a positive even integer ≥ 4. If E_{2k} the classical Eisenstein series of weight $2k$ and level $\Gamma_0(p)$, then associated to E_{2k} there is another Eisenstein series, the *evil twin* E^{evil}_{2k}. The Eisenstein series E^{evil}_{2k} is classical, cuspidal at the cusp at infinity and has slope $2k-1$. Explicitly, the evil twin is given by the following formula:

$$E^{\text{evil}}_{2k}(\tau) = E_{2k}(\tau) - E_{2k}(p\tau). \tag{2.11}$$

We consider the specialization E_{-2k} of our Eisenstein family. Let θ be the operator on q-expansions that acts as $q \cdot d/dq = (2\pi i)^{-1} d/d\tau$. Then one can easily compute that

$$\theta^{2k+1} E_{-2k} = E^{\text{evil}}_{2k+2}. \tag{2.12}$$

Thus we can "almost" reconstruct E_{-2k} from E^{evil}_{2k+2} by considering

$$E'_{-2k}(\tau) = (2\pi i)^{2k+1} \int \cdots \int E^{\text{evil}}_{2k+2} (d\tau)^{2k+1} \in \mathbb{Q}[[q]]. \tag{2.13}$$

The function E'_{-2k} is holomorphic in a neighbourhood of the cusp $i\infty$, but is not modular (although similar modified forms satisfy some form of functional equation—see [3, page 273]). Actually, for our purposes, we could have introduced the function E'_{-2k} directly without reference to E^{evil}_{2k+2}. However, by writing down the connection we stress the ties between our method and that of Beukers [3]. Let $\eta \in \mathbb{C}_p$ and consider the expression

$$H = E^*_{2k}\big(E'_{-2k} + \eta\big). \tag{2.14}$$

If $\eta = \zeta_p(1 + 2k)/2$ then H is equal to $E^*_{2k} E_{-2k}$ and is thus an overconvergent modular function of weight 0. For all other η, however, H is not overconvergent since otherwise E^*_{2k} would be overconvergent of weight 0, which is impossible (this follows from [10, Corollary 4.4.1]). Thus we obtain a strictly analytic (over $\mathbb{C}_p$) characterization of $\zeta_p(1 + 2k)$.

Suppose that $X_0(p)$ has genus zero. Then the ordinary locus of $X_0(p)$ containing the cusp at infinity is a rigid analytic disc.

Suppose that z is a classical meromorphic modular form that is a local parameter over the cusp $i\infty$ and has no poles on the component of the ordinary locus of $X_0(p)$ containing ∞ (so z is overconvergent). Viewing z first as a complex analytic function, by the inverse function theorem, we may expand

$$H = \sum_{n=0}^{\infty} \big(a_n - b_n \eta\big) z^n. \tag{2.15}$$

Now considering this sum p-adically, we know that H is overconvergent if and only if $\eta = \zeta_p(1 + 2k)/2$. Thus we expect the radius of convergence to jump at this point, and correspondingly the sequence a_n/b_n to converge p-adically to $\zeta_p(1 + 2k)/2$. If we can estimate both the p-adic and Archimedean radii of convergence for various η, we may be able to apply our criterion of irrationality. In the next section, we carry this out in detail for $p = 2$.

3 The irrationality of $\zeta_p(3)$ for $p = 2$ and $p = 3$

3.1 $p = 2$

Let $p = 2$. Then $X_0(2)$ has genus zero, and is uniformized by the function

$$f = \frac{\Delta(2\tau)}{\Delta(\tau)} = q \prod_{n=1}^{\infty} \left(1 + q^n\right)^{24}. \tag{3.1}$$

Moreover, we note the following facts about f and $X_0(2)$. The curve $X_0(2)$ has two cusps, $i\infty$ and 0. The value of f at these cusps is equal to 0 and ∞, respectively. In particular, f is a local uniformizer at the cusp at infinity. The curve $X_0(2)$ has one elliptic point, at $(1 + i)/2$. The value of f at this point is equal to -2^{-6}. This can be proved by noting that $f'/f = E_2^*$ and that

$$\frac{E_2^{*6}}{\Delta} = \frac{\left(1 + 2^6 f\right)^3}{f}. \tag{3.2}$$

The ordinary locus of (the 2-adic rigid analytic curve) $X_0(2)$ has two components, given by

$$\|f\|_2 \leq 1, \qquad \|f\|_2 \geq 2^{12}. \tag{3.3}$$

The Fricke involution permutes these two spaces, as it sends $2^{12}f$ to $1/f$.

