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Even Galois representations and the Fontaine–Mazur conjecture

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Abstract We prove, under mild hypotheses, that there are no irreducible two-dimensional ordinary *even* Galois representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with distinct Hodge–Tate weights. This is in accordance with the Fontaine–Mazur conjecture. If $K/\mathbf{Q}$ is an imaginary quadratic field, we also prove (again, under certain hypotheses) that $\mathrm{Gal}(\overline{\mathbf{Q}}/K)$ does not admit irreducible two-dimensional ordinary Galois representations of non-parallel weight.

1 Introduction

Potential modularity has proved to be a powerful tool for studying arithmetic questions in the Langlands program [19, 27]. In this note, we show how this circle of ideas can be employed in a novel way to deduce some new instances of the Fontaine–Mazur conjectures.

The conjecture of Fontaine and Mazur ([15, p. 41], first edition, p. 190 second edition) is the following:

Conjecture 1.1 (Fontaine–Mazur) *Let*

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

be an irreducible representation which is unramified except at a finite number of primes and which is not the Tate twist of an even representation which fac-

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tors through a finite quotient group of $G_{\mathbf{Q}}$. Then ρ is associated to a cuspidal newform f if and only if ρ is potentially semi-stable at p.

Let $D_p \subset G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ denote a decomposition group at p. Assume that the Hodge–Tate weights of ρ are distinct, and so, in particular, the Hodge–Tate weights of any twist of ρ are also distinct. Any two-dimensional representation of D_p with finite image is Hodge–Tate with Hodge–Tate weights $(0, 0)$. Hence, no twist of ρ can have finite image, and Conjecture 1.1 predicts that ρ is modular. It is well known that Galois representations arising from classical modular forms are *odd*, namely, $\det(\rho(c)) = -1$ for a complex conjugation $c \in G_{\mathbf{Q}}$. We deduce that if ρ is *even* (that is, $\det(\rho(c)) = 1$), and the Hodge–Tate weights of ρ are distinct, then the conjecture of Fontaine and Mazur predicts that ρ does not exist.

Up to conjugation, the image of ρ lands in $\mathrm{GL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of some finite extension $L/\mathbf{Q}_p$ (see Lemme 2.2.1.1 of [8]). Let $\mathbf{F}$ denote the residue field of $\mathcal{O}$. We prove the following.

Theorem 1.2 *Let E be a totally real field, and let $\rho : G_E \to \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Suppose that $p > 7$, and, furthermore, that*

(1) *$\rho | D_v$ is ordinary with distinct Hodge–Tate weights for all $v | p$.*
(2) *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*
(3) $[E(\zeta_p)^+ : E] > 2$.

Then $\det(\rho(c)) = -1$ *for any complex conjugation c.*

In light of the recent proof of Serre's conjecture [19, 20] and modularity lifting theorems for ordinary representations [16], we immediately deduce the following corollary.

Corollary 1.3 *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Suppose that $p > 7$, and, furthermore, that*

(1) *$\rho | D_p$ is ordinary, with distinct Hodge–Tate weights.*
(2) *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$.*

Then ρ is modular.

Remark It should be remarked that *some* p-adic Hodge theory condition is necessary for the proof Theorem 1.2. For example, Corollary 1(b) of Ramakrishna [23] shows that there exist infinitely many even surjective representations $\rho : G_{\mathbf{Q}} \to \mathrm{SL}_2(\mathbf{Z}_7)$ unramified outside a finite set of primes.

The idea behind this theorem is simple: It suffices to show that ρ is *potentially* modular over some totally real field. Since modular representations are odd, the theorem follows immediately. (The actual argument is somewhat more circuitous.)

There are other circumstances in which one expects (following Fontaine-Mazur) the nonexistence of semistable Galois representations and for which the methods of this paper also apply. Let $K/\mathbf{Q}$ be an imaginary quadratic field, and let

$$\rho : G_K \to \mathrm{GL}_2(\mathcal{O})$$

be a continuous irreducible geometric Galois representation. The most general modularity conjectures (following Fontaine–Mazur, Langlands, Clozel, and others) predict the existence of a cuspidal automorphic representation π for $G = \mathrm{GL}(2)/K$ such that for all finite places $v \nmid p$ of K, π_v is determined by $\rho|_{K_v}$ via the local Langlands correspondence. Suppose that p splits in K, and that the local representations $\rho|_{D_v}$ for $v|p$ have Hodge–Tate weights $(0, m)$ and $(0, n)$ respectively, for positive integers m, n. The Hodge–Tate weights conjecturally determine the corresponding infinity type π_∞ of π. If $m \neq n$, however, a vanishing theorem of Borel and Wallach implies that no such cuspidal π can exist. We prove the following result in this direction.

Theorem 1.4 *Let $\rho : G_K \to \mathrm{GL}_2(\mathcal{O})$ be a continuous irreducible geometric Galois representation. Suppose that $p > 7$ splits in K, and, furthermore, that*

(1) *$\rho|D_v$ is ordinary for $v|p$, with Hodge–Tate weights $(0, m)$ and $(0, n)$, where $m, n > 0$.*
(2) *The residual representation $\overline{\rho}$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$, and the projective representation $\mathrm{Proj}(\overline{\rho}) : G_K \to \mathrm{PGL}_2(\mathbf{F})$ does not extend to $G_{\mathbf{Q}}$.*

Then $m = n$.

