SEMISTABLE MODULARITY LIFTING OVER IMAGINARY QUADRATIC FIELDS

FRANK CALEGARI

1. Introduction

In this paper, we prove non-minimal modularity lifting theorem for ordinary Galois representations over imaginary quadratic fields. Our first theorem is as follows:

Theorem 1.1. *Let $F/\mathbf{Q}$ be an imaginary quadratic field, and let $p > 2$ be a prime which is unramified in F. Let E/F be a semistable elliptic curve with ordinary reduction at all $v|p$. Suppose that the mod p Galois representation:*

$$\overline{\rho}_{E,p} : G_F \to \mathrm{Aut}(E[p]) = \mathrm{GL}_2(\mathbf{F}_p)$$

is absolutely irreducible over $G_{F(\zeta_p)}$ and is modular. Assume that the Galois representations attached to ordinary cohomology classes for Bianchi groups are ordinary — see Conjecture A. Then E is modular.

The modularity hypotheses is satisfied, for example, when $\overline{\rho}_{E,p}$ extends to an odd representation $\overline{\rho}$ of $G_{\mathbf{Q}}$. In particular, if 3 or 5 is unramified in F, this theorem implies — conditionally on Conjecture A — the modularity of infinitely many j invariants in $F \setminus \mathbf{Q}$, because one can take any E/F such that $E[p] \simeq A[p]$ where A/F is the base change of an elliptic curve over $\mathbf{Q}$. If $p = 3$, then the representation associated to $E[3]$ has solvable image ($\mathrm{PGL}_2(\mathbf{F}_3) \simeq S_4$). However, unlike in the case of totally real fields, the automorphic form π associated to the corresponding Artin representation does not in any obvious way admit "congruences" to modular forms of cohomological weight, and hence the modularity hypothesis cannot be deduced from Langlands–Tunnell [Lan80, Tun81] (as in the deduction of Theorem 0.3 of [Wil95] from Theorem 0.2). We deduce Theorem 1.1 from the following:

Theorem 1.2. *Assume conjecture A. Suppose that $p > 2$ is unramified in F, and let*

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

be a continuous irreducible Galois representation unramified outside finitely many primes. Assume that:

1. *The determinant of ρ is ϵ^{k-1}, where ϵ is the cyclotomic character.*
2. *If $v|p$, then $\rho|D_v$ is ordinary and crystalline with Hodge–Tate weights $[0, k-1]$ for some $k \geq 2$.*
3. *$\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible. If $p = 5$, then the projective image of $\overline{\rho}$ is not $\mathrm{PGL}_2(\mathbf{F}_5)$.*
4. *$\overline{\rho}$ is modular.*
5. *If ρ is ramified at $v \nmid p$, then $\rho|D_v$ is semistable, that is, $\rho|I_v$ is unipotent.*

Then ρ is modular.

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The main idea of this paper is to combine the modularity lifting theorems of [CG] with the techniques on level raising developed in [CV]. Wiles' original argument for proving modularity in the non-minimal case required two ingredients: the use of a subtle numerical criterion concerning complete intersections which were finite flat over a ring of integers $\mathcal{O}$, and Ihara's Lemma. Although Ihara's Lemma (in some form) is available for imaginary quadratic fields (see [CV], Ch IV), it seems tricky to generalize the numerical criterion to this setting — the Hecke rings are invariably not torsion free, and are rarely complete intersections even in the minimal case (the arguments of [CG] naturally present the minimal deformation ring as a quotient of a power series in $q-1$ variables by q elements). Instead, the idea is to work in a context in which the "minimal" deformations are all Steinberg at a collection of auxiliary primes S. It turns out that a natural setting where one expects this to be true is in the cohomology of the S-arithmetic group $\mathrm{PGL}_2(\mathcal{O}_F[1/S])$. In order to apply the methods of [CG], one requires two main auxiliary hypotheses to hold. The first is that the range of cohomology which doesn't vanish after localizing at a suitable maximal ideal $\mathfrak{m}$ has length $\ell_0 = 1$. When the number of primes m dividing S is zero, this is an easy lemma, and was already noted in [CV] (Lemma 5.9). When $m = 1$, the required vanishing follows from the congruence subgroup property of $\mathrm{PGL}_2(\mathcal{O}_F[1/S])$ as proved by Serre [Ser70]. When $m > 1$, however, the problem is more subtle. The cohomology in this range may well be non-trivial and is related to classes arising from the algebraic K-theory of $\mathcal{O}_F$ (as explained in [CV]). Nevertheless, if one first completes at a non-Eisenstein maximal ideal $\mathfrak{m}$, the necessary vanishing required for applications to modularity is expected to hold, and indeed was conjectured in [CV]. We do not, however, prove this vanishing conjecture in this paper. Instead, we prove that the patched cohomology in these lower degrees is sufficiently small (as a module over the patched diamond operator ring $S_\infty = \mathcal{O}[\![x_1, \ldots, x_q]\!]$) that a modified version of the argument of [CG] still applies.

There are three further technical obstacles which must be dealt with. We now discuss them in turn.

The methods of [CG] require that the Galois representations (constructed in much greater generality than used here by [Sch15]) satisfy the expected local properties at $v|p$ and $v \nmid p$. The required local–global compatibility for $v \nmid p$ was established by [Var]. The required local–global compatibility for $v|p$ in the ordinary case is still open. We do not resolve this issue here, but instead make the weakest possible assumption necessary for applications — namely that cohomology classes on which the operator U_v is invertible give rise to Galois representations which admit an unramified quotient on which Frobenius at v acts by U_v. We believe that this formulation (Conjecture A) might be amenable to current technology.

A second issue that we must deal with is relating the modularity assumption on $\overline{\rho}$ for $\mathrm{PGL}_2(\mathcal{O}_F)$ to the required modularity for the group $\mathrm{PGL}_2(\mathcal{O}_F[1/S])$. This is a form of level raising, and to prove it we use the level raising spectral sequence of [CV]. This part of the paper is not conditional on any conjectures, and may be viewed as a generalization of Ribet's level raising theorem in this context. Many of the ideas here are already present in [CV].

The final issue which must be addressed is that Scholze's Galois representations are only defined over the ring $\mathbf{T}/I$ for some nilpotent ideal I with a fixed level of nilpotence (depending on the group). Moreover, some of the constructions here also require increasing the degree of nilpotence. Thus we are also required to explain how the methods of [CG] may be adapted to this context. This last point requires only a technical modification. The essential point is that if a finitely generated $S_\infty = \mathcal{O}[\![x_1, \ldots, x_q]\!]$-module M is annhilated by I^2, then M/I has the same co-dimension over S_∞ as M.

