EVEN GALOIS REPRESENTATIONS

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0.1. **Introduction.** These are edited notes for some talks I gave during the Fontaine Trimester at the Institut Henri Poincaré in March 2010. The exposition of U-oddness follows closely the work of Bellaïche and Chenevier [4]. I would like to thank Matthew Emerton and Florian Herzig for helpful remarks. I have updated these notes to include references to the recent work of Thorne [19], which allows for some simplifications in the exposition.

1. Motives over $\mathbf{Q}$

Let M be a pure Grothendieck motive over $\mathbf{Q}$ with (pure) weight w. M admits a Hodge decomposition

$$M_B \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=w} H^{p,q}.$$

The action of complex conjugation (induced by the action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on M considered as a motive over $\mathbf{R}$) on this decomposition sends $H^{p,q}$ to $H^{q,p}$. It follows that the trace of complex conjugation on $M_B \otimes \mathbf{C}$ has absolute value at most $h^{w/2,w/2} := \dim H^{w/2,w/2}$. The compatibility of étale and de Rham realizations of M implies that if $r : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_n(\mathbf{Q}_p)$ is the p-adic Galois representation associated to M, then $|\mathrm{Trace}(r(c))| \le h^{w/2,w/2}$. For example, if M is regular ($h^{p,q} \le 1$ for each p, q) then $|\mathrm{Trace}(\rho(c))| \le 1$.

Suppose that

$$\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_n(\mathbf{Q}_p)$$

is continuous, absolutely irreducible, unramified outside finitely many primes, and potentially semistable at p with distinct Hodge–Tate weights. The conjecture of Fontaine and Mazur [15] (together with the Tate conjecture) predicts that ρ arises from some pure motive M. In particular, it should be the case that

$$|\mathrm{Trace}(\rho(c))| \le^{?} 1.$$

In some cases, it is possible to prove this inequality without first proving the Fontaine–Mazur conjecture. Using Taylor's notion of *potential modularity*, we can deduce certain cases of this inequality under additional hypotheses, including that ρ is ordinary and that ρ is self dual up to twist. The general question appears to be very hard for $n \ge 3$.

2. Odd Representations

Suppose that $G/\mathbf{Q}$ is an inner form of a split algebraic group (this assumption is merely to avoid mentioning the L-group). If π is an L-algebraic cuspidal automorphic form for $G/\mathbf{Q}$, then one expects (see, for example, [8]) to associate to π a Galois representation[1]

$$\rho := \rho_\pi : G_{\mathbf{Q}} \to G^\vee(\overline{\mathbf{Q}}_p).$$

If (up to twist) π is cohomological, then one might also expect that ρ arises from a regular motive. In particular, if $G = \mathrm{GL}_n$, then (as in the previous discussion) one would expect the image of complex conjugation $\rho(c)$ to have trace ± 1. When $G = \mathrm{GL}_2$, the Galois representations associated to classical modular forms of weight ≥ 2 constructed by Deligne can be seen to have this property. Explicitly, the determinant of the representation associated to a classical modular form f is given by $\epsilon^{k-1} \cdot \eta$, where ϵ is the cyclotomic character, k is the weight of f, and η is the Nebentypus character of f. Since $\eta(-1) = (-1)^k$, the determinant of complex conjugation is always $(-1)^{k-1}\eta(-1) = -1$. This leads to the following definition: a Galois representation into $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is *odd* if complex conjugation has determinant -1 and *even* if it has determinant $+1$. The elements of order two in $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ with determinant minus one form a unique conjugacy class. For a semi-simple simply connected algebraic group G over $\mathbf{R}$, we may consider the involutions of G acting on the adjoint representation. By [6], Proposition 6.1 (see also [11]), all such involutions have trace at least

$$\mathrm{rank}(G) - 2 \cdot \mathrm{rank}(K),$$

and there is a unique conjugacy class such that equality holds. One may think of this involution as acting on the adjoint representation with "as many minus ones as possible". Recall that for a semi-simple real algebraic group G with maximal compact K and Borel B, the quantity $l_0 = \mathrm{rank}(G) - \mathrm{rank}(K) \geq 0$ is a measure of the failure (or not) of G to have discrete series, and $\dim(G) = 2 \cdot \dim(B) - \mathrm{rank}(G)$.

