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Potential automorphy over CM fields

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Abstract

Let F be a CM number field. We prove modularity lifting theorems for regular n-dimensional Galois representations over F without any self-duality condition. We deduce that all elliptic curves E over F are potentially modular, and furthermore satisfy the Sato–Tate conjecture. As an application of a different sort, we also prove the Ramanujan Conjecture for weight zero cuspidal automorphic representations for $\mathrm{GL}_2(\mathbf{A}_F)$.

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1. **Introduction**

In this paper, we prove the first unconditional modularity lifting theorems for n-dimensional regular Galois representations without any self-duality conditions. A version of these results were proved in [CG18] *conditional* on two conjectures. The first conjecture was that the Galois representations constructed by Scholze in [Sch15] satisfy a strong form of local-global compatibility at all primes. The second was a vanishing conjecture for the mod-p cohomology of arithmetic groups localized at non-Eisenstein primes that mirrored the corresponding (known) vanishing theorems for cohomology corresponding to tempered automorphic representations in characteristic zero. We prove many cases of the first of these conjectures in this paper. Our arguments crucially exploit work of Caraiani and Scholze [CS19] on the cohomology of non-compact Shimura varieties. (See also [CS17] for the compact version of these results.) The details of this argument are carried out in Sections 4 and 5. (It turns out that, in the easier case when $l \neq p$, one can argue more directly using the original construction in [Sch15], and this is done in Section 3.) On the other hand, we do *not* resolve the second conjecture concerning the vanishing of mod-p cohomology in this paper. Rather, we sidestep this difficulty by a new technical innovation: a derived version of "Ihara avoidance" that simultaneously generalizes the main idea of [Tay08] as well as a localization in characteristic zero idea first used in [KT17]. This argument, together with the proofs of the main automorphy lifting theorems, is given in Section 6. The result is that we are able to prove quite general modularity lifting theorems in both the

ordinary and Fontaine–Laffaille case for general n-dimensional representations over CM fields, in particular Theorems 6.1.1 and 6.1.2. Instead of reproducing those theorems here (which require a certain amount of notation), we instead reproduce here a few corollaries of our main theorems that are worked out in Section 7. The first theorem is a special case of Corollaries 7.1.13 and 7.1.14:

THEOREM 1.0.1. *Let E be an elliptic curve over a CM number field F. Then E and all the symmetric powers of E are potentially modular. Consequently, the Sato–Tate conjecture holds for E.*

For an application of a different sort, we also have the following special case of the Ramanujan conjecture (see Corollary 7.1.15):

THEOREM 1.0.2. *Let F be a CM field, and let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight 0. Then, for all primes v of F, the representation π_v is tempered.*

This is, to our knowledge, the first case of the Ramanujan conjecture to be proved for which neither the underlying Galois representation V nor some closely related Galois representation (such as $V^{\otimes 2}$ or $\mathrm{Symm}^2 V$) is known to occur as a summand of the étale cohomology of some smooth proper algebraic variety over a number field; in such cases temperedness (at unramified primes) is ultimately a consequence of Deligne's purity theorem. Our proof, in contrast, follows more closely the original strategy proposed by Langlands. Langlands explained [Lan70] how one could deduce Ramanujan from functoriality; namely, functoriality implies the automorphy of $\mathrm{Symm}^n(\pi)$ and $\mathrm{Symm}^n(\pi^\vee)$ as well as the product $\mathrm{Symm}^n(\pi) \boxtimes \mathrm{Symm}^n(\pi^\vee)$. Then, by considering standard analytic properties of the standard L-function associated to $\mathrm{Symm}^n(\pi) \boxtimes \mathrm{Symm}^n(\pi^\vee)$ (and exploiting a positivity property of the coefficients of this L-function) one deduces the required bounds. As an approximation to this, we show that all the symmetric powers of π (and $\pi^\vee$) are *potentially* automorphic, and we then invoke analytic properties of the Rankin–Selberg L-function (in the guise of the Jacquet–Shalika bounds [JS81b]) as a replacement for the (potential) automorphy of their product.

1.1. *A brief overview of the argument.* Let F/F^+ be an imaginary CM field, let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a compact open subgroup, let X_K denote the corresponding (non-Hermitian) locally symmetric space, let $E/\mathbf{Q}_p$ denote a finite extension with ring of integers $\mathcal{O}$, and let $\mathcal{V} = \mathcal{V}_\lambda$ denote a local system on X_K that is a lattice inside an algebraic representation of weight λ defined over E. (For example, $\mathcal{V}$ could be the trivial local system $\mathcal{O}$.) After omitting a finite set of primes S containing the p-adic places (and satisfying some further hypotheses), one may define a Hecke algebra $\mathbf{T} = \mathbf{T}^S$ as the image of a formal ring of Hecke operators in $\mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}))$, where $\mathbf{D}(\mathcal{O})$ is the

derived category of $\mathcal{O}$-modules. (This is isomorphic to the usual ring of Hecke operators acting on $H^*(X_K, \mathcal{V})$ up to a nilpotent ideal, but for technical reasons it is better to work in the derived setting; cf. [NT16].) For a non-Eisenstein maximal ideal $\mathfrak{m}$, the main result of [Sch15] guarantees the existence of a Galois representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}_{\mathfrak{m}}/J)$$

characterized, up to conjugation, by the characteristic polynomials of Frobenius elements at places $v \notin S$, where J is a nilpotent ideal whose exponent depends only on n and $[F : \mathbf{Q}]$. It is crucial for applications to modularity lifting theorems (following the strategy outlined in [CG18]) to know that this Galois representation satisfies local-global compatibility at *all* primes. (As usual, in order to talk about local-global compatibility at a prime in S, one has to work with variants of $\mathbf{T}$ including Hecke operators at these primes — we ignore all such distinctions here). Since $\mathbf{T}_{\mathfrak{m}}/J$ is (in general) not flat over $\mathcal{O}$, it is not exactly clear what one should expect to mean by local-global compatibility. For example, for primes $l \neq p$, a (torsion) representation that is Steinberg at l need not be ramified at l. Instead, we ask that the characteristic polynomials of $\rho_{\mathfrak{m}}(\sigma)$ for $\sigma \in I_v$ for $v|l \in S$ and $l \neq p$ have the expected shape. Such a condition is amenable to arguments using congruences, and we prove a version of this compatibility in Section 3 (see Theorem 3.1.1). Note that our theorem only applies to a limited range of l; in particular, we assume that the level K_v (for $v|l \in S$ and $l \neq p$) satisfies the inclusions $\mathrm{Iw}_{v,1} \subset K_v \subset \mathrm{Iw}_v$ (where Iw_v and $\mathrm{Iw}_{v,1}$ are the Iwahori and pro-l Iwahori respectively) and additionally l satisfies various splitting conditions relative to the field F. This suffices for applications to modularity, where we make a soluble base change to ensure that Theorem 3.1.1 applies to both Taylor–Wiles primes and the ramified primes S away from p. This part of the argument requires only the construction of Galois representations in [Sch15].

Local–global compatibility for $l = p$ is more subtle. Indeed, we are not confident enough to formulate a precise conjecture of what local-global compatibility means in general in the torsion setting. Instead, we restrict to two settings where the conjectural formulation of local-global compatibility is more transparent: the case when $\rho_{\mathfrak{m}}$ should be Fontaine–Laffaille (assuming, in particular, that p is unramified in F) and the ordinary case (with no restriction on F); Sections 4 and 5 are devoted to proving such theorems. In both of these cases, the underlying strategy is as follows. Associated to our data is a quasi-split unitary group $\widetilde{G}$ over F^+ that is a form of GL_{2n} that splits over F/F^+. There is a parabolic subgroup P of $\widetilde{G}$ whose Levi subgroup G over F^+ may be identified with $\mathrm{Res}_{F/F^+} \mathrm{GL}_n$, and hence associated with the locally symmetric spaces X_K as above. The point of this construction is that $\widetilde{G}$ may be associated to a Shimura variety $\widetilde{X}_{\widetilde{K}}$ (and thus to Galois representations of known provenance)

whereas the cohomology of X_K appears inside (in some non-trivial way) a spectral sequence computing the cohomology of the boundary $\partial\widetilde{X}_{\widetilde{K}}$ of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$. One now faces several complications. The first is that the cohomology of the boundary involves different parabolic subgroups of $\widetilde{G}$ besides P. This is resolved by the assumption that $\mathfrak{m}$ is non-Eisenstein. The second is separating inside the boundary cohomology (associated to P) the contribution coming from G and that coming from the unipotent subgroup U of P. Fortunately, the unipotent subgroup U is abelian and well understood, and we show (for $p > n^2$) that the relevant cohomology we are interested in occurs as a direct summand of the cohomology of $\partial\widetilde{X}_{\widetilde{K}}$ (see Theorem 4.2.1). Note that for a general coefficient system $\mathcal{V} = \mathcal{V}_\lambda$ on X_K, there are a number of different coefficient systems $\mathcal{V}_{\widetilde{\lambda}}$ on $\widetilde{X}_{\widetilde{K}}$ for which $H^*(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$ can be related to $H^*(X_K, \mathcal{V}_\lambda)$, and this freedom of choice will be important in what follows. By these arguments, we may exhibit $R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}$ up to shift as a direct summand of $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$. (Here $\widetilde{\mathfrak{m}}$ is the corresponding ideal of the Hecke algebra $\widetilde{\mathbf{T}}$ for $\widetilde{G}$, and $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is the corresponding (reducible) $2n$-dimensional representation associated to $\mathfrak{m}$, from which $\overline{\rho}_{\mathfrak{m}}$ was constructed.) Now suppose that d is the complex (middle) dimension of $\widetilde{X}_{\widetilde{K}}$. We now make crucial use of the following theorem, which is the main theorem of [CS19]. (See Theorem 4.3.3 for a more general statement.)

THEOREM 1.1.1 (Caraiani–Scholze [CS19, Th. 1.1]). *Assume that $F^+ \neq \mathbf{Q}$, that $\mathfrak{m}$ is non-Eisenstein, and that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic in the sense of Definition 4.3.1. Assume that for every prime l that is the residue characteristic of a prime dividing S or Δ_F, there exists an imaginary quadratic field $F_0 \subset F$ in which l splits. Then*

$$H^i(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi)_{\widetilde{\mathfrak{m}}} = 0 \text{ if } i < d, \text{ and } H^i_c(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi)_{\widetilde{\mathfrak{m}}} = 0 \text{ if } i > d.$$

This immediately gives a diagram as follows:

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}[1/p])_{\widetilde{\mathfrak{m}}} \hookleftarrow H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}},$$

where the leftmost term can be understood in terms of automorphic forms on Shimura varieties and, in particular, (under appropriate assumptions) gives rise to Galois representations having the desired p-adic Hodge theoretic properties, and the rightmost term (by construction) now sees the part of $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$ that (after shifting) contributes *in degree* d, at least up to a fixed level of nilpotence.