Remark 1.3. Our theorem and its proof may be generalized to allow other ramification types at auxiliary primes $v \nmid p$, *providing* that this new ramification is of minimal type, e.g $\rho(I_v) \simeq \overline{\rho}(I_v)$. This can presumably be achieved using the modification found by Diamond [Dia97] and also developed in [CDT99]. The required change would be to modify the corresponding local system at such primes. We avoid this in order to clarify exactly the innovative aspects of this paper.

Suppose that $\overline{\rho}$ satisfies the conditions of Theorem 1.2. The assumption that $\overline{\rho}$ is modular is defined to mean that the localization $H_1(Y, \mathcal{L})_{\mathfrak{m}} \neq 0$ for a certain arithmetic quotient Y and a local system $\mathcal{L}$ corresponding to $\overline{\rho}$ and maximal ideal $\mathfrak{m}$ of the corresponding anaemic Hecke algebra. (This is a weaker property than requiring $\overline{\rho}$ to be the mod-p reduction of a representation associated to an automorphic form of minimal level.) This is equivalent to asking that $H_1(Y, \mathcal{L}/\varpi)_{\mathfrak{m}}$ is non-zero and also to asking that $H_2(Y, \mathcal{L}/\varpi)_{\mathfrak{m}}$ is non-zero. (If H_2 vanishes, then H_1 is torsion free, which implies that there exists a corresponding automorphic form, which then must contribute to H_2.)

1.1. **Notation.** We fix an imaginary quadratic field $F/\mathbf{Q}$, and an odd prime p which is unramified in F. Let $\mathcal{O}$ denote the ring of integers in a finite extension of $\mathbf{Z}_p$. We shall assume that $\mathcal{O}$ is sufficiently large that it admits inclusions $\mathcal{O}_{F,v} \to \mathcal{O}$ for each $v|p$, and that the residue field $k = \mathcal{O}/\varpi$ contains sufficiently many eigenvalues of any relevant representation $\overline{\rho}$. Let N denote a tame level prime to p. Let S denote a finite set of primes disjoint from N and p. Let m denote the number of primes in S. By abuse of notation, we sometimes use S to denote the ideal of $\mathcal{O}_F$ which is the product of the primes in S.

Let $\mathbb{G} = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{PGL}(2)/F)$, and write $G_\infty = \mathbb{G}(\mathbf{R}) = \mathrm{PGL}_2(\mathbf{C})$. Let K_∞ denote a maximal compact of G_∞ with connected component K^0_∞, so $G_\infty/K^0_\infty = \mathcal{H}$ is hyperbolic 3-space. Let $\mathbb{A}$ be the adele ring of $\mathbf{Q}$, and $\mathbb{A}^{(\infty)}$ the finite adeles. For any compact open subgroup K of $\mathbb{G}(\mathbb{A}^{(\infty)})$, we may define an "arithmetic manifold" (or rather "arithmetic orbifold") $Y(K)$ as follows:

$$Y(K) := \mathbb{G}(F)\backslash(\mathcal{H} \times \mathbb{G}(\mathbb{A}^{(\infty)}))/K = \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K^0_\infty K.$$

The orbifold $Y(K)$ is not compact but has finite volume; it may be disconnected.

Let $K_0(v)$ denote the Iwahori subgroup of $\mathrm{PGL}_2(\mathcal{O}_{F,v})$, and let $K_1(v)$ denote the pro-v Iwahori, which is the kernel of the map $K_0(v) \to k_v^\times$.

Definition 1.4. *Let R be an ideal of $\mathcal{O}_F$. If we choose K to consist of the level structure $K_0(v)$ for $v|R$ and maximal level structure elsewhere, then we write $Y_0(R)$ for $Y(K)$. If K has level $K_0(v)$ for $v|R$ and $K_1(v)$ for $v|Q$ for some auxiliary Q, we write $Y_1(Q;R)$ for $Y(K)$.*

Given S, we may similarly define S-arithmetic locally symmetric spaces (directly following §3.6 and §4.4 of [CV]) as follows. Let $\mathscr{B}_S$ be the product of the Bruhat–Tits buildings of $\mathrm{PGL}_2(F_v)$ for $v \in S$; we regard each building as a contractible simplicial complex, and so $\mathscr{B}_S$ is a contractible square complex. In particular, $\mathscr{B}_S$ has a natural filtration:

$$\mathscr{B}^0_S \subset \mathscr{B}^1_S \subset \mathscr{B}^2_S \subset \ldots$$

where $\mathscr{B}^{(j)}_S$ comprises the union of cells of dimension $\leq j$. Consider the quotient

$$Y(K[\frac{1}{S}]) := \mathbb{G}(F)\backslash\left(G_\infty/K_\infty \times \mathscr{B}_S \times \mathbb{G}(\mathbb{A}^{\infty,S})/K^S\right).$$

This has a natural filtration by spaces Y^j_S defined by replacing $\mathscr{B}_S$ with $\mathscr{B}^j_S$. The space $Y^j_S - Y^{j-1}_S$ is a smooth manifold of dimension $\dim(Y_{\{\infty\}}) + j$. When K has type $K_0(v)$ for $v|R$, we write $Y_0(R)[1/S]$ for these spaces, and, with additional level $K_1(v)$ for $v|Q$ and Q prime to R and S, we write $Y_1(Q;R)[1/S]$. The cohomology of $Y[1/S]$ and its covers will naturally recover spaces of automorphic forms which are Steinberg at primes dividing S. In order to deal with representations which correspond to a quadratic unramified twist of the Steinberg representation, we need to introduce a local system as follows.