2.1. **Definition.** *A representation $\rho : G_{\mathbf{Q}} \to G^\vee(\overline{\mathbf{Q}}_p)$ is $G^\vee$-odd if the action of c on $\mathrm{ad}(\mathfrak{g}^\vee)$ has trace $\mathrm{rank}(G) - 2 \cdot \mathrm{rank}(K)$. Equivalently, ρ is odd if and only if*

$$\begin{aligned}\dim(\mathrm{ad}(\mathfrak{g}^\vee))^{c=-1} &= \frac{1}{2}\left(\dim(\mathrm{ad}(\mathfrak{g}^\vee)) - \left(\dim(\mathrm{ad}(\mathfrak{g}^\vee))^{c=+1} - \dim(\mathrm{ad}(\mathfrak{g}^\vee))^{c=-1}\right)\right)\\ &= \frac{1}{2}\left(\dim(G) - (\mathrm{rank}(G) - 2\cdot \mathrm{rank}(K))\right)\\ &= \frac{1}{2}\left(\dim(G) + \mathrm{rank}(G) - 2l_0\right) = \dim(B) - l_0.\end{aligned}$$

One expects that if π is regular up to twist, then ρ is odd [16]. One reason to expect this might be true is that it is in accordance with the most general "$R = \mathbf{T}$" conjectures [11]. Given a residual representation $\overline{\rho} : G_{\mathbf{Q}} \to G^\vee(\mathbf{F})$, the universal $G^\vee$-deformation ring of $\overline{\rho}$ has expected relative (over $\mathbf{Z}_p$) dimension

$$\dim H^1(G_{\mathbf{Q}}, \mathrm{ad}(\mathfrak{g}^\vee)) - \dim H^2(G_{\mathbf{Q}}, \mathrm{ad}(\mathfrak{g}^\vee)) = \dim \mathrm{ad}(\mathfrak{g}^\vee) - \dim H^0(G_{\mathbf{R}}, \mathrm{ad}(\mathfrak{g}^\vee)) = \dim(\mathrm{ad}(\mathfrak{g}^\vee))^{c=-1},$$

where $\mathfrak{g}^\vee$ is the Lie algebra of $G^\vee$ considered as a representation of $G_{\mathbf{Q}}$ via $\overline{\rho}$. Thus "odd" representations are those whose deformation rings are as large as possible, namely, of dimension $\dim(B) - l_0$. For example, if $G^\vee = \mathrm{PGL}_2$, then deformations of $\overline{\rho}$ correspond to deformations of some GL_2 lift with fixed determinant. If $\overline{\rho}$ is odd, the deformation space is $\dim(B) - l_0 = 2 - 0$ dimensional, whereas for even representations the corresponding deformation space should be zero dimensional.

[1] Note that ρ may not be uniquely defined up to conjugacy. For example, A_6 has two pairs of projective representations r_1, r_2 such that $r_1(g)$ and $r_2(g)$ are conjugate for all $g \in A_6$ but r_1 and r_2 are not themselves conjugate. The corresponding Maass form π for SL_3 (presuming it exists) should therefore be associated to "both" representations, in which multiplicity one would fail for SL_3. This idea can be found in [7] where it is used it to prove that multiplicity one fails for SL_n for $n \geq 3$.