Theorem 1.2 is proven in Sect. 3, and Theorem 1.4 is proven in Sect. 2. Recall the abbreviations RAESDC and RACSDC for an automorphic form π for $\mathrm{GL}(n)$ stand for regular, algebraic, essentially-self-dual, and cuspidal and regular, algebraic, conjugate-self-dual, and cuspidal, respectively.

2 The tensor representation

In this section, we prove Theorem 1.4. The following proof is inspired by an idea of Ramakrishnan [24] to construct the Galois representations associated to $\mathrm{Sym}^2\pi_K$, where π_K is a regular algebraic automorphic form for $\mathrm{GL}(2)/K$ for an imaginary quadratic field K, without assuming any conditions on the central character.

Let ρ satisfy the first condition of Theorem 1.4. Our proof is by contradiction. Assume that the Hodge–Tate weights of ρ at the primes $v|p$ are $(0,m)$ and $(0,n)$ respectively, where $m,n>0$ and $m\neq n$. Let us consider the representation $\rho\otimes\rho^c : G_K\to \mathrm{GL}_4(\mathcal{O})$; it lifts to a representation of $G_{\mathbf{Q}}$ that is unique up to twisting by the quadratic character η of $\mathrm{Gal}(K/\mathbf{Q})$.

Lemma 2.1 *The extension ψ of $\rho\otimes\rho^c$ to $G_{\mathbf{Q}}$ satisfies $\psi\simeq\psi^\vee\chi$, where $\chi|_K=\det(\rho)\det(\rho^c)$ and $\chi(c)=\varepsilon=+1$ for any complex conjugation $c\in G_{\mathbf{Q}}$.*

Proof Since $\psi|_K$ and $\psi^\vee\det(\rho)\det(\rho^c)|_K$ are isomorphic and extend to $\mathbf{Q}$ uniquely up to twisting by η, the lemma is obvious up to the sign of ε. If $\varepsilon=-1$, then $\psi^\vee(c)=-\psi(c)$ and hence $\mathrm{Tr}(\psi(c))=0$. From the definition of ψ, however,

$$\psi(c)\sim\pm\begin{pmatrix}1&0&&\\0&1&&\\&&0&1\\&&1&0\end{pmatrix},$$

and hence $\varepsilon=+1$. □

Theorem 2.2 *The representation ψ is modular for $\mathrm{GL}(4)/F^+$ for some totally real field F^+.*

Proof The representation ψ is ordinary with Hodge–Tate weights $(0,m,n,m+n)$, which are distinct by assumption. The isomorphism $\psi\simeq\psi^\vee\chi$ gives rise to a pairing $\langle x,y\rangle$ on the vector space L^4 associated to ψ such that $\langle\sigma x,\sigma y\rangle=\chi(\sigma)\langle x,y\rangle$. Because $\wedge^2\psi$ is irreducible, this pairing is symmetric. Thus $\langle y,x\rangle=\varepsilon\langle x,y\rangle$, where $\varepsilon=+1=\chi(c)$. Hence, we may apply Theorem 7.5 of [5] to deduce the existence of a RAESDC representation Π_{F^+} for $\mathrm{GL}(4)/F^+$ associated to ψ. In order to apply this theorem, we also need to assume that $p>8$, and that, extending scalars if necessary, $\overline{\psi}$ has 2-big image, a calculation that we relegate to Sect. 6. □

2.1 A digression about normalizations

The modularity lifting theorems of [5, 13] associate to certain Galois representations $\rho: G_{F^+}\to\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ an algebraic automorphic form Π_{F^+} for $\mathrm{GL}(n)/F^+$. Here *algebraic* means C-algebraic (cf. [9] and §4.2 of [12]; here C refers either to "cohomological" or "Clozel"). On the other hand, it is sometimes useful to consider the normalized twist $\Pi_{F^+}|\cdot|^{(n-1)/2}$. Each normalization has its advantage; the former arises naturally when one constructs automorphic forms via cohomology, whereas the latter has the property that the restriction of the corresponding Langlands parameter at a place $v|\infty$ to

$W_{\mathbf{C}} = \mathbf{C}^{\times}$ is an algebraic character—that is, the normalization $\Pi_{F^+}|\cdot|^{(n-1)/2}$ is L-algebraic (where L refers either to "L-group" or "Langlands"). If v is a finite place of F^+ not dividing p, then the automorphic form Π_{F^+} associated to ρ by the main theorems of [5] and [13] (see §3.1 of [13]) are related as follows:

$$\iota \circ (\rho|_{W_{F_v^+}})^{\mathrm{ss}} = \mathrm{rec}\big(\Pi^{\vee}_{F^+,v}|\cdot|^{\frac{1-n}{2}}\big)^{\mathrm{ss}},$$

where rec is the local Langlands correspondence (see [17]), ss denotes semi-simplification, and ι is an isomorphism $\overline{\mathbf{Q}}_p \simeq \mathbf{C}$. In particular, the reciprocity map associates the normalization $\Pi_{F^+}|\cdot|^{(n-1)/2}$ to the dual representation $\rho^{\vee}$. In the sequel, when we refer to the *Langlands parameters* at infinity associated to Π_{F^+}, we shall literally mean the Archimedean Langlands parameters associated to the twist $\Pi_{F^+}|\cdot|^{(n-1)/2}$. This allows us to work with parameters that more naturally reflect the properties of the Galois representation ρ, moreover, these parameters are somewhat more conveniently behaved under functoriality. If ϵ is the p-adic cyclotomic character of $G_{\mathbf{Q}}$, we follow the convention that the Hodge–Tate weight of $\epsilon|_{G_{\mathbf{Q}_p}}$ is 1. The corresponding automorphic representation of $\mathrm{GL}(1)/\mathbf{Q}$ is $|\cdot|^{-1}$, and we have

$$\iota \circ (\epsilon|_{W_{\mathbf{Q}_p}})^{\mathrm{ss}} = \mathrm{rec}\big(|\cdot|^{-1,\vee}\big) = \mathrm{rec}(|\cdot|).$$