Let $\epsilon : S \rightarrow \{\pm 1\}$ be a choice of sign for every place $v \in S$. Associated to ϵ there is a natural character $\chi_\epsilon : \mathbb{G}(F) \rightarrow \{\pm 1\}$, namely $\prod_{v \in S: \epsilon(v) = -1} \chi_v$; here χ_v is the "parity of the valuation of determinant," obtained via the natural maps

$$\mathbb{G}(F) \xrightarrow{\det} F^\times/(F^\times)^2 \rightarrow \prod_v F_v^\times/(F_v^\times)^2 \xrightarrow{v} \pm 1,$$

where the final map is the parity of the valuation. Correspondingly, we obtain a *sheaf of $\mathcal{O}$-modules*, denoted $\mathcal{F}_\epsilon$, on the space $Y[1/S]$. Namely, the total space of the local system $\mathcal{F}_\epsilon$ corresponds to the quotient of

$$\left(G_\infty/K_\infty \times \mathscr{B}_S \times \mathbb{G}(\mathbb{A}^{\infty,S}))/K^S\right)$$

by the action of $\mathbb{G}(F)$: the natural action on the first factor, and the action via χ_ϵ on the second factor. Finally, let $\mathcal{F}$ be the direct sum of $\mathcal{F}_\epsilon$ over all $2^m = 2^{|S|}$ choices of sign ϵ.

1.2. **Local Systems.** For a pair (m,n) of integers at least two, one has the representation

$$\mathrm{Sym}^{m-2}\mathbf{C}^2 \otimes \overline{\mathrm{Sym}^{n-2}\mathbf{C}^2}$$

of $\mathrm{GL}_2(\mathbf{C})$. These representations give rise to local systems of $Y[1/S]$ (and its covers) defined over $\mathcal{O}_F[1/S]$, and hence also to $\mathcal{O}$. Similarly, for any S and any ϵ as above, there are corresponding local systems $\mathcal{L}$ obtained by tensoring this local system with $\mathcal{F}$.

Remark 1.5 (Amalgams)**.** The structure of the groups $\mathrm{PGL}_2(\mathcal{O}_F[1/S])$ and its congruence subgroups for $S = T \cup \{v\}$ as amalgam of $\mathrm{PGL}_2(\mathcal{O}_F[1/T])$ with itself over the Iwahori subgroup of level v implies, by the long exact sequence associated to an amalgam, that there is an exact sequence:

$$\ldots \rightarrow H_n(Y_1(Q;R)[1/T], \mathcal{L}/\varpi^r)^2 \rightarrow H_n(Y_1(Q_N;R)[1/S], \mathcal{L}/\varpi^r) \rightarrow H_{n-1}(Y_1(Q_N;Rv)[1/T], \mathcal{L}/\varpi^r) \rightarrow \ldots$$

This simple relationship between S arithmetic groups is special to the case $n = 2$, and is crucial for our inductive arguments.

Remark 1.6 (Orbifold Cohomology)**.** Whenever we write $H_*(Y, \mathcal{L})$ for an orbifold Y, we mean the cohomology as orbifold cohomology rather than the cohomology of the underlying space.

1.3. **Hecke Operators.** We may define Hecke operators T_v for primes v not dividing S acting on $H_*(Y_1(Q;R)[1/S], \mathcal{L})$ in the usual way. For primes $v|S$, one also has the operators U_v. The action of U_v on the cohomology of $\mathcal{F}_\epsilon$ is by $U_v = \epsilon(v) \in \{\pm 1\}$. More generally, on $H_*(Y_1(Q;R)[1/S], \mathcal{L})$, we have (cf. the proof of Lemma 9.5 of [CG]):

$$U_v^2 = U_{v^2} = 1.$$

For primes $v|RQ$, there is also a Hecke operator we denote by U_v. We denote by $\mathbf{T}_Q$ be the $\mathcal{O}$-algebra of endomorphisms generated by the action of these Hecke operators on the direct sum of cohomology groups $H_*(Y_1(Q;R)[1/S],\mathcal{L}/\varpi^r)$ for any given m, and let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$.

2. Galois Representations

Suppose that $\mathcal{L}$ has parallel weight (k,k) for some integer $k \geq 2$. Our main assumption on the existence of Galois representations is as follows:

Conjecture A (Ordinary $\Rightarrow$ Ordinary)**.** *Assume that $\mathfrak{m}$ is non-Eisenstein of residue characteristic $p>2$ and is associated to a Galois representation $\overline{\rho}$, and assume that $T_v \notin \mathfrak{m}$ for $v|p$. Then there exists a continuous Galois representation $\rho = \rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(\mathbf{T}_{Q,\mathfrak{m}})$ with the following properties:*

(1) *If $\lambda \notin R \cup Q \cup \{v|p\} \cup S$ is a prime of F, then ρ is unramified at λ, and the characteristic polynomial of $\rho(\mathrm{Frob}_\lambda)$ is*
$$Y^2 - T_\lambda X + N_{F/\mathbf{Q}}(\lambda)^{k-1} \in \mathbf{T}_{Q,\mathfrak{m}}[X].$$
(2) *For $v|p$, the representation $\rho|D_v$ is ordinary with eigenvalue the unit root of $X^2 - T_v X + N(v)^{k-1}$.*
(3) *If $v \in R$, then $\rho|I_v$ is unipotent.*
(4) *If $v \in S$, then $\rho|I_v$ is unipotent, and moreover the characteristic polynomial of (any) lift of Frobenius is*
$$X^2 - U_v(N(v)^{k-1} + N(v)^k) + N(v)^{2k-1}.$$
(5) *If $v \in Q$, the operators T_α for $\alpha \in F_v^\times \subset \mathbb{A}_F^{\infty,\times}$ are invertible. Let ϕ denote the character of $D_v = \mathrm{Gal}(\overline{F}_v/F_v)$ which, by class field theory, is associated to the resulting homomorphism:*
$$F_v^\times \to \mathbf{T}_{Q,\mathfrak{m}}^\times$$
given by sending x to T_x. By assumption, the image of $\phi \mod \mathfrak{m}$ is unramified, and so factors through $F_v^\times/\mathcal{O}_v^\times \simeq \mathbf{Z}$, and so $\phi(\mathrm{Frob}_v) \mod \mathfrak{m}$ is well defined; assume that $\phi(\mathrm{Frob}_v) \not\equiv \pm 1 \mod \mathfrak{m}$. Then $\rho|D_v \sim \phi\epsilon \oplus \phi^{-1}$.
(6) *Suppose that $k=2$, and that the level is prime to $v|p$. Then $\rho|D_v$ is finite flat.*

Remark 2.1. If one drops the assumption that $T_v \notin \mathfrak{m}$ for $v|p$ and still assumes the corresponding version of assumption 6, one can also expect to prove a modularity lifting theorem in weight $k=2$ without an ordinary hypothesis. However, it seems plausible that one might be able to prove the weaker form of Conjecture A without assuming the finite flatness condition. If we drop this assumption, our arguments apply verbatim in all situations except when $k=2$ and $\overline{\rho}|_{D_v}$ for some $v|p$ has the very special form that it is finite flat but also admits non-crystalline semistable lifts. One may even be able to handle this case as well by a trick using Hida families (see Remark 5.1) but we do not attempt to fill in the details.