The idea is then to choose the weight $\widetilde{\lambda}$ so that $\mathcal{V}_{\widetilde{\lambda}}$ on $\widetilde{X}_{\widetilde{K}}$ is related to $\mathcal{V}_\lambda$ on X_K by the action of a Weyl group as in Kostant's formula [Kos61, Th. 5.14] (to do this integrally, we need to assume that p is sufficiently large), and that by varying $\widetilde{\lambda}$ we may see *all* of the cohomology of $R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}$ in the degree d

cohomology of $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$. This idea only works for some weights and degrees, so to get around this, we first deepen the levels K and $\widetilde{K}$ at some other place above p that allows us to modify the weight λ at the corresponding embeddings without changing the Hecke algebra. For the modified λ, we can then find $\widetilde{\lambda}$ and a Weyl group element giving us to access to $H^q(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}$ for $q \geq \lfloor \frac{d}{2} \rfloor$ (see Proposition 4.4.1), and we handle the remaining degrees by taking duals. This part of the argument (including the invocation of Theorem 1.1.1) requires various local assumptions on F that can always be achieved after a soluble base change but are not generally satisfied. (In particular, they are not satisfied when $F^+ = \mathbf{Q}$.) We then extract the relevant properties of $\rho_{\mathfrak{m}}$ from those of the determinant associated to $\widetilde{\mathfrak{m}}$. This summarizes the argument of Section 4.

In Section 5, we prove a different local-global compatibility theorem in the ordinary case. Although not strictly necessary for our main theorems (for compatible families, by taking sufficiently large primes, one can always reduce to the Fontaine–Laffaille case), this allows us to prove a modularity lifting theorem that may have wider applicability — in particular, the main local-global compatibility result of this section (Theorem 5.5.1) applies to any prime p, provided F contains an imaginary quadratic field in which p splits. The general approach in this section is similar to that of Section 4. However, instead of exhibiting $R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}$ up to shift as a direct summand (as a Hecke module) of $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$ (whose proof in Section 4 required $p > n^2$), we make arguments on the level of completed cohomology and exploit a version of Emerton's ordinary parts functor. A key computation is that of the ordinary part of a parabolic induction from P to $\widetilde{G}$ in Section 5.3 following arguments of Hauseux [Hau16]. Because only part of the cohomology of the unipotent radical U is ordinary, only relative Weyl group elements appear in the degree shifts (see Theorem 5.4.3) and consequently we only obtain shifts by multiples of $[F^+ : \mathbf{Q}]$ in this way. We get around this by a trick using the center of G, showing that the Hecke algebra acting on $H^*(X_K, \mathcal{V}_\lambda)$ can be understood in terms of the Hecke algebra acting only in degrees that are multiples of $[F^+ : \mathbf{Q}]$ (Lemma 5.4.16). As in the Fontaine–Laffaille case, we can then extract the relevant properties of $\rho_{\mathfrak{m}}$ from those of the determinant associated to $\widetilde{\mathfrak{m}}$.

We now turn to the modularity lifting theorems of Section 6. A key hypothesis of [CG18] was the truth of a vanishing conjecture for integral cohomology localized at a non-Eisenstein maximal ideal $\mathfrak{m}$ outside a prescribed range (mirrored by the characteristic zero vanishing theorems of Borel and Wallach [BW00]). This conjecture remains unresolved. Instead, we exploit a localization in characteristic zero idea first employed in [KT17]. This requires a slightly stronger residual modularity hypothesis — namely, that $\overline{\rho}_{\mathfrak{m}}$ actually comes from an automorphic representation rather than one merely associated to a torsion class — but this will be satisfied for our applications, and it is at any

rate required at other points at the argument (for example to know that the residual modularity hypothesis is preserved under soluble base change). Two points remain. The first, which is mostly technical, is to show that the approach of [CG18] and [KT17] is compatible with the fact that we only have Galois representations to $\mathbf{T}/J$ for some nilpotent ideal J. The second, which is more serious, is to show that the localization argument of [KT17] is compatible with the "Ihara avoidance argument" of [Tay08] and the (essentially identical) $l_0 > 0$ version of this argument in [CG18]. (Here l_0 is the parameter of [BW00] that measures the failure of the underlying real group to admit discrete series and that plays plays a fundamental role in [CG18].) To explain the problem, we briefly recall the main idea of [Tay08] in the $l_0 = 0$ setting. (The difficulties are already apparent in this case.) One compares two global deformation problems that (for exposition) differ only at an auxiliary prime v with $l = N(v) \equiv 1 \mod p$, and that at all other primes have smooth local deformation conditions. The corresponding local deformation rings $R_v^{(1)}$ and $R_v^{(2)}$ at the prime v are taken to be tame local deformation rings that the image of tame inertia has minimal polynomial $(X-1)^n$ or $(X-\zeta_1)\dots(X-\zeta_n)$ respectively for distinct roots of unity $\zeta_i \equiv 1 \mod v$. The corresponding patched modules $H_\infty^{(1)}$ and $H_\infty^{(2)}$ constructed via the Taylor–Wiles method ([TW95], [Kis09]) have the expected depth over S_∞. On the one hand, the generic fibre of $R_v^{(2)}$ is geometrically irreducible, which forces $H_\infty^{(2)}$ to have full support over $R_v^{(2)}$. On the other hand, there is an isomorphism $R_v^{(1)}/\varpi = R_v^{(2)}/\varpi$, and this gives an identification $H_\infty^{(1)}/\varpi \simeq H_\infty^{(2)}/\varpi$. But now, the ring $R_v^{(1)}$ has the convenient property that any irreducible component of its special fibre comes from a unique irreducible component of the generic fibre, and from this a modularity result is deduced in [Tay08] using commutative algebra. Suppose we now drop the hypothesis that the integral cohomology is concentrated in a single degree (still in our $l_0 = 0$ setting), but we continue to assume this holds after inverting p. Now we can no longer control the depth of the S_∞-modules $H_\infty^{(1)}$ and $H_\infty^{(2)}$, and so knowing $H_\infty^{(2)}[1/p] \neq 0$ and $H_\infty^{(1)}/\varpi = H_\infty^{(2)}/\varpi$ does not imply that $H_\infty^{(1)}[1/p] \neq 0$. For example, it could happen that $H_\infty^{(1)} = H_\infty^{(1)}/\varpi = H_\infty^{(2)}/\varpi$. The resolution of this difficulty is not to simply compare the patched modules in fixed (final) degree, but the entire patched complex in the derived category. The point is now that these complexes in characteristic p (which are derived reductions of perfect S_∞-complexes for the ring of diamond operators S_∞) remember information about characteristic zero. As a simple avatar of this idea, if M is a finitely generated $\mathbf{Z}_p$-module, then $M[1/p]$ is non-zero if and only if $M \otimes^{\mathbf{L}} \mathbf{F}_p$ has non-zero Euler characteristic over $\mathbf{F}_p$. The main technical formulation of this principle that allows us to prove a version of Ihara avoidance in our setting is Lemma 6.3.4.

Finally, in Section 7, we apply the results of previous sections to prove Theorems 1.0.1 and 1.0.2. We begin with some preliminaries on compatible systems in order to show there are enough primes such that the corresponding residual representations satisfy hypotheses of our modularity lifting theorems. As expected, the arguments of this section make use of the p-q switch ([Wil95], but first exploited in the particular context of potential automorphy in [Tay02]) and a theorem of Moret-Bailly [MB89].

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1.2. *Notation.* We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. We will write char_A for the characteristic polynomial of a matrix A. We write GL_n for the usual general linear group (viewed as a reductive group scheme over $\mathbf{Z}$) and $T_n \subset B_n \subset \mathrm{GL}_n$ for its subgroups of diagonal and of upper triangular matrices, respectively. We will write $O(n)$ (resp. $U(n)$) for the group of matrices $g \in \mathrm{GL}_n(\mathbf{R})$ (resp. $\mathrm{GL}_n(\mathbf{C})$) such that ${}^tg^cg = 1_n$.

If R is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of R.

If Δ is an abelian group, we will let Δ^{tor} denote its maximal torsion subgroup and Δ^{tf} its maximal torsion free quotient. If Δ is profinite and abelian, we will also write $\Delta(l)$ for its Sylow pro-l-subgroup, which is naturally isomorphic to its maximal pro-l continuous quotient. If Γ is a profinite group, then Γ^{ab} will denote its maximal abelian quotient by a closed subgroup. If $\rho : \Gamma \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$ is a continuous homomorphism, then we will let $\overline{\rho} : \Gamma \to \mathrm{GL}_n(\overline{\mathbf{F}}_l)$ denote the *semi-simplification* of its reduction, which is well defined up to conjugacy (by the Brauer–Nesbitt theorem). If M is a topological abelian group with a continuous action of Γ, then by $H^i(\Gamma, M)$ we shall mean the continuous cohomology.

If R is a (possibly non-commutative) ring, then we will write $\mathbf{D}(R)$ for the derived category of R-modules. By definition, an object of $\mathbf{D}(R)$ is a cochain complex of R-modules. An object of $\mathbf{D}(R)$ is said to be perfect if it is isomorphic in this category to a bounded complex of projective R-modules.

If R is a complete Noetherian local ring, $C \in \mathbf{D}(R)$ is a perfect complex, and $T \to \mathrm{End}_{\mathbf{D}(R)}(C)$ is a homomorphism of R-algebras, then the image $\overline{T}$ of T in $\mathrm{End}_{\mathbf{D}(R)}(C)$ is a finite R-algebra, which can therefore be written as a product $\overline{T} = \prod_{\mathfrak{m}} \overline{T}_{\mathfrak{m}}$ of its localizations at maximal ideals. There is a corresponding decomposition $1 = \sum_{\mathfrak{m}} e_{\mathfrak{m}}$ of the unit in $\overline{T}$ as a sum of idempotents. Since $\mathbf{D}(R)$ is idempotent complete, this determines a decomposition $C = \oplus_{\mathfrak{m}} C_{\mathfrak{m}}$ in $\mathbf{D}(R)$. The direct summands $C_{\mathfrak{m}}$ are well defined up to unique isomorphism. We usually reserve the symbol $C^\bullet$ to refer to an element in the category of cochain complexes, although hopefully statements of the form $C^\bullet = 0$ in $\mathbf{D}(R)$ will not cause any confusion.

If G is a locally profinite group, and $U \subset G$ is an open compact subgroup, then we write $\mathcal{H}(U, G)$ for the algebra of compactly supported, U-biinvariant functions $f : G \to \mathbf{Z}$, with multiplication given by convolution with respect to the Haar measure on G that gives U volume 1. If $X \subset G$ is a compact U-biinvariant subset, then we write $[X]$ for the characteristic function of X, an element of $\mathcal{H}(U, G)$.

If G is a reductive group over a field k and $T \subset G$ is a split maximal torus, then we write $W(G, T)$ for the Weyl group (the set of k-points of the quotient $N_G(T)/T$). For example, if $F/\mathbf{Q}$ is a number field, then we may identify

$W((\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)_{\mathbf{C}}, (\mathrm{Res}_{F/\mathbf{Q}}\,T_n)_{\mathbf{C}})$ with $S_n^{\mathrm{Hom}(F,\mathbf{C})}$. If $P \subset G$ is a parabolic subgroup that contains T, then there is a unique Levi subgroup $L \subset P$ that contains T. We write $W_P(G,T)$ for the absolute Weyl group of this Levi subgroup, which may be identified with a subgroup of $W(G,T)$.