2.2 The proof of Theorem 1.4

Let Π_{F^+} denote the automorphic form for $\mathrm{GL}(4)/F^+$ associated to ψ whose existence was established by Theorem 2.2. For any infinite place $v|\infty$ of F^+, the associated Archimedean Langlands parameter is a 4-dimensional representation σ_v of the real Weil group $W_{\mathbf{R}}$ for each infinite place v of F^+. Since Π_{F^+} is algebraic, the restriction of σ_v to $\mathbf{C}^{\times}$ is of the form $z \mapsto z^p\overline{z}^q$ for integers p and q. By purity (Lemma 4.9 of [12]), the sum $p+q$ only depends on π and v. (Since the Galois representation descends to $\mathbf{Q}$, σ_v only depends on π, as we shall see below.) Since π is regular and ψ has Hodge–Tate weights $(0, m, n, m+n)$ at every $v|p$, it follows (with our normalizations) that σ_v is of the form

$$\sigma_v \simeq I\big(z^{m+n}\big) \oplus I\big(z^n\overline{z}^m\big),$$

where $I(\xi)$ denotes the induced representation of ξ from $\mathbf{C}^{\times}$ to $W_{\mathbf{R}}$. Here we have invoked the compatibility at $v|p$ between the infinity type of Π_{F^+} and the corresponding Hodge–Tate weights (see Theorem 1.1(4) of [5]). Since $I(\xi)\otimes I(\zeta) = I(\xi\zeta)\oplus I(\xi\overline{\zeta})$, it follows (cf. (5.8) of [26]) that

$$\wedge^2(\sigma_v) \simeq I\big(z^n\overline{z}^{n+2m}\big) \oplus I\big(z^m\overline{z}^{2n+m}\big) \oplus \mathrm{sgn}^{m+n+1}|\cdot|^{m+n} \oplus \mathrm{sgn}^{m+n+1}|\cdot|^{m+n}.$$

A result of Kim [21] implies the existence of the exterior square $\wedge^2\Pi_{F^+}$ for $\mathrm{GL}(6)/F^+$ whose construction is compatible with functoriality at all places except possibly those dividing 2 and 3. In particular, the infinity type of $\wedge^2\Pi_{F^+}$ at v is $\wedge^2(\sigma_v)$. Moreover, the work of Kim also implies that the representation $\wedge^2\Pi_{F^+}$ can be taken to be isobaric.

Lemma 2.3 *The base change of Π_{F^+} to any solvable Galois extension of F^+ is cuspidal.*

Proof The existence of the base change to any solvable extension follows by repeated applications of Theorem 4.2 (p. 202) of [1]. By assumption, the representation $\overline{\psi}$ and hence ψ remains irreducible after restricting to any solvable extension, and thus the restriction of ψ to any solvable extension admits no isomorphisms to any non-trivial self-twist. Thus, we are always in the setting of Theorem 4.2(a) of *ibid*, and we deduce the corresponding base changes are cuspidal. □

Lemma 2.4 *The automorphic form $\wedge^2\Pi_{F^+}$ is cuspidal for* GL(6).

Proof If $\wedge^2\Pi_{F^+}$ is not cuspidal, then, since it is isobaric, we may write

$$\wedge^2\Pi_{F^+} \simeq \boxplus_{j=1}^{m}\beta_j, \quad m > 1,$$

where each β_j is a cuspidal automorphic representation of $\mathrm{GL}(n_j)/F^+$ with $\sum_j n_j = 6$ and $n_i \le n_j$ if $i \le j$. By comparison with the infinity type $\wedge^2\sigma_v$, the forms β_j will necessarily be algebraic. Theorem 1.1 of [2] gives a list of necessary (and sufficient) conditions for $\wedge^2\Pi_{F^+}$ to be non-cuspidal. We show why none of these possibilities may occur.

(1) $\Pi_{F^+} = \pi_1 \boxtimes \pi_2$, where π_1 and π_2 are cuspidal for $\mathrm{GL}(2)/F^+$. If either π_1 or π_2 does not remain cuspidal over some quadratic extension of F^+, then Π_{F^+} does not remain cuspidal either, contradicting Lemma 2.3. Hence, we may assume that $\mathrm{Sym}^2\pi_1$ and $\mathrm{Sym}^2\pi_2$ are both cuspidal. It follows that

$$\wedge^2\Pi_{F^+} = \beta_1 \boxtimes \beta_2 = \chi_2 \otimes \mathrm{Sym}^2\pi_1 \boxtimes \chi_1 \otimes \mathrm{Sym}^2\pi_2,$$

where χ_1 and χ_2 are characters. In particular, the infinity type of β_1 and β_2 must be

$$I\left(z^n\overline{z}^{n+2m}\right) \oplus \mathrm{sgn}^{m+n+1}|\cdot|^{m+n} \quad \text{and} \quad I\left(z^m\overline{z}^{2n+m}\right) \oplus \mathrm{sgn}^{m+n+1}|\cdot|^{m+n}$$

respectively. Both these characters are algebraic and regular. Moreover, β_1 and β_2 are essentially self-dual (via their identifications up to twist with symmetric squares), and F^+ is totally real. It follows from the main result of [22] that β_1 and β_2 may be associated to three dimensional (essentially self-dual) Galois representations of G_{F^+}. Yet $\wedge^2 \Pi_{F^+}$ is associated to the Galois representation $\wedge^2 \psi|_{G_{F^+}}$, which is easily seen to be irreducible, and these facts are incompatible.