The use of the word "odd" in this generality is, perhaps, slightly unfortunate. Firstly, it leads to ambiguity as to what "even" should mean, since it might either mean "not odd", or that the image of $\rho(c)$ is trivial modulo center (although for GL_2 there is no ambiguity). Secondly, applied to $G = \mathrm{GL}_1$, it leads to the somewhat inconsistent description of all characters as odd, contrary to the usual usage. (Perhaps it would be better to describe such representations as G-odd "up to twist", since, by definition, oddness only depends on the image of complex conjugation in G modulo the center.) It is also not clear (and, in fact, not true, as we shall see later) that functoriality respects oddness. Thus, given a representation into $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ whose image lies inside some subgroup $G(\overline{\mathbf{Q}}_p)$, it is important to specify whether one is talking about oddness with respect to G or to GL_n.

In this talk, we will be interested in oddness mainly for the groups GL_n, the generalized symplectic and orthogonal groups, and a twisted generalized unitary group that has no name but is ubiquitous in recent progress towards modularity results for GL_n [12]. However, in these cases, it is possible to talk about these notions quite concretely, without referring specifically to Galois representations; this is how we shall proceed.

2.1. **Some Formalism.** Let G be a group admitting an involution c. Given such a pair, we may form the semi-direct product $\widetilde{G} = G \ltimes \langle c \rangle$. Of course, the main example to keep in mind is as follows:

2.2. **Example.** Suppose that $F^+/\mathbf{Q}$ is a totally real field, and that F/F^+ is a degree two CM extension. Let $G = G_F := \mathrm{Gal}(\overline{\mathbf{Q}}/F)$, and let c be complex conjugation induced by for some embedding of F into $\mathbf{C}$. In this case, $\widetilde{G} = G_{F^+}$.

There are two related (but different) notions of "oddness" for representations of G and $\widetilde{G}$.

Let W be a finite dimensional representation of $\widetilde{G}$. Let $\rho : \widetilde{G} \to \mathrm{GL}_n(L)$ denote the corresponding homomorphism induced by a choice of basis. The trace of $\rho(g)$ for any element $g \in \widetilde{G}$ does not depend on any choice of basis.

2.3. **Definition** (GL-oddness)**.** *Say that W is GL-odd if* $|\mathrm{Trace}(\rho(c))| \le 1$.

Let V be an absolutely irreducible finite dimensional representation of G over L. If we choose a basis of V, we may represent V by a homomorphism $\rho : G \to \mathrm{GL}_n(L)$. Given V, we may form a representation V^c on the same underlying vector space as V by applying the involution c. We obtain a corresponding representation $\rho^c : G \to \mathrm{GL}_n(L)$. Let us suppose that:

(1) The contragradient $V^\vee$ of V is isomorphic to $V^c \otimes \chi$ for a character χ of G.
(2) The character χ extends to a character of $\widetilde{G}$, equivalently, $\chi^c = \chi$.

Having chosen a basis for V, the contragradient representation has a natural basis, and the corresponding representation $\rho^\vee$ is given by the conjugate transpose of ρ, explicitly, $\rho^\vee(g) = (\rho(g)^T)^{-1} = \rho(g^{-1})^T$. Under our assumptions, we deduce that there is an isomorphism

$$\rho^\vee \simeq \rho^c \chi.$$

By Schur's Lemma, there exists a unique invertible matrix A up to scalar such that $\rho^\vee = A\rho^c A^{-1}\chi$. Let us take the inverse transpose of both sides, and replace g by $c(g)$. Using the fact that $\chi^c = \chi$, we find that $\rho^c = (A^T)^{-1}\rho^\vee A^T \chi^{-1}$, and thus

$$\rho^c = (A^T)^{-1}\rho^\vee A^T\chi^{-1} = (A^T)^{-1}(A\rho^c A^{-1}\chi)A^T\chi^{-1} = ((A^T)^{-1}A)\rho^c((A^T)^{-1}A)^{-1}.$$

By Schur's lemma, we deduce that $(A^T)^{-1}A$ is a scalar and hence $A^T = \lambda A$. Taking the transpose of this relation, we deduce that $\lambda = \pm 1$.