2.1. **Assumptions.** Let k be a finite field of characteristic p. We shall assume, from now one, that the representation:
$$\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_2(k)$$
satisfies all the hypotheses of Theorem 1.2. In particular, it has determinant ϵ^{k-1}, the restriction $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible, and there exist suitable collections of Taylor–Wiles primes.

2.2. **Patched Modules.** Using the methods of [CG], we may patch together for any T and R (and any non-Eisenstein $\mathfrak{m}$) the homology groups $H_*(Y_1(Q_N;R)[1/T],\mathcal{L}/\varpi^N)$ to obtain a complex P_∞ such that:

(1) P_∞ is a perfect complex of finite S_∞-modules supported in degrees $m+2$ to 1, where $S_\infty = W(k)[\![x_1,\ldots,x_q]\!]$ is the patched module of diamond operators, where $q-1$ is the dimension of the minimal adjoint Selmer group $H^1_\emptyset(F,\mathrm{ad}^0(\overline{\rho}))$, and q is the dimension of the minimal dual Selmer group $H^1_{\emptyset^*}(F,\mathrm{ad}^0(\overline{\rho})(1))$.

(2) Let $\mathfrak{a} = (x_1,\ldots,x_q)$ be the augmentation ideal of S_∞, and let $\mathfrak{a}_N = ((1+x_1)^{p^n} - 1,\ldots,(1+x_q)^{p^n}-1)$ be the ideal with $S_\infty/\mathfrak{a}_N = \mathbf{Z}_p[(\mathbf{Z}/p^n\mathbf{Z})^q]$. Then
$$H_*(P_\infty\otimes S_\infty/(\mathfrak{a}_N,\varpi^N)) = H_*(Y_H(Q_N;R)[1/T],\mathcal{L}/\varpi^N)_{\mathfrak{m}}$$
for infinitely many sets of suitable Taylor-Wiles primes Q_N which are $1 \mod p^n$, and Y_H is the quotient of Y_1 which is a cover Y_0 with Galois group $\Delta = (\mathbf{Z}/p^n\mathbf{Z})^q$. Moreover,
$$H_*(P_\infty) = \operatorname{proj\,lim} H_*(Y_H(Q_N;R)[1/T],\mathcal{L}/\varpi^N)_{\mathfrak{m}}.$$
We denote these patched homology groups by $\widetilde{H}_*(Y_0(R)[1/T],\mathcal{L})$.

Note that we can do this construction with the addition of some auxiliary level structure, and also simultaneously for any finite set of different auxiliary level structures.

3. The Galois action in low degrees

Let $\mathbf{t}_{Q,\mathfrak{m}}$ be the quotient of $\mathbf{T}_{Q,\mathfrak{m}}$ which acts faithfully in degrees $\le m$, namely on
$$\bigoplus_k \bigoplus_{i\le m} H_i(Y_1(Q_N;R)[1/S],\mathcal{L}/\varpi^k)_{\mathfrak{m}}.$$

Proposition 3.1. *There exists an integer k depending only on $m=|S|$ such that there exists a representation*
$$\rho^{\mathbf{t}}: G_F \to \mathrm{GL}_2(\mathbf{t}_{Q,\mathfrak{m}}/I)$$
where $I^k = 0$ and such that $\rho^{\mathbf{t}}$ is Steinberg or unramified quadratic twist of Steinberg at primes dividing S.

Proof. We proceed by induction. Suppose that $S = vT$, where T has $m-1$ prime divisors. From the amalgam sequence of Remark 1.5, we find that there is an exact sequence:
$$H_n(Y_1(Q_N;R)[1/T],\mathcal{L}/\varpi^r)^2_{\mathfrak{m}} \to H_n(Y_1(Q_N;R)[1/S],\mathcal{L}/\varpi^r)_{\mathfrak{m}} \to H_{n-1}(Y_1(Q_N;Rv)[1/T],\mathcal{L}/\varpi^r)_{\mathfrak{m}}.$$
We have $U_v^2 - 1 = 0$ for $v|S$ on $H_*(Y_1(Q_N;R)[1/S],\mathcal{L}/\varpi^r)$. It follows that, for the Galois representation associated to the image of the LHS, the eigenvalues of Frob_v are precisely $N(v)^{k-1}$ and $N(v)^k$, or $-N(v)^{k-1}$ and $-N(v)^k$, depending only on $\overline{\rho}$ (note that $p\ne 2$, so the eigenvalue of $U_v \in \{\pm 1\}$ is determined by $\overline{\rho}$). Moreover, by induction, the Galois representation associated to the RHS is Steinberg at v. Hence, again after possibly increasing the ideal of nilpotence, it follows that the middle term also gives rise to a Steinberg representation. $\square$

The key part of the argument is to show that the action of Galois in low degrees is unramified "up to a small error."

Following [CG], we may, by finding suitably many sequences of Taylor–Wiles primes, patch all these homology groups (localized at $\mathfrak{m}$) for all time. (We need only work with a finite fixed set of auxiliary level structures.) The corresponding patched modules will be, assuming local–global compatibility conjectures, modules over a framed local deformation ring R_{loc}, which will be a power series over the tensor product of local framed deformation rings R_v for $v|RS$. We choose the local deformation ring R_v for $v|p$ to be the ordinary crystalline deformation ring. This coincides with the ordinary deformation ring unless $k = 2$ and the semi-simplification of $\overline{\rho}|D_v$ is a twist of $\epsilon \oplus 1$. In the former case, the ordinary deformation ring is irreducible. In the latter case, the additional finite flat condition also means that R_v is irreducible. The local deformation rings R_v for $v|S$ have two components corresponding to the unramified and Steinberg representations respectively, and two corresponding equi-dimensional quotients R^{st}_v and R^{ur}_v. Their intersection $R^{\mathrm{st,ur}}_v$ is also equi-dimensional with $\dim(R^{\mathrm{st,ur}}_v) = \dim(R^{\mathrm{st}}_v) - 1 = \dim(R^{\mathrm{ur}}_v) - 1$. The ring R_{loc} correpsondingly has 2^m quotients on which one chooses a component of R_v for $v|S$. The common quotient $R^{\mathrm{st,ur}}_{\mathrm{loc}}$ has dimension $\dim(R_{\mathrm{loc}}) - m$.