(2) Π_{F^+} is the Asai transfer of a dihedral cuspidal automorphic representation π_E of $\mathrm{GL}(2)/E$ for some quadratic extension E/F^+. We deduce that the base change of π_E to some CM extension H/E of degree two is no longer cuspidal, and thus that the base change of Π_{F^+} to the (solvable) Galois closure of H over F^+ is also not cuspidal. This contradicts Lemma 2.3.

(3) Π_{F^+} is of symplectic type. By the main theorem of [6] (cf. [10]), we may deduce that the symplectic/orthogonal alternative for the automorphic form determines and is determined by the Galois representation. Yet the Galois representation is of orthogonal type.

(4) Π_{F^+} is the automorphic induction from some quadratic extension E/F^+. If Π_{F^+} arises via automorphic induction, then it is isomorphic to a self twist by a quadratic character. It follows that the base change of Π_{F^+} to E is not cuspidal, contradicting Lemma 2.3. □

On the level of Galois representations, if we let $F = F^+.K$ then

$$\wedge^2 \psi|_F = \mathrm{Sym}^2(\rho)\det(\rho^c) \oplus \mathrm{Sym}^2(\rho^c)\det(\rho).$$

By multiplicity one for GL(6) [18], it follows that $\wedge^2 \Pi_{F^+} \simeq \wedge^2 \Pi_{F^+} \otimes \eta$, where η is the quadratic character of F/F^+. In particular, from Theorem 4.2 of [1], $\wedge^2 \Pi_{F^+}$ is the automorphic induction of an automorphic form for $\mathrm{GL}(3)/F$, which we shall denote by $S(\pi_F)$. From the description of the infinity type $\wedge^2 \sigma_v$ of $\wedge^2 \Pi_{F^+}$ given above, we deduce that the infinity type of $S(\pi_F)$ at v is one of four possibilities:

$$\begin{aligned}
&z \mapsto z^n \overline{z}^{n+2m} \oplus z^m \overline{z}^{2n+m} \oplus z^{m+n} \overline{z}^{m+n},\\
&z \mapsto z^n \overline{z}^{n+2m} \oplus z^{2n+m} \overline{z}^{m} \oplus z^{m+n} \overline{z}^{m+n},\\
&z \mapsto z^{n+2m} \overline{z}^{n} \oplus z^m \overline{z}^{2n+m} \oplus z^{m+n} \overline{z}^{m+n},\\
&z \mapsto z^{n+2m} \overline{z}^{n} \oplus z^{2n+m} \overline{z}^{m} \oplus z^{m+n} \overline{z}^{m+n}.
\end{aligned}$$

These correspond to the only four partitions of (the six characters occurring in) $\wedge^2 \sigma_v|_{\mathbf{C}^\times}$ into two sets of three characters which are permuted by the involution $z \mapsto \overline{z}$. This calculation uses the fact that $\{0, m, n, m+n\}$ are all distinct. The notation $S(\pi_F)$ is meant to suggest the existence of an automorphic form π_F for $\mathrm{GL}(2)/F$ associated to $\rho|_{G_F}$; if such a π_F existed then $\mathrm{Sym}^2 \pi_F$ would be isomorphic (up to twist) to $S(\pi_F)$. It is not necessary for our arguments, however, to establish the existence of such a π_F. We see explicitly that the infinity type of $S(\pi_F)$ is regular algebraic and cohomological. Moreover, we deduce that the infinity type is not preserved by the Cartan involution (which effectively replaces z by $\overline{z}$ in this case—once more noting that $\{0, m, n, m+n\}$ are all distinct by hypothesis), and thus $S(\pi_F)$ cannot exist, by Borel–Wallach (Theorem 6.7, VII, p. 226 [7]). As stated, this theorem applies only in the compact case. However, the same proof (via vanishing of $(\mathfrak{g}, K)$-cohomology) applies more generally to the *cuspidal* cohomology in the non-compact case, which also may be computed via $(\mathfrak{g}, K)$-cohomology. □

3 Even representations

Let $E/\mathbf{Q}$ be a totally real field. Recall that a representation $r : G_E \to \mathrm{GL}_n(\mathcal{O})$ is *odd* if, for any complex conjugation $c \in G_E$, $\mathrm{Tr}(\rho(c)) = -1$, 0, or 1. We start by noting the following:

Theorem 3.1 *Let n be odd. Let E be a totally real field, and let $r : G_E \to \mathrm{GL}_n(\mathcal{O})$ be a continuous irreducible Galois representation unramified outside finitely many primes. Let χ be a finite order character of G_E that is unramified at all $v|p$. Suppose that $p > 2n$, and, furthermore, that*

(1) *r is self-dual up to twist: $r \simeq r^\vee \epsilon^{1-n}\chi$.*
(2) *r is ordinary for all $v|p$ with distinct Hodge–Tate weights.*
(3) *$\overline{r}$ has 2-big image, in the sense of* [5], *Definition* 7.2.
(4) *The fixed field of* $\mathrm{ad}(\overline{r})$ *does not contain* $E(\zeta_p)^+$.
(5) $(\det \overline{r})^2 = \epsilon^{n(1-n)} \bmod p$.