2.4. **Definition** (U-oddness). *The U-sign of V is given by λ. We say that V is U-odd if $\lambda = +1$.*

Suppose now that W is a finite dimensional representation of $\widetilde{G}$ such that $W^\vee \simeq W \otimes \chi$ for a character χ of $\widetilde{G}$. Suppose, moreover, that $V = W|_G$ is absolutely irreducible. Since V extends to $\widetilde{G}$, there is a natural isomorphism $V^c \simeq V$. Thus $V^\vee \simeq V^c \otimes \chi$. Hence it makes sense to consider both the GL-sign of W and the U-sign of V.

2.5. **Lemma.** *Suppose that $\chi(c) = -1$. Then W is GL-odd.*

Proof. Since $\mathrm{Trace}(\rho^\vee(c)) = \mathrm{Trace}((\rho(c^{-1})^T) = \mathrm{Trace}(\rho(c)^T) = \mathrm{Trace}(\rho(c))$, we deduce that

$$\mathrm{Trace}(\rho(c)) = \mathrm{Trace}(\rho^\vee(c)) = \chi(c)\mathrm{Trace}(\rho(c)) = 0$$

if $\chi(c) = 0$. □

If $\chi(c) = +1$, then W may or may not be GL-odd.

2.2. **The symplectic/orthogonal alternative.** There is a natural G-equivariant pairing $W \times W^\vee \to L$. Given an isomorphism $W^\vee \simeq W \otimes \chi$ as above, we deduce the existence of a $\widetilde{G}$-equivariant pairing: $W \times (W \otimes \chi) \to L$, which we may write as

$$W \times W \to \chi^{-1}.$$

This map factors through $\mathrm{Sym}^2(W)$ or $\bigwedge^2(W)$, and the pairing consequently defines a (generalized) orthogonal or symplectic pairing. By abuse of notation, we refer to W as either orthogonal or symplectic. Let us explicitly identify the pairing on W. By Schur's lemma, we may write $\rho^\vee = B\rho B^{-1}\chi$ for some matrix B. Taking the inverse transpose of this equation, we deduce that $\rho = (B^T)^{-1}\rho^\vee(B^T)\chi^{-1}$, and hence $(B^T)^{-1}B$ commutes with ρ, and thus $B^T = \mu B$ for some $\mu = \pm 1$. It follows that for all $g \in \widetilde{G}$, we have a relation

$$\rho(g)^T B \rho(g) = B\chi^{-1}(g).$$

Consider the pairing $\langle \cdot, \cdot \rangle$ defined by $\langle x, y \rangle = xBy^T$. We compute that

$$g\langle x, y \rangle = \langle \rho(g)x, \rho(g)y \rangle = (\rho(g)x)B(\rho(g)y)^T = x(\rho(g)^T B\rho(g))y^T = xB\chi^{-1}(g)y^T = \chi^{-1}(g)\langle x, y \rangle.$$

This pairing is symmetric if $\mu = 1$ and $B^T = B$, and alternating if $\mu = -1$ and $B^T = -B$. In particular, if $\mu = 1$, then W is orthogonal, and if $\mu = -1$, then W is symplectic.

2.6. **Lemma.** *If W is orthogonal, then $V = W|_G$ is U-odd if and only if $\chi(c) = +1$. If W is symplectic, then V is U-odd if and only if $\chi(c) = -1$.*

Proof. There exists a matrix B as above such that $(\rho(g)^T)^{-1} = B\rho(g)B^{-1}\chi(g)$ and $B^T = \mu B$. In the notation of the previous section, however, we also know there exists a matrix A such that

$$(\rho(g)^T)^{-1} = A\rho^c(g)(A)^{-1}\chi(g) = (A\rho(c))\rho(g)(A\rho(c))^{-1}\chi(g).$$