The patched modules $\widetilde{H}_i$ are also naturally modules over a patched ring of diamond operators $S_\infty = \mathcal{O}[\![x_1, \ldots, x_q]\!]$. In the context of [CG], we have $\ell_0 = 1$, or that $\dim(S_\infty) = \dim(R_{\mathrm{loc}}) - 1$. We have an exact sequence as follows:

$$\ldots \widetilde{H}_i(Y_1(Q_N; Rv)[1/T]) \to \widetilde{H}_i(Y_1(Q_N; R)[1/T])^2 \to \widetilde{H}_i(Y_1(Q_N; R)[1/S]) \to \widetilde{H}_{i-1}(Y_1(Q_N; Rv)[1/T]) \ldots .$$

For a finitely generated S_∞-module M, let the co-dimension of M denote the co-dimension of the support of M as an S_∞-module.

Proposition 3.2. *Let S be divisible by m primes. We have the following estimate:*

$$\mathrm{codim}_{S_\infty} \widetilde{H}_i(Y_0(R)[1/S]) \geq \begin{cases} m - i + 3 & i \leq m \\ 1, & i = m + 1. \end{cases}$$

Proof. The claim for $i = m + 1$ follows by considering dimensions of deformation rings, because these modules are finite over R_{loc}. For For $i \leq m$, we proceed via induction on m. Write $S = vT$, where T has $m - 1$ prime factors. There is an exact sequence:

$$\widetilde{H}_i(Y_0(R)[1/T])^2 \to \widetilde{H}_i(Y_0(R)[1/S]) \to \widetilde{H}_{i-1}(Y_0(vR)[1/T]).$$

Assuming that $i \leq m$, we have $i - 1 \leq m - 1$. In the Serre category of S_∞-modules modulo those of co-dimension at least $(m-1)-(i-1)+3 = m-i+3$, we therefore have a surjection:

$$\widetilde{H}_i(Y_0(R)[1/T])^2 \to \widetilde{H}_i(Y_0(R)[1/S]).$$

This implies that the Galois representation associated to the latter module is, (in this category) unramfied at v; and, using other v, for all $v|S$. It suffices to show that the RHS is zero, or equivalently, that it does not have co-dimension at most $m - i + 2$. We would like to claim that, by Proposition 3.1, the action of R_{loc} in these degrees factors through the quotient $R^{\mathrm{st}}_{\mathrm{loc}}$. This is not precisely true, since Proposition 3.1 only says the Galois representation is Steinberg after taking the quotient by a nilpotent ideal. If M is an S_∞-module, then the support of M/J for a nilpotent ideal J will be the same as the support of M (see also the discussion in §6). Hence, passing to a suitable quotient of $\widetilde{H}_i$, we may assume the module acquires an action of R_{loc} which factors through $R^{\mathrm{st}}_{\mathrm{loc}}$. Yet by what we have just shown above, the corresponding Galois representations are also unramified at $v|S$, and so are quotients of $R^{\mathrm{st,ur}}_{\mathrm{loc}}$. Since $\dim(R^{\mathrm{st,ur}}_{\mathrm{loc}}) = \dim(R_{\mathrm{loc}}) - m = \dim(S_\infty) - m - 1$, we deduce that $\widetilde{H}_i$ has co-dimension at least

$$m + 1 > m - i + 2$$

providing that $i \geq 2$. If $i = 0$, the module is trivial, because $\mathfrak{m}$ is not Eisenstein and H_0 is Eisenstein. If $i = 1$, we are done by the congruence subgroup property, which also implies that H_i vanishes after localization at $\mathfrak{m}$. $\square$

4. Level Raising

4.1. **Ihara's Lemma and the level raising spectral sequence.** We recall some required constructions and results from [CV]. The following comes from Chapter IV of [CV]. Let $S = T \cup \{v\}$. Let $\mathcal{L}$ be a local system (which could be torsion). We assume that $\mathcal{L}/\varpi$ is self-dual. For example, we could take $\mathcal{L} = \mathcal{O}/\varpi^k$ for some k.

Let $Y = Y(K)$ for some K of level prime to S. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$.

Lemma 4.1 (Ihara's Lemma)**.** *If $\mathfrak{m}$ is not Eisenstein, then*

$$H_1(Y_0(v)[1/T], \mathcal{L})_{\mathfrak{m}} \to H_1(Y[1/T], \mathcal{L})^2_{\mathfrak{m}}$$

is surjective.

Proof. It suffices to show that $H_1(Y[1/S], \mathcal{L})_{\mathfrak{m}}$ for $S = Tv$ is trivial. From the amalgam sequence 1.5, we see the cokernel is a quotient of the group $H_1(Y[1/S], \mathcal{L})_{\mathfrak{m}}$, and hence it suffices to show that this is trivial. The homology of $Y[1/S]$ can be written as the direct sum of the homologies of S-arithmetic groups commensurable with $\mathrm{GL}_2(\mathcal{O}_F[1/S])$, and, by [Ser70], these groups satisfy the congruence subgroup property (this crucially uses the fact that S is divisible by at least one prime v, and that the lattice is non-cocompact). The congruence kernel has order dividing the group of roots of unity μ_F. Since $p > 2$ is unramfied in F, this is trivial after tensoring with $\mathbf{Z}_p$. An easy computation then shows that the relevant cohomology group is Eisenstein. (See [CV], § 4.) $\square$

In order to prove the required level raising result (Theorem 4.2), we also need the level raising spectral sequence of [CV] (Theorem 4.4.1). If $\mathfrak{m}$ is non-Eisenstein, then the E^1-page of the spectral sequence is:

$$\begin{tikzcd}
0 & \ldots \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \\
H_2(Y, \mathcal{L})^{2^{|S|}}_{\mathfrak{m}} & \ldots \arrow[l] & \displaystyle\bigoplus_{v|S} H_2(Y_0(S/v), \mathcal{L})^2_{\mathfrak{m}} \arrow[l] & H_2(Y_0(S), \mathcal{L})_{\mathfrak{m}} \arrow[l] \\
H_1(Y, \mathcal{L})^{2^{|S|}}_{\mathfrak{m}} & \ldots \arrow[l] & \displaystyle\bigoplus_{v|S} H_1(Y_0(S/v), \mathcal{L})^2_{\mathfrak{m}} \arrow[l] & H_1(Y_0(S), \mathcal{L})_{\mathfrak{m}} \arrow[l] \\
0 & \ldots \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd} \tag{1}$$

The vanishing of the zeroeth and third row follow from the assumption that $\mathfrak{m}$ is not Eisenstein. This spectral sequence converges to $H_*(Y[1/S], \mathcal{L})_{\mathfrak{m}}$. Tautologically, it degenerates on the E^2-page. After tensoring with $\mathbf{Q}$, the sequences above are exact at all but the final term, corresponding to the fact that $H_*(Y[1/S], \mathcal{L})_{\mathfrak{m}} \otimes \mathbf{Q}$ vanishes outside degrees $[m+1, m+2]$.