Suppose that G comes equipped with a Borel subgroup B containing T. Then we can form $X^*(T)^+ \subset X^*(T)$, the subset of B-dominant characters. If P is a parabolic subgroup of G that contains B, with Levi L as above, then $B \cap L$ is a Borel subgroup of L and we write $X^*(T)^{+,P}$ for the subset of $(B \cap L)$-dominant characters. The set

$$W^P(G,T) = \{w \in W(G,T) \mid w(X^*(T)^+) \subset X^*(T)^{+,P}\}$$

is a set of representatives for the quotient $W_P(G,T)\backslash W(G,T)$.

Galois representations. If F is a perfect field, we let $\overline{F}$ denote an algebraic closure of F and G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. We will use ζ_n to denote a primitive n^{th}-root of 1. Let ϵ_l denote the l-adic cyclotomic character and $\bar{\epsilon}_l$ its reduction modulo l. We will also let $\omega_l : G_F \to \mu_{l-1} \subset \mathbf{Z}_l^\times$ denote the Teichmüller lift of $\bar{\epsilon}_l$. If E/F is a separable quadratic extension, we will let $\delta_{E/F}$ denote the non-trivial character of $\mathrm{Gal}(E/F)$. We will write Br_F for the Brauer group of F.

We will write $\mathbf{Q}_{l^r}$ for the unique unramified extension of $\mathbf{Q}_l$ of degree r and $\mathbf{Z}_{l^r}$ for its ring of integers. We will write $\mathbf{Q}_l^{\mathrm{nr}}$ for the maximal unramified extension of $\mathbf{Q}_l$ and $\mathbf{Z}_l^{\mathrm{nr}}$ for its ring of integers. We will also write $\widehat{\mathbf{Z}}_l^{\mathrm{nr}}$ for the l-adic completion of $\mathbf{Z}_l^{\mathrm{nr}}$ and $\widehat{\mathbf{Q}}_l^{\mathrm{nr}}$ for its field of fractions.

If K is a finite extension of $\mathbf{Q}_p$ for some p, we write K^{nr} for its maximal unramified extension, I_K for the inertia subgroup of G_K, $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius, and W_K for the Weil group. If K'/K is a Galois extension, we will write $I_{K'/K}$ for the inertia subgroup of $\mathrm{Gal}(K'/K)$. We will write $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ for the Artin map normalized to send uniformizers to geometric Frobenius elements. We will write $\omega_{l,r}$ for the character $G_{\mathbf{Q}_{l^r}} \to \mathbf{Z}_{l^r}^\times$ such that $\omega_{l,r} \circ \mathrm{Art}_{\mathbf{Q}_{l^r}}$ sends l to 1 and sends $a \in \mathbf{Z}_{l^r}^\times$ to the Teichmüller lift of $a \bmod l$. This is sometimes referred to as "the fundamental character of niveau r." (Thus $\omega_{l,1} = \omega_l$.)

We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Frobenius semi-simple Weil–Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. We write rec_K^T for the arithmetic normalization of the local Langlands correspondence, as defined in, e.g., [CT14, §2.1]; it is defined on irreducible admissible representations of $\mathrm{GL}_n(K)$ defined over any field that is abstractly isomorphic to $\mathbf{C}$ (e.g., $\overline{\mathbf{Q}}_l$).

If (r,N) is a Weil–Deligne representation of W_K, we will write $(r,N)^{F\text{-ss}}$ for its Frobenius semisimplification. If ρ is a continuous representation of G_K over

$\overline{\mathbf{Q}}_l$ with $l \neq p$, then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K. (See, for instance, Section 1 of [TY07].) By a *Steinberg* representation of $\mathrm{GL}_n(K)$ we will mean a representation $\mathrm{Sp}_n(\psi)$ (in the notation of Section 1.3 of [HT01]), where ψ is an unramified character of $K^\times$. If π_i is an irreducible smooth representation of $\mathrm{GL}_{n_i}(K)$ for $i = 1, 2$, we will write $\pi_1 \boxplus \pi_2$ for the irreducible smooth representation of $\mathrm{GL}_{n_1+n_2}(K)$ with $\mathrm{rec}(\pi_1 \boxplus \pi_2) = \mathrm{rec}(\pi_1) \oplus \mathrm{rec}(\pi_2)$. If K'/K is a finite extension and if π is an irreducible smooth representation of $\mathrm{GL}_n(K)$, we will write $\mathrm{BC}_{K'/K}(\pi)$ for the base change of π to K' that is characterized by $\mathrm{rec}_{K'}(\mathrm{BC}_{K'/K}(\pi)) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If ρ is a de Rham representation of G_K over $\overline{\mathbf{Q}}_p$, then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil–Deligne representation of W_K, and if $\tau : K \hookrightarrow \overline{\mathbf{Q}}_p$ is a continuous embedding of fields, then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to τ. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact if W is a de Rham representation of G_K over $\overline{\mathbf{Q}}_p$ and if $\tau : K \hookrightarrow \overline{\mathbf{Q}}_p$, then the multiset $\mathrm{HT}_\tau(W)$ contains i with multiplicity $\dim_{\overline{\mathbf{Q}}_l}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus, for example, $\mathrm{HT}_\tau(\epsilon_p) = \{-1\}$.

If G is a reductive group over K and P is a parabolic subgroup with unipotent radical N and Levi component L, and if π is a smooth representation of $L(K)$, then we define $\mathrm{Ind}_{P(K)}^{G(K)} \pi$ to be the set of locally constant functions $f : G(K) \to \pi$ such that $f(hg) = \pi(hN(K))f(g)$ for all $h \in P(K)$ and $g \in G(K)$. It is a smooth representation of $G(K)$ where $(g_1 f)(g_2) = f(g_2 g_1)$. This is sometimes referred to as "natural" or "un-normalized" induction. We let δ_P denote the determinant of the action of L on $\mathrm{Lie}\, N$. Then we define the "normalized" or "unitary" induction $\text{n-Ind}_{P(K)}^{G(K)} \pi$ to be $\mathrm{Ind}_{P(K)}^{G(K)}(\pi \otimes |\delta_P|_K^{1/2})$. If P is any parabolic in $\mathrm{GL}_{n_1+n_2}$ with Levi component $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, then $\pi_1 \boxplus \pi_2$ is a sub-quotient of $\text{n-Ind}_{P(K)}^{\mathrm{GL}_{n_1+n_2}(K)} \pi_1 \otimes \pi_2$.

We will let c denote complex conjugation on $\mathbf{C}$. We will write $\mathrm{Art}_{\mathbf{R}}$ (resp. $\mathrm{Art}_{\mathbf{C}}$) for the unique continuous surjection

$$\mathbf{R}^\times \twoheadrightarrow \mathrm{Gal}(\mathbf{C}/\mathbf{R})$$

(resp. $\mathbf{C}^\times \twoheadrightarrow \mathrm{Gal}(\mathbf{C}/\mathbf{C})$). We will write $\mathrm{rec}_{\mathbf{C}}$ (resp. $\mathrm{rec}_{\mathbf{R}}$), or simply rec, for the local Langlands correspondence from irreducible admissible $(\mathrm{Lie}\, \mathrm{GL}_n(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}, O(n))$-modules (resp. $(\mathrm{Lie}\, \mathrm{GL}_n(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}, U(n))$-modules) to continuous, semi-simple n-dimensional representations of the Weil group $W_{\mathbf{R}}$ (resp. $W_{\mathbf{C}}$) (see [Lan89]). If π_i is an irreducible admissible $(\mathrm{Lie}\, \mathrm{GL}_{n_i}(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}, O(n_i))$-module (resp. $(\mathrm{Lie}\, \mathrm{GL}_{n_i}(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}, U(n_i))$-module) for $i = 1, \ldots, r$ and if $n = n_1 + \cdots + n_r$, then we define an irreducible admissible $(\mathrm{Lie}\, \mathrm{GL}_n(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}, O(n))$-module (resp. $(\mathrm{Lie}\, \mathrm{GL}_n(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}, U(n))$-module) $\pi_1 \boxplus \cdots \boxplus \pi_r$ by

$$\mathrm{rec}(\pi_1 \boxplus \cdots \boxplus \pi_r) = \mathrm{rec}(\pi_1) \oplus \cdots \oplus \mathrm{rec}(\pi_r).$$

If π is an irreducible admissible $(\mathrm{Lie}\,\mathrm{GL}_n(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}, O(n))$-module, then we define $\mathrm{BC}_{\mathbf{C}/\mathbf{R}}(\pi)$ to be the irreducible admissible $(\mathrm{Lie}\,\mathrm{GL}_n(\mathbf{C}) \otimes_{\mathbf{R}} \mathbf{C}, U(n))$-module defined by

$$\mathrm{rec}_{\mathbf{C}}(\mathrm{BC}_{\mathbf{C}/\mathbf{R}}(\pi)) = \mathrm{rec}_{\mathbf{R}}(\pi)|_{W_{\mathbf{C}}}.$$

If π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and $\xi \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\mathbf{C})}$, we say that π is regular algebraic of weight ξ if the infinitesimal character of π_∞ is the same as that of $V_\xi^\vee$, where V_ξ is the algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n$ of highest weight ξ (see Section 2.2.1). We say that it is regular algebraic if it is regular algebraic of some weight.

We will write $||\ \ ||$ for the continuous homomorphism

$$||\ \ || = \prod_v |\ \ |_v : \mathbf{A}^\times/\mathbf{Q}^\times \longrightarrow \mathbf{R}^\times_{>0},$$

where each $|\ \ |_v$ has its usual normalization, i.e., $|p|_p = 1/p$.

Now suppose that $K/\mathbf{Q}$ is a finite extension. We will write $||\ \ ||_K$ (or simply $||\ \ ||$) for $||\ \ || \circ \mathbf{N}_{K/\mathbf{Q}}$. We will also write

$$\mathrm{Art}_K = \prod_v \mathrm{Art}_{K_v} : \mathbf{A}_K^\times/\overline{K^\times(K_\infty^\times)^0} \xrightarrow{\sim} G_K^{\mathrm{ab}}.$$

If v is a finite place of K, we will write $k(v)$ for its residue field, q_v for $\#k(v)$, and Frob_v for Frob_{K_v}. If v is a real place of K, then we will let $[c_v]$ denote the conjugacy class in G_K consisting of complex conjugations associated to v. If K'/K is a quadratic extension of number fields, we will denote by $\delta_{K'/K}$ the nontrivial character of $\mathbf{A}_K^\times/K^\times\mathbf{N}_{K'/K}\mathbf{A}_{K'}^\times$. (We hope that this will cause no confusion with the Galois character $\delta_{K'/K}$. One equals the composition of the other with the Artin map for K.) If K'/K is a soluble, finite Galois extension and if π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$, we will write $\mathrm{BC}_{K'/K}(\pi)$ for its base change to K', an (isobaric) automorphic representation of $\mathrm{GL}_n(\mathbf{A}_{K'})$ satisfying

$$\mathrm{BC}_{K'/K}(\pi)_v = \mathrm{BC}_{K'_v/K_{v|_K}}(\pi_{v|_K})$$

for all places v of K'. If π_i is an automorphic representation of $\mathrm{GL}_{n_i}(\mathbf{A}_K)$ for $i = 1, 2$, we will write $\pi_1 \boxplus \pi_2$ for the automorphic representation of $\mathrm{GL}_{n_1+n_2}(\mathbf{A}_K)$ satisfying

$$(\pi_1 \boxplus \pi_2)_v = \pi_{1,v} \boxplus \pi_{2,v}$$

for all places v of K.