Consider the series

$$H = E_{2k}^*\left(E'_{-2k} + \eta\right) =: \sum_{n=0}^{\infty} \left(a_n - b_n \eta\right) f^n. \tag{3.4}$$

Since $f = q + \cdots$, it follows that $b_n \in \mathbb{Z}$ for $n \geq 1$ (the nonconstant terms of E_{2k}^* are integral). Since the q^n coefficient of E'_{-2k} lies in $\mathbb{Z}/n^{2k-1}$, it also follows that $[1, 2, \ldots, n]^{2k+1} a_n \in \mathbb{Z}$, where $[1, 2, \ldots, n]$ is the greatest common divisor of 1 up to n.

We establish the 2-adic convergence of this series for various values of η.

Lemma 3.1. If $p = 2$ and $\eta = \zeta_p(1 + 2k)/2$, then the radius of convergence of H is at least 2^{12}. If $\eta \neq \zeta_p(1 + 2k)/2$, then the radius of convergence is at most 1. □

Proof. If $\eta = \zeta_p(1 + 2k)/2$, then $H = E_{2k}^* E_{-2k}$. Since E_{-2k} is overconvergent, it extends as a rigid analytic function somewhere into the supersingular annuli. However, it is also a finite slope eigenform of level $\Gamma_0(2) = \Gamma_0(p)$, and such sections extend far into the supersingular annuli. In particular, by Buzzard [4, Theorem 5.2], it extends entirely over the supersingular annuli. Thus the radius of convergence is at least 2^{12}. If $\eta \neq \zeta_p(1 + 2k)/2$, then H is not overconvergent. Thus it cannot extend into the supersingular annuli and the radius of convergence is at most 1. ■

From this lemma, we may approximate $\zeta_p(1+2k)/2$, since when $\eta = \zeta_p(1+2k)/2$, the radius of convergence guaranteed by the previous lemma implies that

$$\left\|a_n - b_n\eta\right\|_2 \ll 2^{-(12-\epsilon)n} \tag{3.5}$$

for any $\epsilon > 0$ and sufficiently large n. We therefore expect that the rational numbers a_n/b_n provide good rational approximations to η. For this, it suffices to show that the b_n are not (in the 2-adic sense) *too small*. Since E^*_{2k} is not overconvergent of weight 0, it follows as in the proof of Lemma 3.1 that if $\epsilon > 0$, then $\|b_n\|_2 \geq 2^{-\epsilon n}$ for infinitely many n. We show that when $k = 1$ this is the case for *all* sufficiently large n.

Lemma 3.2. Suppose that $k = 1$. Let $\varphi(n)$ denote the number of occurrences of the digit one in the binary expansion of n. Then the 2-adic valuation of b_n is $3(\varphi(n)-1)$. In particular, given $\epsilon > 0$, for all sufficiently large n, one has $\|b_n\|_2 \geq 2^{-\epsilon n}$. □

Proof. The coefficients b_n are obtained by writing a modular form of weight $2k = 2$ in terms of the inverse of some modular function. As remarked in [11, Fact 1, Section 2.3], the generating function for b_n therefore satisfies a linear differential equation of order $2+1=3$. The proof of this fact also leads to an explicit construction of this differential equation, and we may derive in this way the following recurrence relation satisfied by b_n:

$$\begin{aligned} n^3 b_n = &-3\cdot 2^3\big(8(n-1)^3 - 2(n-1) - 1\big)b_{n-1} \\ &- 3\cdot 2^9\big(8(n-2)^3 - 2(n-2) + 1\big)b_{n-2} - 2^{18}(n-3)^3 b_{n-3}. \end{aligned} \tag{3.6}$$

The lemma is easily established for small k. Assume the valuation of b_k is equal to $3(\varphi(k)-1)$ for all $k < n$. Since $3\varphi(k) \leq 3\varphi(k-1)+3$, the recurrence relation for b_n implies that $\|n^3 b_n\|_2 = \|2^3 b_{n-1}\|_2$ by the triangle inequality. The result follows immediately. ■

From this lemma, it follows that (for $\epsilon > 0$ and all $n \gg 0$ as above)

$$\left\|\zeta_2(3) - \frac{2a_n}{b_n}\right\|_2 \ll 2^{-(12-\epsilon)n}. \tag{3.7}$$