We now establish a level–raising result.

Theorem 4.2. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}$ with residue field k of characteristic p. Let S be a product of m primes v so that $T_v^2 - (1 + N(v))^2 \in \mathfrak{m}$. Then*

$$H_*(Y, \mathcal{L}/\varpi)_{\mathfrak{m}} \neq 0 \Rightarrow H_*(Y[1/S], \mathcal{L}/\varpi)_{\mathfrak{m}} \neq 0.$$

Proof. Consider the spectral sequence of [CV] in equation 1 above. It is clear that the upper right hand corner term remains unchanged after one reaches the E^2-page. Assuming, for the sake of contradition, that $H_{m+2}(Y[1/S], \mathcal{L}/\varpi)_\mathfrak{m}$ vanishes, it follows that the map

$$H_2(Y_0(S), \mathcal{L}/\varpi)^2_\mathfrak{m} \to \bigoplus_{v|S} H_2(Y_0(S/v), \mathcal{L}/\varpi)_\mathfrak{m}$$

is injective. By Poincaré duality, there is an isomorphism $H_2(Y, \mathcal{L}/\varpi)_\mathfrak{m} \simeq H^1_c(Y, \mathcal{L}/\varpi)_\mathfrak{m}$. Here we use the fact that $\mathcal{L}/\varpi$ is a self-dual local system. Because $\mathfrak{m}$ is non-Eisenstein, there is an isomorphism between $H^1_c(Y, \mathcal{L}/\varpi)_\mathfrak{m}$ and $H^1(Y, \mathcal{L}/\varpi)_\mathfrak{m}$. Finally, by the universal coefficient theorem, $H^1(Y, \mathcal{L}/\varpi)_\mathfrak{m}$ is dual to $H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}$. Hence taking the dual of the injection above yields the surjection:

$$\bigoplus_{v|S} H_1(Y_0(S/v), \mathcal{L}/\varpi)^2_\mathfrak{m} \to H_1(Y_0(S), \mathcal{L}/\varpi)_\mathfrak{m}.$$

It suffices to show that this results in a contradiction. By Ihara's lemma, it follows that the composite map

$$\bigoplus_{v|S} H_1(Y_0(S/v), \mathcal{L}/\varpi)^2_\mathfrak{m} \to H_1(Y_0(S), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y, \mathcal{L}/\varpi)^{2^m}_\mathfrak{m}$$

is also surjective. Our assumption is that, for some choice of signs, the elements $D_v = T_v \pm (1 + N(v)) \in \mathfrak{m}$ for all $v|S$. The map above decomposes into a sum of maps from each individual term, each of which factor as follows

$$H_1(Y_0(S/v), \mathcal{L}/\varpi)^2_\mathfrak{m} \to H_1(Y_0(S), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y_0(S/v), \mathcal{L}/\varpi)^2_\mathfrak{m} \to H_1(Y, \mathcal{L}/\varpi)^{2^m}_\mathfrak{m}$$

An alternative description of this map can be given by replacing every pair of groups by a single term, and replacing the two natural degeneracy maps with either the sum or difference of these maps (depending on a sequence of choice of Fricke involutions, which depend on the sign occurring in D_v) we end up with a map of the form:

$$H_1(Y_0(S/v), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y_0(S), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y_0(S/v), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}.$$

On the other hand, the composite of the first two maps is the map obtained by pushing forward and then pulling back, which (after either adding or subtracting the relevant maps) is exactly the Hecke operator D_v. It follows that the composite of the entire map is then killed if one passes to the quotient $H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}/D_v H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}$. In particular, it follows that the composite

$$\bigoplus_{v|S} H_1(Y_0(S/v), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y_0(S), \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m} \to H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}/I H_1(Y, \mathcal{L}/\varpi)_\mathfrak{m}$$

is zero, where I is the ideal generated by D_v for all $v|S$. This contradicts the surjectivity unless I generates the unit ideal. But this in turn contradicts the assumption that $D_v \in \mathfrak{m}$ for all $\mathfrak{m}$. □

5. The argument

Let ρ be as in Theorem 1.2. By assumption, we have $H_2(Y_0(S), \mathcal{L})_\mathfrak{m} \neq 0$, by the assumption that $\overline{\rho}$ is modular. Hence $H_{m+2}(Y[1/S], \mathcal{L}/\varpi)_\mathfrak{m}$ is modular by Theorem 4.2. As in 2.2, we obtain a complex P_∞ such that:

(1) P_∞ is a perfect complex of finite S_∞-modules supported in degrees $m+2$ to 1.

(2) $H_*(P_\infty \otimes S_\infty/(\mathfrak{a}_N, \varpi^N)) = H_*(Y_1(Q_N)[1/S], \mathcal{L}/\varpi^N)_{\mathfrak{m}}$ for infinitely many sets of suitable Taylor-Wiles primes Q_N, and moreover,

$$H_*(P_\infty) =: \widetilde{H}_*(Y[1/S], \mathcal{L}) = \operatorname{proj}\lim H_*(Y_1(Q_N)[1/S], \mathcal{L}/\varpi^N)_{\mathfrak{m}}.$$