We will call a number field K a CM field if it has an automorphism c such that for all embeddings $i : K \hookrightarrow \mathbf{C}$, one has $c \circ i = i \circ c$. In this case, either K is totally real or a totally imaginary quadratic extension of a totally real field. In either case, we will let K^+ denote the maximal totally real subfield of K.

Suppose that K is a number field and

$$\chi : \mathbf{A}_K^\times/K^\times \longrightarrow \mathbf{C}^\times$$

is a continuous character. If there exists $a \in \mathbf{Z}^{\mathrm{Hom}(K,\mathbf{C})}$ such that

$$\chi|_{(K_\infty^\times)^0} : x \longmapsto \prod_{\tau \in \mathrm{Hom}(K,\mathbf{C})} (\tau x)^{a_\tau},$$

we will call χ algebraic. In this case, we can attach to χ and a rational prime l and an isomorphism $\imath : \overline{\mathbf{Q}}_l \xrightarrow{\sim} \mathbf{C}$, a unique continuous character

$$r_{l,\imath}(\chi) : G_K \longrightarrow \overline{\mathbf{Q}}_l^\times$$

such that for all $v \nmid l$, we have

$$\imath \circ r_{l,\imath}(\chi)|_{W_{K_v}} \circ \mathrm{Art}_{K_v} = \chi_v.$$

There is also an integer $\mathrm{wt}(\chi)$, the weight of χ, such that

$$|\chi| = ||\ \ ||_K^{-\mathrm{wt}(\chi)/2}.$$

(See the discussion at the start of [BLGGT14, §A.2] for more details.)

If K is a totally real field, we call a continuous character

$$\chi : \mathbf{A}_K^\times / K^\times \longrightarrow \mathbf{C}^\times$$

totally odd if $\chi_v(-1) = -1$ for all $v|\infty$. Similarly, we call a continuous character

$$\mu : G_K \longrightarrow \overline{\mathbf{Q}}_l^\times$$

totally odd if $\mu(c_v) = -1$ for all $v|\infty$.

2. Preliminaries on the cohomology of locally symmetric spaces and Galois representations

Our main objects of study in this paper are n-dimensional Galois representations and their relation to the cohomology of congruence subgroups of GL_n (equivalently, the cohomology of the locally symmetric spaces attached to congruence subgroups of GL_n). In this introductory section we establish some basic notation and definitions concerning these objects, and we recall some of their fundamental known properties. In particular, we will define cohomology groups associated to an arbitrary weight and level and also define the Hecke algebras that act on these cohomology groups.

2.1. *Arithmetic locally symmetric spaces: generalities.*

2.1.1. *Symmetric spaces.* Let F be a number field, and let G be a connected linear algebraic group over F. We consider a space of type $S - \mathbf{Q}$ for $\mathbf{G} := \mathrm{Res}_{F/\mathbf{Q}}\mathrm{G}$, in the sense of [BS73, §2] (see also [NT16, §3.1]). This is a pair consisting of a homogeneous space X^{G} for $\mathbf{G}(\mathbf{R})$ and a family of Levi subgroups of $\mathbf{G}_{\mathbf{R}}$ satisfying certain conditions. From [BS73, Lem. 2.1], the homogeneous space X^{G} is determined up to isomorphism. We will refer to X^{G} as *the*

symmetric space for G. For example, if $\mathrm{G} = \mathrm{GL}_{n,F}$, we can take $X^{\mathrm{G}} = \mathrm{GL}_n(F_\infty)/K_\infty \mathbf{R}^\times$ for $K_\infty \subset \mathrm{GL}_n(F_\infty)$ a maximal compact subgroup.

An open compact subgroup $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}_F^\infty)$ is said to be *neat* if all of its elements are neat. An element $g = (g_v)_v \in \mathrm{G}(\mathbf{A}_F^\infty)$ is said to be neat if the intersection $\cap_v \Gamma_v$ is trivial, where $\Gamma_v \subset \overline{\mathbf{Q}}^\times$ is the torsion subgroup of the subgroup of $\overline{F}_v^\times$ generated by the eigenvalues of g_v acting via some faithful representation of G.

We will call a "good subgroup" any neat open compact subgroup $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}_F^\infty)$ of the form $K_G = \prod_v K_{G,v}$, the product running over finite places v of F. If K_G is a good subgroup, then we define

$$X^{\mathrm{G}}_{K_{\mathrm{G}}} := \mathrm{G}(F)\backslash \left(X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)/K_{\mathrm{G}}\right) \text{ and } \mathfrak{X}_{\mathrm{G}} := \mathrm{G}(F)\backslash \left(X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)\right),$$

the latter with the discrete topology on $\mathrm{G}(\mathbf{A}_F^\infty)$.

These topological spaces may be given the structure of smooth manifolds, and $\mathrm{G}(\mathbf{A}_F^\infty)$ acts on $\mathfrak{X}_{\mathrm{G}}$ by right translation. We can identify $X^{\mathrm{G}}_{K_{\mathrm{G}}} = \mathfrak{X}_{\mathrm{G}}/K_{\mathrm{G}}$. Note that the space X^{G} is diffeomorphic to Euclidean space. The neatness condition on K_{G} implies that $X^{\mathrm{G}}_{K_{\mathrm{G}}}$ can be identified with a finite disjoint union of quotients of X^{G} by the action of torsion-free arithmetic subgroups of $\mathrm{G}(F)$.

We let $\overline{X}^{\mathrm{G}}$ denote the partial Borel–Serre compactification of X^{G} (see [BS73, §7.1]). Define

$$\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}} := \mathrm{G}(F)\backslash \left(\overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)/K_{\mathrm{G}}\right) \text{ and } \overline{\mathfrak{X}}_{\mathrm{G}} := \mathrm{G}(F)\backslash \left(\overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)\right).$$

For any good subgroup $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}_F^\infty)$, the space $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$, which can be identified with $\overline{\mathfrak{X}}_{\mathrm{G}}/K_{\mathrm{G}}$, is compact (see [BS73, Th. 9.3]). More precisely, $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ is a compact smooth manifold with corners with interior $X^{\mathrm{G}}_{K_{\mathrm{G}}}$; the inclusion $X^{\mathrm{G}}_{K_{\mathrm{G}}} \hookrightarrow \overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ is a homotopy equivalence. We also define $\partial X^{\mathrm{G}} = \overline{X}^{\mathrm{G}} - X^{\mathrm{G}}$ and

$$\partial X^{\mathrm{G}}_{K_{\mathrm{G}}} := \mathrm{G}(F)\backslash \left(\partial X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)/K_{\mathrm{G}}\right) \text{ and } \partial\mathfrak{X}_{\mathrm{G}} := \mathrm{G}(F)\backslash \left(\partial X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)\right).$$

2.1.2. *Hecke operators and coefficient systems.* If S is a finite set of finite places of F, we set $\mathrm{G}^S := \mathrm{G}(\mathbf{A}_F^{\infty,S})$ and $\mathrm{G}_S := \mathrm{G}(\mathbf{A}_{F,S})$, and similarly $K^S_{\mathrm{G}} = \prod_{v\notin S} K_{\mathrm{G},v}$ and $K_{\mathrm{G},S} = \prod_{v\in S} K_{\mathrm{G},v}$. We also sometimes write $\mathrm{G}^\infty = \mathrm{G}(\mathbf{A}_F^\infty)$.

Let R be a ring, and let $\mathcal{V}$ be an $R[\mathrm{G}(F) \times K_{\mathrm{G},S}]$-module, finite free as R-module. We now explain how to obtain a local system of finite free R-modules, also denoted $\mathcal{V}$, on $X^{\mathrm{G}}_{K_{\mathrm{G}}}$, and how to equip the complex $R\Gamma(X^{\mathrm{G}}_{K_{\mathrm{G}}}, \mathcal{V}) \in \mathbf{D}(R)$ with an action of the Hecke algebra $\mathcal{H}(\mathrm{G}^S, K^S_G)$, following the formalism of [NT16] (in particular, viewing $X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)$ as a right $\mathrm{G}(F) \times \mathrm{G}(\mathbf{A}_F^\infty)$-space).

The $R[\mathrm{G}(F) \times K_{\mathrm{G},S}]$-module $\mathcal{V}$ determines (by pullback from a point) a $\mathrm{G}(F) \times \mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaf, also denoted $\mathcal{V}$, of finite free R-modules on $X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)$, hence (by descent under a free action, as in [NT16, Lem. 2.17]) a $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaf $\mathcal{V}$ on $\mathfrak{X}_{\mathrm{G}}$. By taking derived

global sections we obtain $R\Gamma(\mathfrak{X}_{\mathrm{G}}, \mathcal{V})$, which is an object of the derived category of $R[\mathrm{G}^S \times K_{\mathrm{G},S}]$-modules. By taking derived invariants under K_{G} we obtain $R\Gamma(K_{\mathrm{G}}, R\Gamma(\mathfrak{X}_{\mathrm{G}}, \mathcal{V}_{\mathfrak{X}_{\mathrm{G}}}))$, which is an object of the derived category of $\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R$-modules.