Now we turn to the Archimedean valuations of a_n and b_n. Our arguments here are completely analogous to those of Beukers [3]. By considering f at the cusps and the elliptic points, we see that the radius of convergence of f will be equal to the first branching value, which occurs at $f((1+i)/2) = -1/2^6$. Thus we obtain the estimates

$$|a_n|, |b_n| \ll 2^{(6+\epsilon)n} \tag{3.8}$$

for all $\epsilon > 0$ and sufficiently large n (depending on ϵ). The coefficient a_n is not an integer, however. If we write $a_n = c_n/d_n$, then since $[1, 2, \ldots, n]^{2k+1} a_n \in \mathbb{Z}$, it follows from the prime number theorem that (with the usual restrictions on ϵ and n)

$$|d_n| \leq [1, 2, \ldots, n]^{2k+1} \ll e^{(2k+1+\epsilon)n}. \tag{3.9}$$

Consequently, if we write $2a_n/b_n = p_n/q_n$ where p_n and q_n are integers, then

$$\begin{aligned} |p_n| &\leq |c_n| = |d_n||a_n| \ll 2^{(6+(2k+1)/\log 2+\epsilon)n}, \\ |q_n| &\leq |d_n||b_n| \ll 2^{(6+(2k+1)/\log 2+\epsilon)n}. \end{aligned} \tag{3.10}$$

Combining this with our 2-adic estimates, we have proven the following.

Lemma 3.3. There exist integers p_n, q_n such that q_n approaches infinity, and such that if

$$\theta = \frac{12 \log 2}{6 \log 2 + 3}, \tag{3.11}$$

then

$$0 < \left\| \zeta_2(3) - \frac{p_n}{q_n} \right\|_2 \leq \frac{1}{\left(\max\{|p_n|, |q_n|\}\right)^{\theta - \epsilon}} \tag{3.12}$$

for sufficiently large n. □

Proof. First note that if $n = \ell$ is prime, then $a_\ell \notin \mathbb{Z}_\ell$. Thus $\ell | q_\ell$ and q_n is unbounded. From the estimates we have proven so far, it therefore suffices to prove that $a_n - \eta b_n \neq 0$ for sufficiently large n. Assume otherwise. Then H is a polynomial in f. In particular, $\zeta_p(3) \in \mathbb{Q}$, H has coefficients in $\mathbb{Q}$ and E_{-2} is a classical meromorphic eigenform of weight -2. From the q-expansion, we may determine that E_{-2} has no poles away from the cusps, and a pole of order at most 1 at $\tau = 0$. It follows that H is linear in f, which contradicts the fact that $a_2/b_2 \neq a_3/b_3$. It is also possible to prove the stronger statement that $a_n/b_n \neq \eta$ for almost all n. Indeed one can show that the sequence a_n satisfies the same recurrence relation as b_n, and then an elementary argument (estimating the Archimedean size of the solutions to the linear recurrence) implies that $|a_n/b_n|$ is monotonically *increasing* for $n \geq 2$. The fact that $\|b_n\|_2 \geq 2^{-\epsilon n}$ for infinitely many n would then suffice for irrationality applications. ■

As a corollary of this, we prove the following.

Theorem 3.4. If $p = 2$, then $\zeta_p(3) \notin \mathbb{Q}$. □

Proof. Since $\theta = 1.1618804316\ldots > 1$, we may apply our criterion for irrationality. ■

If $k = 2$, then our method does not work. Even if we proved the analogue of Lemma 3.2, we would still only have $\theta = 0.9081638111 < 1$, which is not sufficient to establish the irrationality of $\zeta_2(5)$ (nor indeed $\zeta_2(1+2k)$ for any other $k \geq 2$).

The explicit values of a_n and b_n can be determined directly through algebraic manipulation. As a guide to the reader, we write down explicitly the first few terms of some relevant series. One sees that

$$\begin{aligned} E_2^* &= \frac{1}{24} + q + q^2 + 4q^3 + q^4 + 6q^5 + \cdots, \\ E'_{-2} &= q + q^2 + \frac{28}{27}q^3 + q^4 + \frac{126}{125}q^5 + \cdots. \end{aligned} \tag{3.13}$$

Note that E_{-2} is a Hecke eigenform with pth Hecke operator having eigenvalue $1 + 1/p^3$ (for p odd) and 1 for U_2. Since

$$f = q \prod_{n=1}^{\infty} (1 + q^n)^{24} = q + 24q^2 + 300q^3 + 2624q^4 + 18126q^5 + \cdots, \tag{3.14}$$

we find that

$$q = f - 24f^2 + 852f^3 - 35744f^4 + 1645794f^5 + \cdots, \tag{3.15}$$

and thus

$$\begin{aligned} 24E_2^* E'_{-2} &= 24 \sum_n a_n f^n = f + f^2 - \frac{8072}{27} f^3 + \frac{160841}{9} f^4 - \frac{1088512616}{1125} f^5 + \cdots, \\ 24E_2^* &= 24 \sum_n b_n f^n = 1 + 24f - 552f^2 + 19392f^3 - 810024f^4 + 37210944f^5 + \cdots. \end{aligned} \tag{3.16}$$

The coefficients of these series are the 2-adic analogue of Apéry's sequences $\{a_n, b_n\}$.