Suppose that the corresponding quotients had actions of Galois representations mapping to the entire Hecke rings $\mathbf{T}$ rather than $\mathbf{T}/I$ for some nilpotent ideals I of fixed order. Then this action would extend to an action of $R^{\mathrm{st}}_{\mathrm{loc}}$ on $H_*(P_\infty)$, where here $R^{\mathrm{st}}_{\mathrm{loc}}$ is defined to have Steinberg conditions at all primes in S, an ordinary condition at $v|p$, and unramified elsewhere. In the special case when $k = 2$ and one of the representations $\overline{\rho}|_{D_v}$ for $v|p$ is twist equivalent to a representation of the form

$$\begin{pmatrix} \epsilon & * \\ 0 & 1 \end{pmatrix}$$

where $*$ is peu ramifiée, we take the local deformation ring at $v|p$ to be the finite flat deformation ring instead. The corresponding ring $R^{\mathrm{st}}_{\mathrm{loc}}$ is reduced of dimension $\dim(S_\infty) - 1$ and has one geometric component. We would be done as long as the co-dimension of $H_*(P_\infty)$ is equal to at most one, because then the action of R_{loc} must be faithful, and we deduce our modularity theorem. As in Lemma 6.2 of [CG], if $H_*(P_\infty)$ has co-dimension at least 2, then it must be the case that, for some $j \ge 2$,

$$\operatorname{codim}_{S_\infty} \widetilde{H}_{m+2-j}(Y[1/S]) \le j,$$

or, with $i = m + 2 - j$ and $i \le m$,

$$\operatorname{codim}_{S_\infty} \widetilde{H}_i(Y[1/S]) \le m + 2 - i.$$

Yet this exactly contradicts Proposition 3.2 (with $R = 1$), and we are done.

Remark 5.1. If one wants to weaken Conjecture A by omitting part 6, then one can instead work over the full Hida family, where the corresponding ordinary deformation ring once more has a single component (in every case). The modularity method in the Hida family case works in essentially the same manner, see [KT], so one expects that the arguments of this paper can be modified to handle this case as well.

5.1. **Nilpotence.** In practice, we only have an Galois representation to $\mathbf{T}/I$ for certain nilpotent ideals I. Equivalently, we only have a Galois representation associated to the action of $\mathbf{T}$ on

$$H_*(Y_1(Q_N)[1/S], \mathcal{L}/\varpi^N)/I$$

for ideals I with some fixed nilpotence. Even if there is no such ideal I when $S = 1$, our inductive arguments for higher S use exact sequences which increases the nilpotence.

It suffices to show that R_{loc} maps to the action of $\mathbf{T}$ on certain sub-quotients of $\widetilde{H}_i(Y[1/S])$ which are "just as large" as the modules $\widetilde{H}_i(Y[1/S])$ themselves. Roughy, the idea is that one can also patch the ideals I to obtain an action of $\mathbf{T}$ and S_∞ on $\widetilde{H}_i(Y[1/S])/I$ for some ideal I of S_∞ with $I^k = 0$ and k depending only on S and $\overline{\rho}$. The Galois deformation rings now give lower bounds for the co-dimension of the modules $\widetilde{H}_i(Y[1/S])/I$. Since $I^k = 0$, these can be promoted to give the same lower bounds for the co-dimension of the modules $\widetilde{H}_i(Y[1/S])$, and then the argument above will go through unchanged. This is (essentially) what we now do.

6. Notes on nilpotent Ideals

6.1. **Passing to finite level.** Let $S = \mathcal{O}[\![\Delta_\infty]\!]$. If I and J are ideals of S, then $\mathrm{Tor}^i(S/I, S/J)$ is an S/I and a S/J-module, hence an $S/(I+J)$-module. So, if $\mathrm{Tor}^0(S/I, S/J) = S/(I+J)$ is finite, then so is $\mathrm{Tor}^i(S/I, S/J)$. Hence, by induction, if M is finitely generated and $\mathrm{Tor}^0(S/I, M)$ is finite, then so is $\mathrm{Tor}^i(S/I, M)$. Moreover, there is a spectral sequence:

$$\mathrm{Tor}^j(S_\infty/\mathfrak{a}, H^i(P_\infty)) \Rightarrow H^{i+j}(P_\infty \otimes_{S_\infty} S_\infty/\mathfrak{a})$$

6.2. **The setup.** Let $\Delta_\infty = \mathbf{Z}_p^q$ and $\Delta_N = (\mathbf{Z}/p^N\mathbf{Z})^q$. Let $S_\infty = \mathcal{O}[\![\Delta_\infty]\!]$, and let $S_N = \mathcal{O}[\Delta_N]$. We begin with the assumption that we have arranged things so that the complexes patch on the level of S_∞-modules. That is, we have a complex P_∞ of finite free S_∞-modules so that, if

$$P_N = P_\infty \otimes_{S_\infty} S_N/\varpi^N,$$

then $H^*(P_N)$ is the complex of cohomology associated to (infinitely many) Taylor–Wiles sets Q_N with coefficients in $\mathcal{O}/\varpi^N$. There is a natural identification

$$H^*(P_\infty) = \operatorname{proj\,lim} H^*(P_N),$$

and a natural map

$$H^*(P_\infty) \otimes_{S_\infty} S_N/\varpi^N \to H^*(P_N).$$

Because everything is finitely generated, and so in particular $H^*(P_\infty) \otimes_{S_\infty} S_N/\varpi^N$ is finite, there exists some function $f(N)$ (which we may take to be increasing and $\geq N$) such that

$$H^*(P_\infty) \otimes_{S_\infty} S_N/\varpi^N = H^*(P_{f(N)}) \otimes S_N/\varpi^N.$$

Having fixed such a function $f(N)$, we define A_N to be $H^*(P_{f(N)}) \otimes S_N/\varpi^N$. By construction, there is a natural *surjective* map

$$A_N \to A_N \otimes S_M/\varpi^M \to A_M$$

for all $N \geq M$, and $\operatorname{proj\,lim} A_N = H^*(P_N)$. For various choices of $Q = Q_N$ giving rise toA_N (really the primes in Q_N are $1 \mod p^{f(N)}$), we get different actions of different Hecke algebras $\mathbf{T}$. We shall construct quotients B_N of A_N on which R_∞ acts on the corresponding quotients of $\mathbf{T}$ which act faithfully on B_N, and then patch to get a quotient B_∞ of $A_\infty = H^*(P_\infty)$ on which R_∞ also acts. The main point is to ensure that B_∞ has the same codimension as A_∞.