On the other hand, if we only think of $\mathcal{V}$ as a K_{G}-equivariant sheaf on $\mathfrak{X}_{\mathrm{G}}$, it is equivalent to a sheaf $\mathcal{V}$ on $X_{K_{\mathrm{G}}}^{\mathrm{G}}$ (applying once again [NT16, Lem. 2.17]). The complex $R\Gamma(X_{K_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})$ is naturally isomorphic in $\mathbf{D}(R)$ to the image of the complex $R\Gamma(K_{\mathrm{G}}, R\Gamma(\mathfrak{X}_{\mathrm{G}}, \mathcal{V}_{\mathfrak{X}_{\mathrm{G}}}))$ under the exact forgetful functor

$$\mathbf{D}(\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R) \to \mathbf{D}(R);$$

cf. [NT16, Prop. 2.18]. In this way, we obtain a canonical homomorphism

$$\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R \to \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(X_{K_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})). \tag{2.1.3}$$

The same formalism applies equally well to the Borel–Serre compactification (because $G(F) \times K_{\mathrm{G}}$ acts freely on $\overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)$). Even more generally, if Y is any right $\mathrm{G}^S \times K_{\mathrm{G},S}$-space and C is any bounded-below complex of $\mathrm{G}^S \times K_{\mathrm{G},S}$-equivariant sheaves of R-modules on Y, there is a homomorphism

$$\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R \to \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(K_{\mathrm{G}}, R\Gamma(Y, C))).$$

Taking $j : X^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty) \to \overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty)$ to be the canonical open immersion and $\mathcal{V}$ to be an $R[\mathrm{G}(F) \times K_{\mathrm{G},S}]$-module, finite free as R-module, this determines an action of the Hecke algebra on the cohomology groups with compact support:

$$\begin{aligned}\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R \to \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(\mathrm{G}(F) \times K_{\mathrm{G}}, R\Gamma(\overline{X}^{\mathrm{G}} \times \mathrm{G}(\mathbf{A}_F^\infty), j_! \mathcal{V}))) \\ = \mathrm{End}_{\mathbf{D}(R)}(R\Gamma_c(X_{K_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})).\end{aligned}$$

We have the following lemma, which is a consequence of the existence of the Borel–Serre compactification (see [BS73, §11]):

Lemma 2.1.4. *Let K_{G} be a good subgroup, let R be a Noetherian ring, and let $\mathcal{V}$ be an $R[\mathrm{G}(F) \times K_{\mathrm{G}}]$-module, finite free as R-module. Then $H^*(X_{K_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})$ is a finitely generated R-module.*

A variant of this construction arises when we are given a normal good subgroup $K'_{\mathrm{G}} \subset K_{\mathrm{G}}$ with the property that $K_{\mathrm{G}}^S = (K'_{\mathrm{G}})^S$. Then we write $R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(X_{K'_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V}) \in \mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}])$ for the complex in this category computing the cohomology of $H^*(X_{K'_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})$ with its natural $K_{\mathrm{G}}/K'_{\mathrm{G}} = K_{\mathrm{G},S}/K'_{\mathrm{G},S}$-action. The image of this complex under the forgetful functor $\mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}]) \to \mathbf{D}(R)$ is $R\Gamma(X_{K'_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})$, and there is a homomorphism

$$\mathcal{H}(\mathrm{G}^S, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R \to \mathrm{End}_{\mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}])}(R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(X_{K'_{\mathrm{G}}}^{\mathrm{G}}, \mathcal{V})) \tag{2.1.5}$$

that recovers (2.1.3) after composition with the map

$$\mathrm{End}_{\mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}])}(R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(X^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V})) \to \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(X^{\mathrm{G}}_{K_{\mathrm{G}}},\mathcal{V})) \tag{2.1.6}$$

given by the functor $R\Gamma(K_{\mathrm{G}}/K'_{\mathrm{G}},?)$.

The following lemma is a strengthening of Lemma 2.1.4:

Lemma 2.1.7. *Let K_{G} be a good subgroup, and let $K'_{\mathrm{G}} \subset K_{\mathrm{G}}$ be a normal subgroup that is also good. Let R be a Noetherian ring, and let $\mathcal{V}$ be an $R[\mathrm{G}(F)\times K_{\mathrm{G}}]$-module, finite free as R-module. Then $R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(X^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V})$ is a perfect object of $\mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}])$; in other words, it is isomorphic in this category to a bounded complex of projective $R[K_{\mathrm{G}}/K'_{\mathrm{G}}]$-modules.*

Proof. Pullback induces an isomorphism

$$R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(\overline{X}^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V}) \to R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(X^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V}),$$

so it suffices to show that $R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(\overline{X}^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V})$ is a perfect complex. As in [BS73, §11], we see that $\overline{X}^{\mathrm{G}}_{K_{\mathrm{G}}}$ admits a finite triangulation; this pulls back to a $\mathrm{G}(F)\times K_{\mathrm{G}}$-invariant triangulation of $\overline{X}^{\mathrm{G}}\times\mathrm{G}(\mathbf{A}_F^\infty)$. Let $C_\bullet$ be the corresponding complex of simplicial chains. It is a bounded complex of finite free $\mathbf{Z}[\mathrm{G}(F)\times K_{\mathrm{G}}]$-modules. The lemma now follows on observing that $R\Gamma_{K_{\mathrm{G}}/K'_{\mathrm{G}}}(\overline{X}^{\mathrm{G}}_{K'_{\mathrm{G}}},\mathcal{V})$ is isomorphic in $\mathbf{D}(R[K_{\mathrm{G}}/K'_{\mathrm{G}}])$ to the complex $\mathrm{Hom}_{\mathbf{Z}[\mathrm{G}(F)\times K'_{\mathrm{G}}]}(C_\bullet,\mathcal{V})$. □

Finally we introduce some notation relevant for relating the Hecke operators of G and of its parabolic subgroups. Let us therefore now assume that G is reductive, and let $\mathrm{P}=\mathrm{MN}$ be a parabolic subgroup with Levi subgroup M. Let $K_{\mathrm{G}}\subset\mathrm{G}(\mathbf{A}_F^\infty)$ be a good subgroup. In this situation, we define $K_{\mathrm{P}}=K_{\mathrm{G}}\cap\mathrm{P}(\mathbf{A}_F^\infty)$, $K_{\mathrm{N}}=K_{\mathrm{G}}\cap\mathrm{N}(\mathbf{A}_F^\infty)$, and we define K_{M} to be the image of K_{P} in $\mathrm{M}(\mathbf{A}_F^\infty)$. We say that K_{G} is decomposed with respect to $\mathrm{P}=\mathrm{MN}$ if we have $K_{\mathrm{P}}=K_{\mathrm{M}}\ltimes K_{\mathrm{N}}$; equivalently, if $K_{\mathrm{M}}=K_{\mathrm{G}}\cap\mathrm{M}(\mathbf{A}_F^\infty)$.

Assume now that K_{G} is decomposed with respect to $\mathrm{P}=\mathrm{MN}$, and let S be a finite set of finite places of F such that for all $v\notin S$, $K_{\mathrm{G},v}$ is a hyperspecial maximal compact subgroup of $\mathrm{G}(F_v)$. In this case, we can define homomorphisms

$$r_{\mathrm{P}}:\mathcal{H}(\mathrm{G}^S,K_{\mathrm{G}}^S)\to\mathcal{H}(\mathrm{P}^S,K_{\mathrm{P}}^S) \text{ and } r_{\mathrm{M}}:\mathcal{H}(\mathrm{P}^S,K_{\mathrm{P}}^S)\to\mathcal{H}(\mathrm{M}^S,K_{\mathrm{M}}^S),$$

given respectively by "restriction to P" and "integration along N"; see [NT16, §§2.2.3, 2.2.4] for the definitions of these maps, along with the proofs that they are indeed algebra homomorphisms and that integration along N preserves integrality. We write

$$\mathcal{S}=r_{\mathrm{M}}\circ r_{\mathrm{P}} \tag{2.1.8}$$

for the composite map, or $\mathcal{S} = \mathcal{S}_{\mathrm{M}}^{\mathrm{G}}$ when we wish to emphasize the ambient groups. By abuse of notation, we also denote by r_{P}, r_{M} and $\mathcal{S} = \mathcal{S}_{\mathrm{M}}^{\mathrm{G}}$ the same maps for the local Hecke algebras at $v \notin S$.

2.1.9. *The Hecke algebra of a monoid.* We in fact need a slight generalization of the discussion in the previous section, which we outline now in a similar way to [NT16, §2.2].

We first discuss the local situation. Let F be a non-archimedean local field, and let G be a reductive group over F. Let q denote the cardinality of the residue field of F. If $U \subset G(F)$ is an open compact subgroup and $\Delta \subset G(F)$ is an open submonoid that is invariant under left and right multiplication by elements of U, then we can consider the subset $\mathcal{H}(\Delta, U) \subset \mathcal{H}(G(F), U)$ of functions $f : G(F) \to \mathbf{Z}$ that are supported in Δ. It follows from the definition of the convolution product that this subset is in fact a subalgebra. If R is a ring and M is an $R[\Delta]$-module (or more generally, a complex of $R[\Delta]$-modules), then there is a corresponding homomorphism $\mathcal{H}(\Delta, U) \to \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(U, M))$. This extends the formalism for the full Hecke algebra described in [NT16, 2.2.5] and recalled in the previous section.

Now let $P \subset G$ be a parabolic subgroup with Levi decomposition $P = MN$, and let $\overline{P} = M\overline{N}$ denote the opposite parabolic. Let $U \subset G$ be an open compact subgroup that admits an Iwahori decomposition with respect to P. By definition, this means that if we define $U_N = U \cap N(F)$, $U_M = U \cap M(F)$, and $U_{\overline{N}} = U \cap \overline{N}(F)$, then the two product maps

$$U_N \times U_M \times U_{\overline{N}} \to U \text{ and } U_{\overline{N}} \times U_M \times U_N \to U$$

are bijective. In this case, we write $\Delta_M \subset M(F)$ for the set of U-positive elements, i.e., those $t \in M(F)$ that satisfy $tU_N t^{-1} \subset U_N$ and $U_{\overline{N}} \subset tU_{\overline{N}}t^{-1}$. We define $\Delta = U_N \Delta_M U_{\overline{N}}$.

Lemma 2.1.10. *Note that Δ_M and Δ are monoids. Moreover, Δ_M is open in $M(F)$, Δ is open in $G(F)$, we have $U\Delta U = \Delta$, and $\Delta \cap M(F) = \Delta_M$.*

Proof. It is clear from the definition that Δ_M is closed under multiplication, and also that Δ_M, Δ are open in $M(F)$ and $G(F)$, respectively. To show that $U\Delta U = \Delta$, we simply observe that if $m \in \Delta_M$, then the definition of positivity gives

$$\begin{aligned} UmU &= UmU_N U_M U_{\overline{N}} = UmU_M U_{\overline{N}} = U_N U_M U_{\overline{N}} m U_M U_{\overline{N}} \\ &= U_N U_M m U_M U_{\overline{N}} \subset U_N \Delta_M U_{\overline{N}} = \Delta. \end{aligned}$$

To show that Δ is closed under multiplication, we must show that $Um_1Um_2U \subset U\Delta_M U$. Using the definition of positivity, we see that

$$Um_1Um_2U = Um_1U_N U_M U_{\overline{N}} m_2 U = Um_1U_M m_2 U,$$

so it is equivalent to show $m_1 U_M m_2 \subset \Delta_M$; and this is true, since $U_M \subset \Delta_M$. Finally, the identity $\Delta \cap M(F) = \Delta_M$ follows from the following observation: if $u_1 t \bar{u}_2 = m \in M(F)$ for $u_1 \in U_N$, $t \in \Delta_M$ and $\bar{u}_2 \in U_{\overline{N}}$, then $\bar{u}_2 = t^{-1} u_1^{-1} m \in P(F) \cap \overline{N}(F)$, so $\bar{u}_2$ must be the identity. Similarly, u_1 must be the identity, so $m = t \in \Delta_M$. □

It follows that the Hecke algebras $\mathcal{H}(\Delta, U)$ and $\mathcal{H}(\Delta_M, U_M)$ are defined. Moreover, $\Delta_P = \Delta \cap P(F)$ is a monoid, and we can consider also the Hecke algebra $\mathcal{H}(\Delta_P, U_P)$.