Then there exits a totally real field $F^+/\mathbf{Q}$ such that the restriction $r|_{F^+}$ is modular, i.e., associated to a RAESDC form Π_{F^+}.

Proof Taking determinants of both sides of the relation $r \simeq r^\vee \epsilon^{1-n}\chi$, we deduce that $\epsilon^{1-n}\chi(c) = +1$ for any complex conjugation c. Since r is irreducible and self-dual up to twist, we may deduce that r preserves a non-degenerate pairing $\langle x, y\rangle$. Since n is odd, moreover, this pairing must be symmetric. Hence, we may apply Theorem 7.5 of [5] (with $\epsilon = +1$) to deduce the result. □

Remark It should be possible to prove this theorem for $p > n$ as follows. Suppose that r corresponds to a RACSDC automorphic Π_F for $\mathrm{GL}(n)/F$ for some CM field F. By Lemma 4.3.3 of [13], we may deduce that Π_F descends to a RAESDC form Π_{F^+} for $\mathrm{GL}(n)/F^+$. Hence, in light of modularity lifting theorems of Geraghty [16] (in particular, Theorem 5.3.2), it suffices to prove that $\overline{r}|_{G_F}$ is modular for some CM field F which is sufficiently disjoint from $E(\zeta_p)^+$. One approach to proving modularity theorems of this type arises in the work of Barnet-Lamb ([4], Proposition 7). As written, this theorem requires some extra assumptions, in particular, that $\overline{r}$ is crystalline and that the residue field of $\mathcal{O}$ is $\mathbf{F}_p$ (the fact that n is odd guarantees that all sign conditions are satisfied). However, combining this approach with recent advances (particularly, generalizing the results about the monodromy of the Dwork family proved in ([5] §4, §5) to the more general setting of [4]), these conditions can presumably be removed in the ordinary case. The reason that one obtains a better bound on p is that this method requires modularity lifting theorems for $\mathrm{GL}(n)$ (over a CM field) rather than modularity theorems for $\mathrm{GL}(2n)$ (over a totally real field) as in the proof of Theorem 7.5 of [5].

Proof of Theorem 1.2 Suppose that ρ satisfies the conditions of Theorem 1.2. I claim that $r = \mathrm{Sym}^2(\rho) \otimes (\epsilon^{-1}\det(\rho)^{-1}) = \mathrm{ad}^0(\rho) \otimes \epsilon^{-1}$ satisfies the conditions of Theorem 3.1. In particular:

(1) r is self-dual up to twist, because $\mathrm{ad}^0(\rho)$ is self-dual.
(2) r is ordinary, because ρ is ordinary (by assumption).
(3) $\overline{r}$ has 2-big image. By corollary 2.5.4 of [13], $\overline{r}$ has big-image. In fact, the same proof applies to show that $\overline{r}$ has 2-big image. The difference in the definition of 2-big image and big image is the requirement that the element $h \in H$ with eigenvalue α arising in the definition of bigness (Definition 2.5.1 of [13], see also Sect. 6 of this paper) has the property that if β is any other generalized eigenvalue of h, then $\alpha^2 \neq \beta^2$. In the proof of Lemma 2.5.2 of [13], the element h is taken to be a generator t of the $\mathbf{F}_p$-split torus of $\mathrm{SL}_2(\mathbf{F}_p)$. The image of t acting via the adjoint representation of $\mathrm{SL}_2(\mathbf{F}_p)$ is

$$\mathrm{diag}\big(\delta^2, 1, \delta^{-2}\big),$$

where δ is a generator of $\mathbf{F}_p^\times$. If α and β are two distinct eigenvalues of this matrix, then $\alpha^2 = \beta^2$ implies that $\delta^8 = 1$. Yet δ generates $\mathbf{F}_p^\times$ and thus has order $p-1$, which does not divide 8 if $p \geq 7$.

It should be noted that Lemma 2.5.2 of [13] is not exactly correct as stated—the requirement on l in *ibid* should be $l > 2n+1$ rather than $l > 2n-1$. The issue is the appeal to [11]; the group $H^1(U, \mathrm{Symm}^{2i})^B$ vanishes for $l > 2n+1$ rather than $l > n+1$. Of relevance to this paper is that $H^1(\mathrm{SL}_2(\mathbf{F}_7), \mathrm{Sym}^4\mathbf{F}^2) \neq 0$; this is why we assume that $p > 7$. In fact, the main theorem of our paper still holds when $p = 7$ under the stronger assumption that $\mathrm{SL}_2(\mathbf{F}_q) \subseteq \mathrm{image}(\overline{\rho})$ for some $q > p$.
(4) The fixed field of $\mathrm{ad}(\overline{r})$ does not contain $E(\zeta_p)^+$. We are assuming the image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$, and in particular that

$$\mathrm{SL}_2(\mathbf{F}_p) \subset \mathrm{image}(\overline{\rho}) \subset \mathrm{GL}_2(\mathbf{F}).$$

It follows that the image of $\mathrm{ad}^0(\overline{\rho})$ contains $\mathrm{PSL}_2(\mathbf{F}_p)$ and is contained in $\mathrm{PGL}_2(\mathbf{F})$. Consequently (by the classification of subgroups of $\mathrm{PGL}_2(\mathbf{F})$), the image is isomorphic to either $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some $\mathbf{F}_p \subset k \subset \mathbf{F}$. Since $\#k \geq 7$, $\mathrm{PSL}_2(k)$ is a simple group. Since $\overline{r}$ is a twist of $\mathrm{ad}^0(\overline{\rho})$, the image of $\mathrm{ad}(\overline{r})$ is also isomorphic to $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$. In particular, the maximal solvable extension E' of E contained inside $E(\mathrm{ad}(\overline{r}))$ has degree at most 2 over E. Hence

$$E\big(\mathrm{ad}(\overline{r})\big) \cap E(\zeta_p)^+ = E' \cap E(\zeta_p)^+ \subset E'.$$

Yet $[E' : E] \leq 2$, and by assumption, $[E(\zeta_p)^+ : E] > 2$. Thus $E(\mathrm{ad}(\overline{r}))$ does not contain $E(\zeta_p)^+$.