6.3. **Hecke Algebras.** For each $Q = Q_N$, let $\Delta = \Delta_N$. Letting Φ run over all the quotients of Δ, and letting k run over all integers at most N, we shall define $\mathbf{T}$ to be the ring of endomorphisms generated by Hecke operators on

$$\bigoplus_{\Phi,k} H^*(Y_1(\Phi), \mathcal{L}/\varpi^k).$$

Localize at a non-Eisenstein ideal $\mathfrak{m}$. On each particular module A in the direct sum above there is a quotient $\mathbf{T}_A$ on which there exists a Galois representation with image in $\mathrm{GL}_2(\mathbf{T}_A/I_A)$ where $I_A^m = 0$ for some universally fixed m. Note that:

(1) One initially knows that m is bounded universally for any fixed piece A. However, there is no problem taking direct sums. The point is as follows; given rings A and B with ideals I_A and I_B such that $I_A^m = I_B^m = 0$, the ideal $(I_A \oplus I_B)$ of $A \oplus B$ satisfies $(I_A \oplus$

$I_B)^m = 0$. In particular, if $\mathbf{T}_\Phi$ is the quotient for a particular Φ and the corresponding ideal is I_Φ, there is a map:

$$\mathbf{T} \hookrightarrow \left(\bigoplus \mathbf{T}_\Phi\right) / \bigoplus I_\Phi,$$

and hence the image is $\mathbf{T}/I$ where $I^m \subset (\bigoplus I_\Phi)^m = 0$.

(2) If there exists a pseudo-representation to $\mathbf{T}/I$ and $\mathbf{T}/J$ there exists one to $\mathbf{T}/I \oplus \mathbf{T}/J$, and the image will be $\mathbf{T}/(I \cap J)$. Hence there exists a minimal such ideal I.

(3) If $N \geq M$, there is a surjective map from $\mathbf{T}_{Q_N} \to \mathbf{T}_{Q_M}$, where the sets Q_N and Q_M are compatible (that is, come from the same set of primes). The reason this is surjective is that we are including all the quotients of Δ in the definition of $\mathbf{T}$. Again, by patching, the map $\mathbf{T}_{Q_N} \to \mathbf{T}_{Q_M}/I_M$ has a Galois representation satisfying local–global, so it factors through a surjection $\mathbf{T}_{Q,N}/I_N \to \mathbf{T}_{Q,M}/I_M$.

In particular, for A_N and A_M drawn from the same set Q, there is a commutative diagram

$$\begin{tikzcd} A_N \arrow[r] \arrow[d] & A_N/I_N =: B_N \arrow[d] \\ A_M \arrow[r] & A_M/I_M =: B_M \end{tikzcd}$$

The point of this construction is that B_N and B_M have actions of the Galois deformation rings R_Q, and hence have actions of R_∞. Moreover, these actions are compatible in the expected way with the action of S_∞ as diamond operators and local ramification operators respectively.

Lemma 6.1. *Suppose that I is an ideal of local ring $(\mathbf{T}, \mathfrak{m})$ such that $I^m = 0$, let $S \to \mathbf{T}$ be a ring homomorphism, let M be a finitely generated $\mathbf{T}$ and S-module with commuting actions compatible with the map from S to $\mathbf{T}$, and let $J = \mathrm{Ann}_S(M/IM)$. Then $J^m M = 0$.*

Proof. The module M has a filtration as $\mathbf{T}$ and S-modules with graded pieces $I^k M/I^{k+1}M$ for $k = 0$ to $m-1$. Hence it suffices to show that each of these graded pieces is annihilated by J. However, there is a surjective homomorphism of $\mathbf{T}$ and S modules given by

$$\bigoplus_{I^k} M/IM \to I^k M/I^{k+1} M,$$

where the sum goes over all generators g of I^k and sends M to $gM \subset I^k M$. Since J annihilates the source, it annihilates the target. $\square$

For each N, we now consider the extra data of a quotient B_N of A_N which carries an action of R_∞. We patch to obtain a pair

$$H^*(P_\infty) = A_\infty \to B_\infty,$$

where B_∞ has an action of R_∞ and S_∞, and there is a natural map $S_\infty \to R_\infty$ which commutes with this action. Let $J = \mathrm{Ann}_{S_\infty}(B_\infty)$. We claim that J^m acts trivially on $H^*(P_\infty)$. To check this, it suffices to check this on A_N for each N. By construction, A_N is a surjective system and hence so is B_N. Thus B_∞ surjects onto B_N, and hence J annihilates B_N, and thus J^m annihilates A_N by Lemma 6.1. Moreover, this same argument works term by term in each degree.

Lemma 6.2. $\operatorname{codim}(B_\infty) = \operatorname{codim}(H^*(P_\infty))$ *(in each degree) as an S_∞-module.*

Proof. Let $I = \operatorname{Ann}_{S_\infty}(H^*(P_\infty))$. Because it is finitely generated, the co-dimension of $H^*(P_\infty)$ is the co-dimension of S_∞/I. Equally, the co-dimension of B_∞ is the co-dimension of S_∞/J (again using finite generation). Hence it suffices to show that

$$J^m \subset I \subset J \Rightarrow \operatorname{codim}(S_\infty/I) = \operatorname{codim}(S_\infty/J).$$

One inequality is obvious. However, the former module has a finite filtration by J^k/J^{k+1}, which is finitely generated and annihilated by k. □

Note that this argument also applies to a submodule $A'_\infty \subset A_\infty$.

Remark 6.3. One way to view the lemma above is follows. The co-dimension of a finitely generated module is defined in terms of the dimension of the support. The dimension of a closed subscheme of S_∞, on the other hand, only depends on its reduced structure.

Let us now remark how to modify the argument of section 5. All bounds on the co-dimensions of $H^*(P_\infty)$ still apply by combining the bounds on the appropriate deformation rings with Lemma 6.3. Hence we deduce that $\widetilde{H}_{m+1}(Y[1/S],\mathcal{L})_\mathfrak{m}$ has co-dimension one and thus (because R_{loc} is reduced has only one geometric component) is nearly faithful as an R_{loc} module. From this we want to deduce that $H_{m+1}(Y[1/S],\mathcal{L})_\mathfrak{m}$ is also nearly faithful as an R module. The module $H_{m+1}(Y[1/S],\mathcal{L})_\mathfrak{m}$ differs from $\widetilde{H}_{m+1}(Y[1/S],\mathcal{L})_\mathfrak{m}/\mathfrak{a}$ by other terms arising from the spectral sequence in 6.1. However, all those terms must be finite — if not, then there must be a smallest degreee j such that $\widetilde{H}_j(Y[1/S],\mathcal{L})_\mathfrak{m}/\mathfrak{a}$ is infinite, which from the spectral sequence will contribute something non-zero to $\widetilde{H}_j(Y[1/S],\mathcal{L})_\mathfrak{m}\otimes\mathbf{Q}$, an impossibility for $j \le m$. Hence we obtain an isomorphism

$$R^{\mathrm{red}} = \mathbf{T}^{\mathrm{red}},$$

as required.

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