Comparing these two identities, we deduce by Schur's lemma that, up to scalar, B is equal to $A\rho(c)$, and hence $A = B\rho(c)$. By definition,

$$\rho(c)^T B\rho(c) = B\chi^{-1}(c).$$

Since c has order two, it follows that $\rho(c)^T B = B\rho(c)\chi(c)$. We deduce that

$$A^T = (B\rho(c))^T = \rho(c)^T B^T = \rho(c)^T(\mu B) = \mu \cdot \rho(c)^T B = \mu \cdot B\rho(c)\chi(c) = \mu \cdot \chi(c)A.$$

Thus $A^T = \lambda A$ where $\lambda = \mu \cdot \chi(c)$. □

2.7. **Remark.** Suppose that $n = \dim(W)$ is odd. Since there are no non-degenerate symplectic forms on odd dimensional vector spaces, W is necessarily orthogonal. On the other hand, by taking the determinant of the equality $W^\vee = W \otimes \chi$, one sees that $\chi(c)^n = 1$, and hence $\chi(c) = +1$. Thus, when n is odd, W is automatically U-odd.

Given the value of $\chi(c)$, the following table summarizes what we may deduce (using Lemmas 2.5 and 2.6) about the GL-oddness and U-oddness of W and $V = W|_G$:

Type	Orthogonal		Symplectic	
$\chi(c)$	-1	1	-1	1
U-odd	No	Yes	Yes	No
GL-odd	Yes	?	Yes	?

Table 1. Relation between U-oddness and GL-oddness

3. Automorphic Representations

Suppose that π is either a RAESDC (regular algebraic essentially self dual cuspidal) automorphic form for $\mathrm{GL}(n)$ over a totally real field F^+, or a RACSDC (regular algebraic conjugate self dual cuspidal) automorphic form π over a CM extension F/F^+. Then, by [13, 17], one may attach (for each prime above p in the field of coefficients) an n-dimensional p-adic Galois representation to ρ. The following theorems were proved by Bellaïche and Chenevier [4] and Taylor [18] respectively.

3.1. **Theorem** (Bellaïche–Chenevier)**.** *Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a p-adic representation associated to a RACSDC form π. Then ρ is U-odd.*

3.2. **Corollary** (Bellaïche–Chenevier)**.** *Let $\rho : G_{F^+} \to \mathrm{GSp}_{2n}(\overline{\mathbf{Q}}_p) \hookrightarrow \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ be a p-adic representation associated to a RAESDC form π of symplectic type. Then ρ is GL-odd.*

3.3. **Theorem** (Taylor)**.** *Let $\rho : G_{F^+} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a p-adic representation associated to a RAESDC form π. Suppose that ρ is absolutely irreducible. Suppose that n is odd. Then ρ is GL-odd.*

3.4. **Conjecture.** *Let $\rho : G_{F^+} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a p-adic representation associated to a RAESDC form π. Then ρ is GL-odd.*

To see why Corollary 3.2 follows from Theorem 3.1, consider a RAESDC form π of symplectic type. After a solvable totally real base change (which does not affect GL-oddness), we may make a base change of π to unitary group associated to a CM extension F/F^+. By Theorem 3.1, we deduce that $\rho|_{G_F}$ is U-odd. We deduce (see Table 1) that ρ is GL-odd.

Note that conjecture 3.4 comes in two flavours, depending on whether the image of ρ is symplectic or orthogonal. In the symplectic case, Conjecture 3.4 follows directly from Corollary 3.2.