Lemma 2.1.11. *Consider the two maps $r_P : \mathcal{H}(\Delta, U) \to \mathcal{H}(\Delta_P, U_P)$ and $r_M : \mathcal{H}(\Delta_P, U_P) \to \mathcal{H}(\Delta_M, U_M)$ given by restriction to $P(F)$ and integration along U_N, respectively. Then both r_P and r_M are algebra homomorphisms.*

Proof. It follows from [NT16, Lem. 2.7] that the map $\mathcal{H}(P(F), U_P) \to \mathcal{H}(M(F), U_M)$ is an algebra homomorphism whenever the condition $U_P = U_N \rtimes U_M$ is satisfied. It remains to show that r_P is an algebra homomorphism. The proof is the same as the proof of [NT16, Lem. 2.4(1)] once we take into account the identity, valid for any function $f : G(F) \to \mathbf{R}$ with compact support contained in $UP(F)$ (and a fortiori, any function $f \in \mathcal{H}(\Delta, U)$):

$$\int_{g \in G(F)} f(g)\, dg = \int_{u \in U} \int_{p \in P(F)} f(pu)\, dp\, du. \qquad \square$$

It will be helpful later to note that the maps r_P and $r_M \circ r_P$ are quite simple, being given on basis elements by the formulae $r_P([UmU]) = [U_P m U_P]$ and $r_M \circ r_P([UmU]) = \#(U_N / m U_N m^{-1})[U_M m U_M] = |\delta_P(m)|_F^{-1} [U_M m U_M]$, respectively. As in the unramified case, we will write $\mathcal{S}$ or $\mathcal{S}_M^G$ for the composite $r_M \circ r_P$.

Lemma 2.1.12. *Consider the map $t : \mathcal{H}(\Delta_M, U_M) \to \mathcal{H}(\Delta, U)$ of $\mathbf{Z}$-modules given on basis elements by $t([U_M m U_M]) = [UmU]$. Then t is an algebra homomorphism.*

Proof. This is [BK98, Cor. 6.12]. □

Thus we have constructed injective algebra homomorphisms

$$t : \mathcal{H}(\Delta_M, U_M) \to \mathcal{H}(\Delta, U),$$
$$\mathcal{S} : \mathcal{H}(\Delta, U) \to \mathcal{H}(\Delta_M, U_M)$$

with the property that for any $m \in \Delta_M$, $t \circ \mathcal{S}([UmU]) = |\delta_P(m)|_F^{-1} [UmU]$ and $\mathcal{S} \circ t([U_M m U_M] = |\delta_P(m)|_F^{-1} [U_M m U_M]$. In certain circumstances, we can extend the domain of definition of these homomorphisms. Following [BK98], we say that an element $z \in \Delta_M$ that lies in the center of M is strongly positive if for any open compact subgroups H_1, H_2 of U_N (resp. $\overline{H}_1, \overline{H}_2$ of $U_{\overline{N}}$), there exists $n \geq 0$ such that $z^n H_1 z^{-n} \subset H_2$ (resp. $z^{-n} \overline{H}_1 z^n \subset \overline{H}_2$).

LEMMA 2.1.13. *Let $z \in \Delta_M$ be strongly positive. Then*

(1) *$[U_M z U_M]$ lies in the center of $\mathcal{H}(\Delta_M, U_M)$, is invertible in $\mathcal{H}(M(F), U_M)$, and $\mathcal{H}(\Delta_M, U_M)[[U_M z U_M]^{-1}] = \mathcal{H}(M(F), U_M)$.*

(2) *Let R be a ring in which q is a unit, and suppose that $[UzU]$ is invertible in $\mathcal{H}(G(F), U) \otimes_{\mathbf{Z}} R$. Then $t \otimes_{\mathbf{Z}} R$ and $\mathcal{S} \otimes_{\mathbf{Z}} R$ extend uniquely to algebra isomorphisms*

$$t : \mathcal{H}(M(F), U_M) \otimes_{\mathbf{Z}} R \to (\mathcal{H}(\Delta, U) \otimes_{\mathbf{Z}} R)[[UzU]^{-1}]$$

and

$$\mathcal{S} : (\mathcal{H}(\Delta, U) \otimes_{\mathbf{Z}} R)[[UzU]^{-1}] \to \mathcal{H}(M(F), U_M) \otimes_{\mathbf{Z}} R.$$

Proof. The element $[U_M z U_M]$ lies in the center of $\mathcal{H}(\Delta_M, U_M)$ because z lies in the center of $M(F)$, by assumption. Its inverse is $[U_M z^{-1} U_M]$. The equality $\mathcal{H}(\Delta_M, U_M)[[U_M z U_M]^{-1}] = \mathcal{H}(M(F), U_M)$ holds because for any $m \in M(F)$, there exists $n \geq 0$ such that $z^n m \in \Delta_M$, hence $[U_M m U_M] = [U_M z U_M]^{-n}[U_M z^n m U_M] \in \mathcal{H}(\Delta_M, U_M)[[U_M z U_M]^{-1}]$. This shows the first part. The second part is elementary. □

LEMMA 2.1.14. *Let R be a ring, let W be an $R[P(F)]$-module, and let $V = \mathrm{Ind}_{P(F)}^{G(F)} W$. Then there is a natural morphism $\phi : V^U \to r_P^* W^{U_P}$ of $\mathcal{H}(\Delta, U) \otimes_{\mathbf{Z}} R$-modules. Moreover, writing $(?)^\sim$ for the forgetful functor from $\mathcal{H}(\Delta, U) \otimes_{\mathbf{Z}} R$-modules to R-modules, the induced morphism $(V^U)^\sim \to (r_P^* W^{U_P})^\sim$ has a functorial splitting.*

Proof. Let $g_1, \ldots, g_n \in G(F)$ be representatives for the double quotient $P(F) \backslash G(F) / U$; we assume that $g_1 = 1$. Then there is an isomorphism of R-modules $V^U \cong \oplus_{i=1}^n W^{g_i U g_i^{-1} \cap P(F)}$, which sends a function $f \in V^U$ to the tuple $(f(g_1), \ldots, f(g_n))$. This is the desired functorial splitting. We claim that the map $V^U \to W^{U_P}$ given by the projection to the first component is in fact Hecke equivariant (with respect to r_P). To see this, choose $f \in V^U$, and let $v = f(1)$, $m \in \Delta_M$. We calculate

$$\begin{aligned}([UmU] \cdot f)(1) &= \int_{g \in UmU} f(g)\, dg = \int_{p \in P(F)} \int_{u \in U} \mathbf{1}_{pu \in UmU} f(pu)\, dp\, du \\ &= \int_{p \in P(F) \cap UmU} f(p)\, dp = [U_P m U_P] \cdot f(1),\end{aligned}$$

as required. □

We now describe how we will apply the above discussion in the global situation. Let F now denote a number field, let G be a reductive group over F, and let $\mathrm{P} \subset \mathrm{G}$ be a parabolic subgroup with Levi decomposition $\mathrm{P} = \mathrm{MN}$. Let $K_{\mathrm{G}} \subset \mathrm{G}(\mathbf{A}_F^\infty)$ be a good subgroup of the form $K = K_{\mathrm{G},S} K_{\mathrm{G},T} K_{\mathrm{G}}^{T \cup S}$, notation and assumptions being as follows:

(1) T, S are finite disjoint sets of finite places of F.
(2) For each place $v \notin S \cup T$ of F, G_{F_v} is unramified and K_v is a hyperspecial maximal subgroup of $\mathrm{G}(F_v)$.
(3) For each place $v \in T$, $K_{\mathrm{G},v}$ admits an Iwahori decomposition with respect to P. We write $\Delta_{\mathrm{G},v} \subset \mathrm{G}(F_v)$ for the corresponding open submonoid and $\Delta_{\mathrm{G},T} = \prod_{v \in T} \Delta_{\mathrm{G},v}$. We define $\Delta_{\mathrm{P},T}$ and $\Delta_{\mathrm{M},T}$ similarly.

We thus have a map

$$\mathcal{S} : \mathcal{H}(\mathrm{G}^{S \cup T} \times \Delta_{\mathrm{G},T}, K_{\mathrm{G}}^S) \to \mathcal{H}(\mathrm{M}^{S \cup T} \times \Delta_{\mathrm{M},T}, K_{\mathrm{M}}^S).$$

Let R be a ring. Applying Lemma 2.1.14 (cf. [NT16, Cor. 2.6]), we see that there is a split morphism in $\mathbf{D}(R)$,

$$R\Gamma([\mathrm{Ind}_{\mathrm{P}_\infty}^{\mathrm{G}_\infty} \mathfrak{X}_{\mathrm{P}}]/K_{\mathrm{G}}, R) \to R\Gamma(X_{K_{\mathrm{P}}}^{\mathrm{P}}, R),$$

which is equivariant for the action of $\mathcal{H}(\mathrm{G}^{S \cup T} \times \Delta_{\mathrm{G},T}, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R$ by endomorphisms on the source and target (the latter action being via the map r_{P}, and induction being in the same sense as in [NT16, §3.1]). The splitting need not be equivariant, but we see that in any case there is a surjective morphism

$$H^*([\mathrm{Ind}_{\mathrm{P}_\infty}^{\mathrm{G}_\infty} \mathfrak{X}_{\mathrm{P}}]/K_{\mathrm{G}}, R) \to r_P^* H^*(X_{K_{\mathrm{P}}}^{\mathrm{P}}, R)$$

of $\mathcal{H}(\mathrm{G}^{S \cup T} \times \Delta_{\mathrm{G},T}, K_{\mathrm{G}}^S) \otimes_{\mathbf{Z}} R$-modules. Similarly [NT16, Prop. 3.4] shows that there is a split morphism in $\mathbf{D}(R)$

$$R\Gamma(X_{K_{\mathrm{M}}}^{\mathrm{M}}, R) \to R\Gamma(X_{K_{\mathrm{P}}}^{\mathrm{P}}, R),$$

which is equivariant for the action of $\mathcal{H}(\mathrm{P}^{S \cup T} \times \Delta_{\mathrm{P},T}, K_{\mathrm{P}}^S) \otimes_{\mathbf{Z}} R$ by endomorphisms on the source and target (the action on the source being via the map r_{M}). Altogether there is no $\mathcal{S}$-equivariant map between the complexes $R\Gamma([\mathrm{Ind}_{\mathrm{P}_\infty}^{\mathrm{G}_\infty} \mathfrak{X}_{\mathrm{P}}]/K_{\mathrm{G}}, R)$ and $R\Gamma(X_{K_{\mathrm{M}}}^{\mathrm{M}}, R)$, these morphisms considered above will together allow us, in the course of proving Theorem 2.4.8 below, to show that $\mathcal{S}$ descends to a map between the Hecke algebras that act faithfully on these complexes. Moreover, in the presence of invertible strongly positive elements as in the statement of Lemma 2.1.13, we will be able to show that this induced map on Hecke algebras is compatible with localization.