(5) Since $\det(\mathrm{ad}^0(\overline{\rho}))$ is trivial, $\det(\overline{r})^2 = \det(\epsilon^{-1})^2 = \epsilon^{-6} = \epsilon^{3(1-3)}$.

We deduce that r is modular over some totally real field F^+. Using the same normalizations discussed in Sect. 2.1, we conclude that there exists an automorphic form Π_{F^+} for $\mathrm{GL}(3)/F^+$ such that, for all finite v in F^+ not dividing p,

$$\iota \circ (r|_{W_{F^+_v}})^{\mathrm{ss}} = \mathrm{rec}\big(\Pi^\vee_{F^+,v} |\cdot|^{-1}\big)^{\mathrm{ss}}.$$

Since $\mathrm{rec}(|\cdot|^{-1}) = \epsilon^{-1}$, we may twist both sides to deduce that

$$\iota \circ \big(\mathrm{ad}^0(\rho)|_{W_{F^+_v}}\big)^{\mathrm{ss}} = \mathrm{rec}\big(\Pi^\vee_{F^+,v}\big)^{\mathrm{ss}}.$$

By multiplicity one for GL(3) [18], we immediately deduce (by considering the Galois representation $\mathrm{ad}^0(\rho)$) that $\Pi^\vee_{F^+} \simeq \Pi_{F^+}$ and that Π_{F^+} has trivial central character. If follows from Theorem A and Corollary B of [25] that Π_{F^+} is the symmetric square of a RAESDC automorphic form π_{F^+} for $\mathrm{GL}(2)/F^+$, that is, a Hilbert modular form. Such automorphic forms are known to admit a p-adic Galois representation, which must be equal to $\rho|_{F^+}$ up to twist. Yet the Galois representations associated to Hilbert modular forms are odd, and thus ρ is odd. □

Instead of appealing to functorial properties of the symmetric square lift, one may appeal to the following theorem:

Theorem 3.2 (Taylor [28]) *Let F^+ be a totally real field, let Π_{F^+} be a RAESDC automorphic form for $\mathrm{GL}(n)/F^+$ for odd n, and let r be a p-adic Galois representation associated to Π_{F^+}. Assume that r is irreducible. Then r is odd.*

Remark By Theorem 3.2, one may deduce immediately from the modularity of r that $\mathrm{Sym}^2(\rho)$ and hence ρ are odd.

Remark If the analog of Theorem 3.2 was known for *even* n, this would also lead to a different proof of Theorem 1.4. Namely, the Galois representation $\psi = \rho \otimes \rho^c$ is shown to be associated to a RAEDSC form Π_{F^+} for $\mathrm{GL}(4)/F^+$, and yet $\mathrm{Tr}(\psi(c)) = \pm 2$.

Remark Although the Fontaine–Mazur conjecture predicts that Theorem 3.1 should continue hold when n is even, the argument above cannot (directly) be made to work. For n even, the representations $\overline{r}$ will *not*, in general, be potentially modular (in the sense we are using) over any CM field F, because the Bellaïche–Chenevier sign of the Galois representation (see [6]) may be -1. Indeed, when $n = 2$, all representations are self-dual up to twist, and so an even representation $\overline{\rho} : G_E \to \mathrm{GL}_2(\mathbf{F})$ is *never* potentially modular over a CM extension F in the sense we are using. (On the other hand, both conjecturally and experimentally (see [14]), $\overline{\rho}$ *is* modular for $\mathrm{GL}(2)/F$ if we omit the self-dual requirement.)

Remark For $n = 4$, suppose that E be a totally real field, and $\rho : G_E \to \mathrm{GSp}_4(\mathcal{O})$ is a continuous irreducible Galois representation unramified outside finitely many primes. Assume, otherwise, that r satisfies all the conditions of Theorem 3.1. Since ρ is symplectic, $\wedge^2\rho$ has a one dimensional summand. Let r be the complementary summand. It is a simple exercise to see that r is self-dual up to twist by a character that is either totally odd or totally even, and that r is ordinary with distinct Hodge–Tate weights. If $\overline{\rho}$ has image containing $\mathrm{GSp}_4(\mathbf{F})$, then the image of $\overline{r}$ in $\mathrm{GL}_5(\mathbf{F})$ is presumably large (this is an unpleasant calculation that the author has no interest in attempting). If so, then one may apply Theorem 3.1 to deduce that r is modular, and deduce from Theorem 3.2 that r is odd. Consequently $\overline{\rho}$ cannot be totally even, that is, $\overline{\rho}(c)$ is not a scalar for any complex conjugation $c \in G_E$. This result seems to be about the natural limit for such arguments—there does not seem to be a way to deduce anything about totally even representations with image in $\mathrm{GSp}_6(\mathcal{O})$, for example.