3.1. **The main idea of this talk.** When π is orthogonal and n is odd (respectively, even), there is a certain friction between Theorem 3.1 and Theorem 3.3 (respectively, Conjecture 3.4) that may be exploited. Namely, suppose that $\rho : G_{E^+} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is continuous, absolutely irreducible, unramified outside finitely many primes, essentially self dual up to twist, potentially semistable with distinct Hodge–Tate weights, and of orthogonal type. If one wants to prove the potential modularity of ρ, it is not obvious how to proceed without assuming that ρ is GL-odd. Nevertheless, the following idea explains how one may sometimes circumvent this assumption, and hence make *a posteriori* deductions about the sign of ρ. Consider the restriction of ρ to a CM field E. The condition that $\rho|_{G_E}$ is U-odd may be possible to verify even when the GL-oddness of ρ is not — for example, when n is odd, U-oddness is automatic (see Remark 2.7). If one deduces that ρ is potentially modular over a CM field F, one may then use cyclic base change [1] to deduce potentially modularity over a totally real field F^+ and hence that ρ is GL-odd. Although this seems somewhat circular, an important point to note is that there will exist *residual* representations $\overline{\rho}$ which are not GL-odd and thus not associated with a RAESDC form over a totally real field, yet their restriction to a CM field will be U-odd and thus (presumably) be modular in the sense of arising from a RACDSC form. However, the modular representations $\overline{\rho}|_{G_F}$ obtained in this way will *not* have any modular lifts that extend to G_{F^+}, even though $\overline{\rho}|_{G_F}$ itself has such an extension.

3.2. **Potential Modularity and Oddness.** The idea described in the previous section can be applied to prove the following result.

3.5. **Theorem.** *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous irreducible representation unramified outside finitely many primes, and suppose that $\rho|D_p$ is ordinary with distinct Hodge–Tate weights. Suppose that:*

(1) *$p > 2(3+1)$ if $n = 2$, $p > 2(5+1)$ if $n = 4$, and $p > 2(n+1)$ in all other cases.*
(2) *$\overline{\rho}$ is absolutely irreducible, and $\mathbf{Q}(\mathrm{ad}\,\overline{\rho}) \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$.*
(3) *If $n = 2$ then $\overline{\rho}$ is absolutely irreducible over any quadratic extension of $\mathbf{Q}$.*
(4) *If $n = 4$ and ρ is symplectic with similitude character ν, the kernel of the map $\wedge^2\overline{\rho} \to \overline{\nu}$ is irreducible.*
(5) *If $n = 4$ and ρ is orthogonal, then the residual representation $\overline{\rho}$ is absolutely irreducible over any quartic extension of $\mathbf{Q}$.*

Then:

(1) *If n is odd, then ρ is GL-odd.*
(2) *If n is even, and ρ is orthogonal, and Conjecture 3.4 holds, then ρ is GL-odd.*
(3) *If $n \leq 5$, then ρ is GL-odd.*

Sketch. If ρ is orthogonal, then by Table 1, either $\chi(c) = -1$, in which case ρ is GL-odd (Lemma 2.5), or $\chi(c) = +1$, in which case ρ is U-odd (Lemma 2.6). The latter assumption allows us to prove the potential modularity of ρ over a CM field F/F^+ by Theorem C of [3]. By [1], we deduce the potential modularity of ρ over F^+, and hence that ρ is GL-odd, either unconditionally if n is odd (Theorem 3.3), or assuming Conjecture 3.4 in general. Note that ρ is always orthogonal if n is odd. This leaves the cases of $n = 2$ and $n = 4$. If $n = 2$, then either ρ is GL-odd or ρ is totally even. In the latter case, we apply the theorem to $\mathrm{Sym}^2(\rho)$, which we deduce is GL-odd, contradicting the assumption that ρ is totally even (See [10] for more details). Thus, it remains to consider the case $n = 4$.

Either ρ is of symplectic or orthogonal type. If ρ is of symplectic type, then, since the maximal torus of $\mathrm{GSp}_4(\overline{\mathbf{Q}}_p)$ is given by diagonal elements such that $ab = cd$, either ρ is GL-odd or $\rho(c)$ is a scalar. In the latter case, we may consider the associated 5-dimensional representation ψ that is a component of $\wedge^2\rho$. It is easy to see that ψ is ordinary with distinct Hodge–Tate weights. By assumption, $\overline{\psi}$ is absolutely irreducible. Hence ψ is modular by Theorem C of [3]. We deduce that ψ is odd by Theorem 3.3, and hence ρ is also odd.