2.2. *Arithmetic locally symmetric spaces*: *the quasi-split unitary group.*

2.2.1. *The quasi-split unitary group*, *the Siegel parabolic*, *and its Levi subgroup*. We now specialize the above discussion to our case of interest. We fix an integer $n \geq 1$. Let F be an (imaginary) CM number field with maximal totally real subfield F^+. Let Ψ_n be the matrix with 1's on the anti-diagonal and 0's elsewhere, and set

$$J_n = \begin{pmatrix} 0 & \Psi_n \\ -\Psi_n & 0 \end{pmatrix}.$$

We write $\widetilde{G}_n = \widetilde{G}$ for the group scheme over $\mathcal{O}_{F^+}$ with functor of points

$$\widetilde{G}(R) = \{g \in \mathrm{GL}_{2n}(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F) \mid {}^t g J_n g^c = J_n\}.$$

Then $\widetilde{G}_{F^+}$ is a quasi-split reductive group over F^+; it is a form of GL_{2n} that becomes split after the quadratic base change F/F^+. If $\overline{v}$ is a place of F^+ that splits in F, then a choice of place $v|\overline{v}$ of F determines a canonical isomorphism $\iota_v : \widetilde{G}(F^+_{\overline{v}}) \cong \mathrm{GL}_{2n}(F_v)$. Indeed, there is an isomorphism $F^+_{\overline{v}} \otimes_{F^+} F \cong F_v \times F_{v^c}$ and ι_v is given by the natural inclusion $\widetilde{G}(F^+_{\overline{v}}) \subset \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_{v^c})$ followed by projection to the first factor.

We write $T \subset B \subset \widetilde{G}$ for the subgroups consisting, respectively, of the diagonal and upper-triangular matrices in $\widetilde{G}$. Similarly we write $G \subset P \subset \widetilde{G}$ for the subgroups consisting, respectively, of the block upper diagonal and block upper-triangular matrices with blocks of size $n \times n$. Then $P = U \rtimes G$, where U is the unipotent radical of P, and we can identify G with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} \mathrm{GL}_n$ via the map

$$g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mapsto D \in \mathrm{GL}_n(R \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F).$$

We observe that after extending scalars to F^+, T and B form a maximal torus and a Borel subgroup, respectively, of $\widetilde{G}$, and G is the unique Levi subgroup of the parabolic subgroup P of $\widetilde{G}$ containing T.

In order to simplify notation, we now write $\widetilde{X} = X^{\widetilde{G}}$ and $X = X^G$. Similarly, we will use the symbols $\widetilde{K}$ and K to denote good subgroups of $\widetilde{G}(\mathbf{A}^\infty_{F^+})$ and $G(\mathbf{A}^\infty_{F^+}) = \mathrm{GL}_n(\mathbf{A}^\infty_F)$, respectively.

We note that the dimensions of these symmetric spaces are

$$\dim_{\mathbf{R}} \widetilde{X} = 2n^2[F^+ : \mathbf{Q}], \qquad \dim_{\mathbf{R}} X = n^2[F^+ : \mathbf{Q}] - 1.$$

We now want to describe some explicit (rational and integral) coefficient systems for these symmetric spaces. The integral coefficient systems we define will depend on a choice of a prime p and a dominant weight for either G or $\widetilde{G}$. We therefore fix a prime p and a finite extension $E/\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$ that contains the images of all embeddings $F \hookrightarrow \overline{\mathbf{Q}}_p$. We write $\mathcal{O}$ for the ring of integers of E, and $\varpi \in \mathcal{O}$ for a choice of uniformizer.

We first treat the case of G. Let Ω be a field of characteristic 0 and large enough such that $\mathrm{Hom}(F, \Omega)$ has $[F : \mathbf{Q}]$ elements. We identify the character group $X^*((\mathrm{Res}_{F^+/\mathbf{Q}} T)_\Omega)$ with $(\mathbf{Z}^n)^{\mathrm{Hom}(F,\Omega)}$ in the usual way, by identifying

$$(\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n)_\Omega = \prod_{\tau \in \mathrm{Hom}(F,\Omega)} \mathrm{GL}_n$$

and by identifying $(\lambda_1, \ldots, \lambda_n) \in \mathbf{Z}^n$ with the character

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{\lambda_1} \ldots t_n^{\lambda_n}$$

of the diagonal maximal torus in GL_n. The $\mathrm{Res}_{F^+/\mathbf{Q}}(B \cap G)_\Omega$-dominant weights are exactly those in the subset $(\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\Omega)}$ given by those tuples $(\lambda_{\tau,i})$ satisfying the condition

$$\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \cdots \geq \lambda_{\tau,n}$$

for each $\tau \in \mathrm{Hom}(F,\Omega)$.

Associated to λ we have the algebraic representation V_λ of $(\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)_\Omega$ of highest weight λ. We may identify $V_\lambda = \otimes_{\tau\in\mathrm{Hom}(F,\Omega)} V_{\lambda_\tau}$, where V_{λ_τ} is the irreducible representation of $\mathrm{GL}_{n,\Omega}$ of highest weight λ_τ. If $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\Omega)}$, we define $\lambda^\vee \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\Omega)}$ by the formula $\lambda^\vee_{\tau,i} = -\lambda_{\tau,n+1-i}$. Then there is an isomorphism $V_\lambda^\vee \cong V_{\lambda^\vee}$, although this is not true for the integral lattices defined below without further hypotheses on λ.

Now take $\Omega = E$. For each $\tau \in \mathrm{Hom}(F,E)$, we let $\mathcal{V}_{\lambda_\tau} \subset V_{\lambda_\tau}$ be the $\mathrm{GL}_n(\mathcal{O})$-invariant $\mathcal{O}$-lattice defined in [Ger19, §2.2] (and called M_{λ_τ} there). We note that this is the integral dual Weyl module of highest weight λ_τ, obtained by evaluating an algebraic induction on $\mathcal{O}$. (Geometrically, the dual Weyl module of highest weight λ_τ is obtained as in the Borel–Weil theorem, by taking global sections of a line bundle determined by λ_τ over the full flag variety associated to GL_n.) We write $\mathcal{V}_\lambda = \otimes_{\tau\in\mathrm{Hom}(F,E)}\mathcal{V}_{\lambda_\tau}$ for the corresponding $\mathcal{O}$-lattice in V_λ. Thus $\mathcal{V}_\lambda$ is an $\mathcal{O}[\prod_{v|p}\mathrm{GL}_n(\mathcal{O}_{F_v})]$-module, finite free as $\mathcal{O}$-module.

We next treat the case of $\widetilde{G}$. Let $\widetilde{I} \subset \mathrm{Hom}(F,\Omega)$ be a subset such that $\mathrm{Hom}(F,\Omega) = \widetilde{I} \sqcup \widetilde{I}^c$. Given $\tau \in \mathrm{Hom}(F^+,E)$, we will sometimes write $\widetilde{\tau}$ for the unique element of $\widetilde{I}$ extending τ. The choice of $\widetilde{I}$ determines an isomorphism

$$(\mathrm{Res}_{F^+/\mathbf{Q}}\,\widetilde{G})_\Omega \cong \prod_{\tau\in\mathrm{Hom}(F^+,\Omega)} \mathrm{GL}_{2n,\Omega}$$

taking $(\mathrm{Res}_{F^+/\mathbf{Q}}\,T)_\Omega$ to the product of the diagonal maximal tori in the GL_{2n}'s, and hence an identification of the character group $X^*((\mathrm{Res}_{F^+/\mathbf{Q}}\,T)_\Omega)$ with $(\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,\Omega)}$. The $(\mathrm{Res}_{F^+/\mathbf{Q}}\,B)_\Omega$-dominant weights are exactly those in the subset $(\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,\Omega)}$. The isomorphism $(\mathbf{Z}^n)^{\mathrm{Hom}(F,\Omega)} \cong (\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,\Omega)}$ identifies a weight λ with the weight $\widetilde{\lambda} = (\widetilde{\lambda}_{\tau,i})$ given by the formula

$$\widetilde{\lambda}_\tau = (-\lambda_{\widetilde{\tau}c,n}, \ldots, -\lambda_{\widetilde{\tau}c,1}, \lambda_{\widetilde{\tau},1}, \lambda_{\tau,2}, \ldots, \lambda_{\widetilde{\tau},n}). \tag{2.2.2}$$

Now let $\Omega = E$. We define integral structures under the assumption that each place $\overline{v}$ of F^+ above p splits in F. Let S_p denote the set of p-adic places of F, and let $\overline{S}_p$ denote the set of p-adic places of F^+. Let $\widetilde{S}_p \subset S_p$ be a subset such that $S_p = \widetilde{S}_p \sqcup \widetilde{S}_p^c$. Let $\widetilde{I} = \widetilde{I}_p$ denote the set of embeddings $\tau : F \hookrightarrow E$ inducing an element of $\widetilde{S}_p$. Given $\overline{v} \in \overline{S}_p$, we will sometimes write $\widetilde{v}$ for the unique element of $\widetilde{S}_p$ lying above $\overline{v}$.

The choice of $\widetilde{S}_p$ determines isomorphisms

$$\widetilde{G} \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_{F^+,p} \cong \prod_{\overline{v} \in \overline{S}_p} \mathrm{GL}_{2n,\mathcal{O}_{F^+_{\overline{v}}}}.$$

The lattice $\mathcal{V}_{\widetilde{\lambda}} \subset V_{\widetilde{\lambda}}$ corresponding to a dominant weight $\widetilde{\lambda} \in X^*((\mathrm{Res}_{F^+/\mathbf{Q}} \widetilde{G})_E)$ is defined as in the previous paragraph. Thus $\mathcal{V}_{\widetilde{\lambda}}$ is an $\mathcal{O}[\prod_{\overline{v}|p} \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})]$-module, finite free as $\mathcal{O}$-module.

We can now define Hecke algebras. Again, we do this first for G. Let S be a finite set of finite places of F containing the p-adic ones, and let K be a good subgroup of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$ such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ if $v \notin S$ and $K_v \subset \mathrm{GL}_n(\mathcal{O}_{F_v})$ if $v|p$. Then for any $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, the complex $R\Gamma(X_K, \mathcal{V}_\lambda)$ is defined (as an object of $\mathbf{D}(\mathcal{O})$, up to unique isomorphism) and comes equipped with an action of Hecke algebras by endomorphisms; see (2.1.3). We define $\mathbf{T}_v = \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v})) \otimes_{\mathbf{Z}} \mathcal{O}$ and $\mathbf{T}^S = \mathcal{H}(\mathrm{GL}_n^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ and, if $\mathcal{V}$ is an $\mathcal{O}[K_S]$-module, finite free as $\mathcal{O}$-module, then we write $\mathbf{T}^S(K, \mathcal{V})$ for the image of the $\mathcal{O}$-algebra homomorphism

$$\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}))$$

constructed in Section 2.1.2. If $\mathcal{V} = \mathcal{V}_\lambda$, then we even write $\mathbf{T}^S(K, \lambda) = \mathbf{T}^S(K, \mathcal{V}_\lambda)$.