3.2 $p = 3$

The same technique can also be applied to other p when $X_0(p)$ has genus zero, where f is chosen to be $(\Delta(p\tau)/\Delta(\tau))^{1/(p-1)}$. In this manner, we prove the following.

Theorem 3.5. If $p = 3$, then $\zeta_p(3) \notin \mathbb{Q}$. □

Proof (sketch). The construction works as for $p = 2$. It suffices to determine the various radii of convergence, and prove that a_n/b_n provide infinitely many distinct convergents to $\zeta_p(3)/2$. For the second fact, one computes that a_n and b_n satisfy the following recurrence:

$$\begin{aligned} n^3 u_n = &-3\big(27(n-1)^3 - 8(n-1) - 4\big)u_{n-1} \\ &- 3^4\big(27(n-2)^3 - 8(n-2) + 4\big)u_{n-2} - 3^9(n-3)^3 u_{n-3}, \end{aligned} \tag{3.17}$$

from which it can be shown by an estimate of the linearly independent solutions to this recurrence that $|a_n/b_n|$ is monotonically increasing for $n \geq 2$.

If $f = (\Delta(p\tau)/\Delta(\tau))^{1/p-1}$, the components of the ordinary locus are given by $\|f\|_3 \leq 1$ and $\|f\|_3 \geq 3^6$. The curve $X_0(3)$ has two cusps 0 and $i\infty$ at which f has a pole and zero, respectively. There is one elliptic point at $1/2 + \sqrt{-3}/6$, at which point f takes the value -3^{-3}. Thus one finds that

$$\theta = \frac{6}{3 + \dfrac{3}{\log 3}} = 1.0469892839\ldots > 1, \tag{3.18}$$

and thus by the criterion of irrationality we are done. ■

Although the proof succeeds for $p = 3$, it fails for the other primes where $X_0(p)$ has genus zero ($p = 5, 7$, and 13). For example, for $p = 5$, we find that

$$\theta = \frac{3}{\dfrac{3}{2} + \dfrac{3}{\log 5}} = 0.8917942081 < 1. \tag{3.19}$$

4 The 2-adic Catalan's constant

There can be no p-adic analogue of Apéry's results for $\zeta(2)$, since $\zeta_p(2) = 0$ for all p. However, we may still study other p-adic L-values, in particular, the analogue of Catalan's constant:

$$G := \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots. \tag{4.1}$$

In this context, we should study Eisenstein series of *odd* weight and nontrivial character. Let $p = 2$. Let χ be the character of conductor 4. Then $L(2,\chi) = G$, whilst for nonnegative k,

$$L(-2k,\chi) = \frac{E_{2k}}{2}, \tag{4.2}$$

where E_{2k} is the $2k$th Euler number. Moreover, there exists for each odd positive integer an Eisenstein series $F_{2k+1} \in S_{2k+1}(\Gamma_1(4),\chi)$ given by the following q-expansion:

$$\begin{aligned} F_{2k+1} &= \frac{L(-2k,\chi)}{2} + \sum_{n=1}^{\infty} q^n \left(\sum_{d|n} d^{2k}\chi(d) \right) \\ &= \frac{L(-2k,\chi)}{2} + \sum_{n=0}^{\infty} \frac{q^{(2n+1)}(-1)^n (2n+1)^{2k}}{(1-q^{2n+1})}. \end{aligned} \tag{4.3}$$

Indeed, F_{2k+1} is nothing but the 2-adic specialization of E_κ to weights of the form $\chi \cdot (a \mapsto a^{2k+1})$, which are even since $\chi(-1) = -1$. Note that

$$F_{-1} = \frac{L_2(2,\chi)}{2} + \sum_{n=0}^{\infty} \frac{q^{(2n+1)}(-1)^n}{(2n+1)^2(1-q^{2n+1})}. \tag{4.4}$$

We will consider the function

$$H = F_1(F'_{-1} + \eta), \tag{4.5}$$

where F'_{-1} is holomorphic on the complex upper half-plane and formally given by the q-expansion

$$F'_{-1} = \sum_{n=0}^{\infty} \frac{q^{(2n+1)}(-1)^n}{(2n+1)^2(1-q^{2n+1})}. \tag{4.6}$$

If $\eta = L_2(2,\chi)/2$, then $F'_{-1} + \eta = F_{-1}$ is overconvergent (and thus so is H), and otherwise it is not. Our arguments now proceed in a very similar manner to those of $\zeta_p(3)$.