4 Compatible families

Let us recall from [29] the notion of a weakly compatible family of Galois representations. (In this section only, $\mathcal{O}$ will be denote a global ring of integers, not a local one.)

Definition 4.1 A weakly compatible family $R = (L, \{\rho_\lambda\}, P_\ell(T), S, \{m, n\})$ of two dimensional Galois representations over $\mathbf{Q}$ consists of:

(1) A number field L with ring of integers $\mathcal{O}$,
(2) A finite set of rational primes S,

(3) For each prime $\ell \notin S$, a monic polynomial $P_\ell(T)$ of degree 2 with coefficients in L,
(4) For each prime λ of $\mathcal{O}$ with residue characteristic p,

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathcal{O}_\lambda)$$

is a continuous representations such that, if $p \notin S$, then $\rho_\lambda|D_p$ is crystalline, and if $\ell \notin S \cup \{p\}$ then ρ_λ is unramified at ℓ and $\rho_\lambda(\mathrm{Frob}_\ell)$ has characteristic polynomial $P_\ell(T)$,
(5) m and n are integers such that for all primes λ of $\mathcal{O}_F$ above p, the representation $\rho_\lambda|D_p$ is Hodge–Tate with Hodge–Tate weights m and n.

Say that R is irreducible if one (respectively, any) ρ_λ is irreducible. We prove the following result (contrast the result of Kisin [20], Corollary (0.5)).

Theorem 4.2 *Let R be an irreducible weakly compatible family of two dimensional Galois representations of $\mathbf{Q}$ with distinct Hodge–Tate weights. Then, up to twist, R arises from a rank 2 Grothendieck motive $M(f)$ attached to a classical modular form f of weight ≥ 2.*

Proof If $\{\rho_\lambda\}$ is odd (for any λ), this is a consequence of [20], Corollary (0.5). Hence, we may assume that $k = n - m > 0$ and that ρ_λ is even for all λ. Without loss of generality, we may assume that $m = 0$. If the projective image of $\overline{\rho}_\lambda$ is either $\{A_4, S_4, A_5\}$ for infinitely many λ, then the projective image of ρ_λ is also finite and we are in case 2 above. If the projective image of $\overline{\rho}_\lambda$ is dihedral for infinitely many λ, then ρ_λ is induced from a one dimensional character of a quadratic extension of $\mathbf{Q}$. In the latter case, the modularity of ρ_λ is an easy consequence of class field theory (see [15]). Hence, we may assume that there exist infinitely many primes $\lambda \in \mathcal{O}$ such that $\mathcal{O}_\lambda = \mathbf{Q}_p$, and such that the image of $\overline{\rho}_\lambda$ contains $\mathrm{SL}_2(\mathbf{F}_p)$. Consider a sufficiently large such prime λ. If $\rho_\lambda|D_p$ is ordinary, then Theorem 1.2 implies that ρ_λ is odd, a contradiction. Otherwise, since we may assume that $p \notin S$, the representation $\rho_\lambda|D_p$ is crystalline at p, and (assuming that p is sufficiently large with respect to k) that $\overline{\rho}_\lambda|D_p \simeq \mathrm{Ind}(\omega_2^k)$ for the fundamental tame character ω_2 of level 2. Consider the representation $r = \mathrm{ad}^0(\overline{\rho}_\lambda) \otimes \epsilon$, where ϵ is the cyclotomic character. The proof of Theorem 3.1 can be modified to show that for p sufficiently large, r is modular and hence $\overline{\rho}_\lambda$ is odd, completing the proof of the theorem. The key adjustment required to prove the potential modularity of r is to replace the appeal to Theorem 7.5 of [5] with Theorem 7.6 of *ibid*, noting that the coefficients of the representation $\overline{\rho}_\lambda$ are $\mathbf{Q}_p$, and that we may take λ sufficiently large so as to deduce from Lemma 7.4 of [5] that $\overline{r}$ has $2k$-big image. $\square$

5 Complements

One idea of this paper is to use potential modularity and functoriality to rule out the existence of Galois representations whose infinity type has the same infinitesimal character as a non-unitary finite dimensional representation. This method, however, cannot be applied in all such situations. Consider a representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_3(E)$ with three distinct Hodge–Tate weights that are *not* in arithmetic progression. The 8 dimensional irreducible subrepresentation ψ of $\rho \otimes \rho^{\vee}$, which one might hope to prove is automorphic for $\mathrm{GL}(8)/F^+$, does not have distinct Hodge–Tate weights. Moreover, even if one knew that ψ arose from some automorphic form Π_{F^+}, functoriality is not sufficiently developed to reconstruct the π_{F^+} (associated to ρ) from Π_{F^+}. Another natural question that falls outside the scope of our methods is the following.

Question 5.1 *Let K be an imaginary quadratic field, and let E/K be an elliptic curve. Does there exist a totally real field F^+ such that E is potentially modular over $F := F^+.K$? That is, does there exist an automorphic representation π for* $\mathrm{GL}(2)/F$ *such that $L(\pi_F, s) = L(E/F, s)$?*

6 $\overline{\psi}$ has big image

In this final, technical section, we verify that the residual representation $\overline{\psi}$ occurring in Lemma 2.1 has 2-big image. The definition of 2-big image depends on the residue field $\mathbf{F}$. However, in all modularity lifting theorems it is harmless to extend scalars. (In particular, in Theorem 7.5 of [5], one is free to replace $\mathcal{O} = \mathcal{O}_L$ by $\mathcal{O}_M$ for any finite extension M/L.)