It remains to consider the case when ρ is orthogonal, $n = 4$, and we do not assume Conjecture 3.4. We deduce as above that ρ is potentially modular. Recall that there is an exact sequence as follows:

$$1 \to \overline{\mathbf{Q}}_p^{\times} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p) \times \mathrm{GL}_2(\overline{\mathbf{Q}}_p) \to \mathrm{GO}_4(\overline{\mathbf{Q}}_p) \to \mathbf{Z}/2\mathbf{Z} \to 1,$$

where the first map sends z to $\left\{\begin{pmatrix} z & \\ & z \end{pmatrix}, \begin{pmatrix} z^{-1} & \\ & z^{-1} \end{pmatrix}\right\}$. Suppose the image of ρ does not surject onto $\mathbf{Z}/2\mathbf{Z}$. I claim that we can lift ρ to a representation of $G_{\mathbf{Q}}$ into $\mathrm{GL}_2(\overline{\mathbf{Q}}_p) \times \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. The obstruction to such a lifting lands in $H^2(\mathbf{Q}, \overline{\mathbf{Q}}_p^{\times})$, which vanishes by a theorem of Tate (see, for example, Theorem 5.4 of [9]; here $\overline{\mathbf{Q}}_p^{\times}$ has the discrete topology). We deduce that the underlying representation of ρ is of the form $r_V \otimes r_W$ for two 2-dimensional representations V and W. Since $V \otimes W$ is ordinary, the same is true of V and W up to a local twist by Lemma 3.6 below. After a global twist, we may assume that V and W are also ordinary. Since $\overline{\rho}$ is irreducible over any quadratic extension, the same is true for $\overline{r}_V$ and $\overline{r}_W$. Hence, we may apply the $n = 2$ case of our theorem to deduce that r_V and r_W are GL-odd, and hence $\rho = r_V \otimes r_W$ is also GL-odd. Suppose that the image of ρ surjects onto $\mathbf{Z}/2\mathbf{Z}$. Then arguing as above, ρ is, up to twist, of the form $r \otimes r^c$ where $r : G_K \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is (up to a twist) an ordinary two dimensional representation (with distinct Hodge–Tate weights) over a quadratic extension $K/\mathbf{Q}$, and $c \in \mathrm{Gal}(K/\mathbf{Q})$ denotes the involution. The representation $\mathrm{ad}^0(r)$ (which does not depend on r up to twist) is thus ordinary with distinct integral Hodge weights for both embeddings of K into $\overline{\mathbf{Q}}_p$. If K is real, we deduce that $\mathrm{ad}^0(r)$ is potentially modular over a totally real field F^+, and thus (as in the proof of Theorem 1.2 of [10]) we deduce that $\mathrm{ad}^0(r)$ is GL-odd, and thus r and ρ are GL-odd. In order to apply this argument, we must assume that $\overline{r}$ is neither irreducible nor dihedral, or equivalently that $\overline{r}$ is not reducible over some quadratic extension L/K. Yet if $\overline{r}$ was reducible over such an L, then $\overline{\rho}|G_L$ would be reducible, and we have assumed otherwise. Suppose that K is imaginary. Since $r \otimes r^c$ has distinct Hodge–Tate weights, we may argue as in the proof of Theorem 1.3 of [10]. In particular, we deduce that $\wedge^2 \rho$ is the induction of an automorphic form for $\mathrm{GL}(3)/F$ for some CM field $F = F^+.K$ of an infinity type whose existence is incompatible with the vanishing theorems of Borel–Wallach (and whose infinity type corresponds to the Galois representation $\mathrm{ad}^0(r)$). $\square$

3.6. **Lemma.** *If V and W are two continuous representations of $G = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ such that $V \otimes W$ is ordinary, then V and W are ordinary up to twist.*