The curve $X_1(4)$ has genus zero, no elliptic points, and three cusps corresponding to $i\infty$, $1/2$, and 0. A uniformizer is given by the function

$$z = \left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/3} = q \prod_{n=1}^{\infty} (1+q^n)^8 (1+q^{2n})^8. \tag{4.7}$$

The function z vanishes at the cusp at infinity, has a pole at the cusp at 0, and equals -2^{-4} at the cusp $1/2$. The Fricke involution sends $2^8 z$ to $1/z$, and the component of the ordinary locus containing infinity is $\|z\|_2 \le 1$. If we write

$$H = \sum_{n=0}^{\infty} (a_n - b_n \eta) z^n, \tag{4.8}$$

the singularity obtained by inverting z occurs at $z = -2^{-4}$, and thus we obtain the Archimedean estimates

$$|a_n|, |b_n| \ll 2^{(4+\epsilon)n} \tag{4.9}$$

for all $\epsilon > 0$ and $n \gg 0$ depending on ϵ. Furthermore, it is clear that (for $n \ge 1$)

$$b_n \in \mathbb{Z}, \qquad [1, 2, \ldots, n]^2 a_n \in \mathbb{Z}. \tag{4.10}$$

Thus if $2a_n/b_n = p_n/q_n$, one obtains the estimates

$$|p_n|, |q_n| \ll 2^{(4+2/\log 2+\epsilon)n}. \tag{4.11}$$

We now study the 2-adic radii of convergence for various η.

Lemma 4.1. If $\eta = L_2(2, \chi)/2$, then the radius of convergence of H is at least 2^8. If $\eta \neq L_2(2, \chi)/2$, then the radius of convergence is at most 1. □

Proof. If $\eta \neq L_2(2, \chi)/2$, then H is not overconvergent so the radius of convergence is at most 1. Suppose that $\eta = L_2(2, \chi)/2$. Then $F_{-1} = F'_{-1} + \eta$ is an overconvergent finite slope eigenform of level $\Gamma_1(4)$. Once more we may appeal to the convergence results of Buzzard [**4**], in particular, Corollary 6.2, to conclude that F_{-1} extends not only over all of the supersingular region but everywhere over of $X_1(4)$ except (possibly) the component of the ordinary locus containing 0, which is $\|z\|_2 \ge 2^8$. ■

One finds as in **Lemma 3.2** that a_n and b_n satisfy the Apéry-like recurrences

$$(n+1)^2 u_{n+1} = \left(4 - 32n^2\right) u_n - 256(n-1)^2 u_{n-1}. \tag{4.12}$$

If $u_1 = -4$ and $u_2 = 28$, then $u_n = b_n$, whilst if $u_1 = 1$ and $u_2 = -3$, then $u_n = a_n$. From this, the valuation of b_n is easily seen to equal $2\varphi(n)$.

Combining these results, we conclude the following.

Theorem 4.2. There exist integers p_n, q_n such that q_n approaches infinity, and such that if

$$\theta = \frac{8 \log 2}{4 \log 2 + 2} = 1.1618804316\ldots > 1, \tag{4.13}$$

then

$$0 < \left\| L_2(2,\chi) - \frac{p_n}{q_n} \right\|_2 \le \frac{1}{\left(\max\left\{|p_n|, |q_n|\right\}\right)^{\theta-\epsilon}} \tag{4.14}$$

for sufficiently large n. In particular, $L_2(2,\chi)$ is irrational. □

We write down the first few terms a_n, b_n:

$$\begin{aligned} &a_n : 0, 1, -3, \frac{116}{9}, \frac{-331}{9}, \frac{-99116}{225}, \frac{3133076}{225}, \cdots, \\ &b_n : -1, -4, 28, -272, 3036, -36624, 464368, \cdots. \end{aligned} \tag{4.15}$$

Note that

$$2 \cdot \frac{a_6}{b_6} = \frac{783269}{13060350} = 2^{-1} + 1 + 2^2 + 2^3 + 2^5 + 2^6 + 2^7 + 2^9 + 2^{13} + 2^{18} + \cdots \tag{4.16}$$

already agrees with $L_2(2,\chi)$ to order 2^{34}.

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