Lemma 6.1 *If $\mathbf{F}_{p^2} \subset \mathbf{F}$, the residual representation $\overline{\psi}$ of Sect.* 2 *has* 2*-big image in the sense of* ([5], *Definition* 7.2).

Proof Write V for the 4-dimensional vector space underlying $\overline{\psi}$. Recall that $G = \mathrm{im}(\psi) \subseteq \mathrm{GL}_4(\mathbf{F})$ has 2-big image if:

(1) G has no p-power quotient.
(2) $H^i(G, \mathrm{ad}^0(V)) = 0$ for $i = 0$ and $i = 1$.
(3) For every irreducible G-submodule W of $\mathrm{ad}^0(V) \subset \mathrm{Hom}(V, V)$, there exists an $g \in G$ such that:
    (a) The α-generalized eigenspace $V_{g,\alpha}$ for g is one dimensional for some $\alpha \in \mathbf{F}$.
    (b) For every other eigenvalue β of g, $\beta^2 \neq \alpha^2$.
    (c) If $v \in V_{g,\alpha}$, there is an inclusion $v \in W(v)$.

If this property holds for a finite index subgroup $H \subseteq G$ of index co-prime to p, then it holds for G.

By assumption, if G is the image of $\overline{\psi}$, then G contains, with index co-prime to p, the group $H = \mathrm{SL}_2(k)^2 \rtimes \mathbf{Z}/2\mathbf{Z}$ for some field $\mathbf{F}_p \subset k \subset \mathbf{F}$. Condition (1) holds for H.

Write V for the 4-dimensional vector space underlying $\overline{\psi}$. There is a natural decomposition $V = X \otimes Y$ where the first $\mathrm{SL}_2(k)$ factor acts on X and the second on Y. We find that

$$\mathrm{ad}^0(V) = W_6 \oplus W_9$$

for irreducible representations W_6 and W_9 of dimensions 6 and 9, respectively. Explicitly,

(1) $W_6 = \mathrm{ad}^0(X) \oplus \mathrm{ad}^0(Y)$,
(2) $W_9 = \mathrm{ad}^0(X) \otimes \mathrm{ad}^0(Y)$.

Let C denote a split Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$, and N a non-split Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$. There are isomorphisms

$$C \simeq \mathbf{F}_p^\times, \qquad N \simeq (\mathbf{F}_{p^2}^\times)^{\sigma=-1},$$

where σ denotes Frobenius in $\mathbf{F}_{p^2}^\times$. Let δ denote a generator of C and γ a generator of N. Then $g = (\delta, \gamma) \subset C \times N \subset \mathrm{SL}_2(\mathbf{F}_p)^2$ is naturally an element of H. The eigenvalues of g on $X \otimes Y$ are $\{\delta\gamma, \delta\gamma^{-1}, \delta^{-1}\gamma, \delta^{-1}\gamma^{-1}\}$. Suppose that the squares of two distinct elements of this set coincide. Then there must be an equality

$$\delta^{4i}\gamma^{4j} = 1$$

where i and j are in $\{-1, 0, 1\}$ and are not both zero. Taking the product (respectively ratio) of this element and its conjugate $\delta^{4i}\gamma^{-4j}$, we deduce that either $\delta^8 = 1$ or $\gamma^8 = 1$. Yet δ and γ have orders $p-1$ and $p+1$ respectively, contradicting the assumption that $p > 7$. In particular, the squares of the eigenvalues (and thus the eigenvalues) of g are distinct. Suppose that $W \subset \mathrm{ad}^0(V)$ is an irreducible representation such that W contains a non-zero g-invariant element w. Then w induces a map $w : V \to V$ which is g-equivariant. Since all the eigenvalues of g are distinct, all the eigenvectors of g are eigenvectors of w. In particular, since w is non-zero, there exists at least one eigenvector of g with eigenvalue α which does not lie in the kernel of w. If $V_{g,\alpha}$ is the line generated by this eigenvector, and $v \in V_{g,\alpha}$, it follows that v is a non-zero multiple of $w(v)$ and thus $v \in W(v)$. By inspection, both W_6 and W_9 contain g-invariant elements, and thus we have verified condition (3).

We now verify condition (2). Since $\mathrm{SL}_2(k)^2$ is contained in G with index co-prime to p, it suffices to show that the cohomology vanishes for this group. This is elementary for H^0. Write $H = \Gamma_1 \times \Gamma_2$ where $\Gamma_1 = \mathrm{SL}_2(k)$ acts on X and $\Gamma_2 = \mathrm{SL}_2(k)$ acts on Y. The modules W_6 and W_9 decompose as H modules of the form $A \otimes B$ where $A = \mathrm{Sym}^{2i}(X)$ and $B = \mathrm{Sym}^{2j}(Y)$ for i and j in $\{0, 1\}$, where at least one of A and B is non-trivial. By symmetry, we may assume that B is non-trivial. Hence $A \otimes B$ has no invariants as an Γ_2-module, and thus, by inflation-restriction,

$$\begin{aligned} H^1(\Gamma_1 \times \Gamma_2, A \otimes B) \hookrightarrow H^1(\Gamma_1, A \otimes B) &= B \otimes H^1(\Gamma_1, A) \\ &= B \otimes H^1(\mathrm{SL}_2(k), \mathrm{Sym}^{2i} X) = 0, \end{aligned}$$

as the latter group vanishes for $i = 0, 1$ and $p > 5$. □

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