Proof. By definition, a representation V is ordinary if and only if

(1) V^{ss} is a sum of characters
(2) V is semistable

Since (1) and (2) are preserved by taking quotients, a quotient of an ordinary representation is ordinary. We proceed via induction on the dimension of V and W. If $\dim(V)$ or $\dim(W)$ is equal to one, then the result is obvious. Suppose that $V \otimes W$ is ordinary. If $W = W' \oplus W''$ is reducible, then $V \otimes W'$ is a quotient of $V \otimes W$, and thus by induction V is ordinary. We may therefore assume that W is irreducible. Since $(V \otimes W)^{\mathrm{ss}}$ is a sum of characters, after twisting, we may assume that there is a $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-invariant map $V \otimes W \to \mathbf{Q}_p$. By Schur's Lemma and the irreducibility of W, it follows that $V \simeq W^*$. Let G denote the Zariski closure of the image of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ inside $\mathrm{GL}(W)$. Since G admits a faithful semisimple linear representation (by construction), G is reductive. Hence $\mathrm{Hom}(W,W) = V \otimes W = (V \otimes W)^{\mathrm{ss}}$ is a sum of characters as a representation of G. Since $\mathrm{Hom}_G(W, W \otimes \chi)$ has dimension at most one (by Schur's Lemma), any character χ can occur to multiplicity at most one in $\mathrm{Hom}(W,W)$. In particular, W admits isomorphisms $W \simeq W \otimes \chi$ for n^2 distinct characters χ. Any such character must satisfy $\chi^n = 1$, as can be seen by taking determinants. Since the semi-simplification of a semistable representation is also semistable, it follows that the characters χ are also semistable. Yet the only semistable characters of finite order are unramified, and there are exactly n unramified characters of order dividing n. Thus we are done, since $n < n^2$ unless $n = 1$. $\square$

3.7. **Remark.** In light of the recent preprint [3], it is possible to extend this theorem to crystalline representations sufficiently deep inside the Fontaine–Laffaille range (that is, there will be some conditions on the Hodge–Tate weights relative to p). The arguments are all the same, except that one has to deduce that V and W are crystalline from the corresponding result for $V \otimes W$. This is a theorem of Di Matteo [14].

3.8. **Question.** *Is it possible to rule out the existence of continuous absolutely irreducible Galois representations* $\rho : G_{\mathbf{Q}} \to \mathrm{GSp}_6(\overline{\mathbf{Q}}_p)$ *with distinct Hodge–Tate weights such that* $\rho(c)$ *is not GL-odd?*

The methods we have described seemingly have little to say about this question — it was a happy accident that for $n = 2$ and $n = 4$ functorial maps existed to GL_n (for $n = 3$ and 5 respectively) which preserved the property of having distinct Hodge–Tate weights. Note that if ρ is symplectic and *not GL*-odd, then ρ restricted to any CM field will not be U-odd either, as one sees from Table 1. It is exactly this fact (which was useful in deducing Corollary 3.2 from Theorem 3.1) that makes Question 3.8 hard.

Let G_2 be the anisotropic group corresponding to $\mathbf{Q}$-automorphisms of the octonions.

3.9. **Example.** *Let* $\rho : G_{\mathbf{Q}} \to G_2(\overline{\mathbf{Q}}_p) \hookrightarrow \mathrm{GL}_7(\overline{\mathbf{Q}}_p)$ *be a continuous irreducible representation unramified outside finitely many primes, and suppose that* $\rho|D_p$ *is ordinary with distinct Hodge–Tate weights under the standard representation. Suppose that* $\overline{\rho}$ *has absolutely irreducible image in* $\mathrm{GL}_7(\mathbf{F})$*, and that* $p > 13$*. Then* ρ *is GL-odd and* G_2*-odd.*

Proof. By Theorem 3.5, we deduce that the compositum of ρ with the standard representation is GL-odd. Since G_2 admits exactly two real forms, the group $G_2(\overline{\mathbf{Q}}_p)$ has a unique non-scalar involution, and thus ρ is automatically G_2-odd. □

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