We now treat the case of $\widetilde{G}$. Let S be a finite set of finite places of F containing the p-adic ones and such that $S = S^c$, and let $\overline{S}$ denote the set of places of F^+ below a place of S. Let $\widetilde{K}$ be a good subgroup of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ such that $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$ for each place $\overline{v} \notin \overline{S}$, and such that $\widetilde{K}_{\overline{v}} \subset \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$ for each place $\overline{v}|p$. In order to simplify notation, we set $\widetilde{G}^S = \widetilde{G}^{\overline{S}}$, $\widetilde{G}_S = \widetilde{G}_{\overline{S}}$, and similarly $\widetilde{K}^S = \widetilde{K}^{\overline{S}}$ and $\widetilde{K}_S = \widetilde{K}_{\overline{S}}$.

For any $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, the complex $R\Gamma(X_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$ is defined, and it comes equipped with an action as in (2.1.3). We define $\widetilde{\mathbf{T}}^S = \mathcal{H}(\widetilde{G}^S, \widetilde{K}^S) \otimes_{\mathbf{Z}} \mathcal{O}$, and if $\widetilde{\mathcal{V}}$ is an $\mathcal{O}[\widetilde{K}_S]$-module, finite free as $\mathcal{O}$-module, then we write $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\mathcal{V}})$ for the image of the $\mathcal{O}$-algebra homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_{\widetilde{K}}, \widetilde{\mathcal{V}}))$$

constructed in Section 2.1.2. If $\widetilde{\mathcal{V}} = \mathcal{V}_{\widetilde{\lambda}}$, then we even write $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda}) = \widetilde{\mathbf{T}}^S(\widetilde{K}, \mathcal{V}_{\widetilde{\lambda}})$. We also denote by

$$\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S \tag{2.2.3}$$

the map induced by (2.1.8).

Note that Lemma 2.1.7 shows that both $\mathbf{T}^S(K, \lambda)$ and $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$ are finite $\mathcal{O}$-algebras. We emphasize that the Hecke algebra $\mathbf{T}^S(K, \lambda)$ is defined only under the assumptions that S contains the p-adic places of F, that K is a

good subgroup such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all $v \notin S$, and that λ is a dominant weight for G. The use of this notation therefore implies that these assumptions are in effect. Similar remarks apply to the Hecke algebra $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$. (In particular, the use of this notation implies that S is stable under complex conjugation, a condition we do not impose for GL_n.)

If $K' \subset K$ is a normal good subgroup with $(K')^S = K^S$, R is an $\mathcal{O}$-algebra, and $\mathcal{V}$ is an $R[K_S]$-module, finite free as R-module, then we write $\mathbf{T}^S(K/K', \mathcal{V}) = \mathbf{T}^S(R\Gamma_{K/K'}(X_{K'}, \mathcal{V}))$ for the image of the homomorphism (cf. 2.1.5):

$$\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(R[K/K'])}(R\Gamma_{K/K'}(X_{K'}, \mathcal{V})).$$

There are canonical surjective homomorphisms

$$\mathbf{T}^S(R\Gamma_{K/K'}(X_{K'}, \mathcal{V})) \to \mathbf{T}^S(K', \mathcal{V}), \quad \mathbf{T}^S(R\Gamma_{K/K'}(X_{K'}, \mathcal{V})) \to \mathbf{T}^S(K, \mathcal{V})$$

induced respectively by the forgetful functor $\mathbf{D}(R[K/K']) \to \mathbf{D}(R)$ and the functor $R\Gamma(K/K', ?) : \mathbf{D}(R[K/K']) \to \mathbf{D}(R)$. If further K/K' is abelian, then we define ${}_{K/K'}\mathbf{T}^S = \mathbf{T}^S \otimes_{\mathcal{O}} \mathcal{O}[K/K']$ and write ${}_{K/K'}\mathbf{T}^S(R\Gamma_{K/K'}(X_{K'}, \mathcal{V}))$ for the image of the homomorphism

$${}_{K/K'}\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(R[K/K'])}(R\Gamma_{K/K'}(X_{K'}, \mathcal{V})).$$

The analogous construction is valid as well for $\widetilde{G}$ but since we will not use it, we do not write down the definition.

We will also occasionally encounter other complexes endowed with actions of the rings $\mathbf{T}^S$ and $\widetilde{\mathbf{T}}^S$. (For example, the cohomology $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$ of the boundary $\partial\widetilde{X}_{\widetilde{K}}$ of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$.) If $C^\bullet \in \mathbf{D}(R)$ and we are given an $\mathcal{O}$-algebra homomorphism $\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(R)}(C^\bullet)$, then we will write $\mathbf{T}^S(C^\bullet)$ for the image of this homomorphism. More generally, if $K' \subset K$ is a normal good subgroup with $(K')^S = K^S$ and $C^\bullet \in \mathbf{D}(R[K/K'])$ is a complex endowed with an $\mathcal{O}$-algebra homomorphism $\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}(R[K/K'])}(C^\bullet)$, then we will write $\mathbf{T}^S(C^\bullet)$ for the image of $\mathbf{T}^S$ in $\mathrm{End}_{\mathbf{D}(R[K/K'])}(C^\bullet)$. If further K/K' is abelian, then we will write ${}_{K/K'}\mathbf{T}^S(C^\bullet)$ for the image of ${}_{K/K'}\mathbf{T}^S$ in $\mathrm{End}_{\mathbf{D}(R[K/K'])}(C^\bullet)$.

If the complex $C^\bullet$ has bounded cohomology, then the map $\mathbf{T}^S(C^\bullet) \to \mathbf{T}^S(H^*(C^\bullet))$ has nilpotent kernel; this is a consequence of the following lemma:

Lemma 2.2.4. *Let R be a (possibly non-commutative) $\mathbf{Z}$-algebra, let $C^\bullet \in \mathbf{D}(R)$ be a complex, and let $T \subset \mathrm{End}_{\mathbf{D}(R)}(C^\bullet)$ be a commutative subring. Let $I = \ker(T \to \mathrm{End}_R(H^*(C^\bullet)))$, and suppose that there exists an integer $d \geq 0$ such that $H^i(C^\bullet) = 0$ if $i \notin [0, d]$. Then $I^{d+1} = 0$.*

Proof. We show by induction on $d \geq 0$ that if $\phi_0, \ldots, \phi_d \in \mathrm{End}_{\mathbf{D}(R)}(C^\bullet)$ satisfy $H^*(\phi_i) = 0$ for $i = 0, \ldots, d$, then $\phi_0 \circ \phi_1 \circ \cdots \circ \phi_d = 0$ in $\mathrm{End}_{\mathbf{D}(R)}(C^\bullet)$. The case $d = 0$ follows because in this case, $C^\bullet$ is isomorphic to $H^0(C^\bullet)$ in $\mathbf{D}(R)$.

In general, we can assume that $\tau_{\leq d-1}(\phi_0 \circ \cdots \circ \phi_{d-1}) = 0$. There is an exact triangle

$$\tau_{\leq d-1} C^\bullet \xrightarrow{f} C^\bullet \xrightarrow{g} H^d(C^\bullet) \longrightarrow.$$

We obtain exact sequences

$$\mathrm{Hom}_{\mathbf{D}(R)}(H^d(C^\bullet), C^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(C^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(\tau_{\leq d-1} C^\bullet, C^\bullet)$$

and

$$\mathrm{Hom}_{\mathbf{D}(R)}(C^\bullet, \tau_{\leq d-1} C^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(C^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_{\mathbf{D}(R)}(C^\bullet, H^d(C^\bullet)).$$

We deduce the existence of elements $\alpha \in \mathrm{Hom}_{\mathbf{D}(R)}(H^d(C^\bullet), C^\bullet)$ and $\beta \in \mathrm{Hom}_{\mathbf{D}(R)}(C^\bullet, \tau_{\leq d-1} C^\bullet)$ such that $\alpha \circ g = \phi_0 \circ \cdots \circ \phi_{d-1}$ and $f \circ \beta = \phi_d$. We thus conclude that $\phi_0 \circ \cdots \circ \phi_d = \alpha \circ g \circ f \circ \beta = 0$. □

As an illustration of the use of this result, suppose that $K' \subset K$ is a normal good subgroup with $(K')^S = K^S$, so that the Hecke algebra $\mathbf{T}^S(K/K', \mathcal{V}_\lambda)$ is defined. We then have a diagram of Hecke algebras

$$\mathbf{T}^S(K', \mathcal{V}_\lambda) \leftarrow \mathbf{T}^S(K/K', \mathcal{V}_\lambda) \rightarrow \mathbf{T}^S(K, \mathcal{V}_\lambda),$$

where the kernel I of the left arrow satisfies $I^{\dim_{\mathbf{R}} X} = 0$ (by Lemma 2.2.4 applied with $R = \mathcal{O}[K/K']$). In particular, if $J \subset \mathbf{T}^S(K, \mathcal{V}_\lambda)$ denotes the image of I, then there exists a canonical map $\mathbf{T}^S(K', \mathcal{V}_\lambda) \rightarrow \mathbf{T}^S(K, \mathcal{V}_\lambda)/J$ of $\mathbf{T}^S$-algebras. Similar statements for the Hecke algebras which act faithfully on cohomology could be proved using the Hochschild–Serre spectral sequence (for the covering $X_{K'} \rightarrow X_K$).

Nilpotent ideals of Hecke algebras will occur frequently throughout this paper, and they often have their origins in applications of the above Lemma 2.2.4. (Compare, for example, the statement and proof of Proposition 2.3.9 below.) We note that the integer $\dim_{\mathbf{R}} X$ depends only on n, and the degree $[F : \mathbf{Q}]$; the exponents of the nilpotent ideals we consider will also usually have this property.

2.2.5. *Some useful Hecke operators.* In this section we define most of the Hecke operators that we will need at various points later in the paper. We fix once and for all a choice ϖ_v of uniformizer at each finite place v of F.

We first define notation for unramified Hecke operators. If v is a finite place of F and $1 \leq i \leq n$ is an integer, then we write $T_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))$ for the double coset operator

$$T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v}) \operatorname{diag}(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1) \mathrm{GL}_n(\mathcal{O}_{F_v})],$$

where ϖ_v appears i times on the diagonal. This is the same as the operator denoted by $T_{M,v,i}$ in [NT16, Prop.-Def. 5.3]. We define a polynomial

$$\begin{aligned} P_v(X) = X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots \\ \cdots + q_v^{n(n-1)/2} T_{v,n} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))[X]. \end{aligned} \tag{2.2.6}$$

It corresponds to the characteristic polynomial of a Frobenius element on $\mathrm{rec}^T_{F_v}(\pi_v)$, where π_v is an unramified representation of $\mathrm{GL}_n(F_v)$. We also find it helpful to introduce, for any $\sigma \in W_{F_v}$, the polynomial $P_{v,\sigma}(X) \in \mathbf{T}_v[X] = (\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v})) \otimes_{\mathbf{Z}} \mathcal{O})[X]$, which equals the characteristic polynomial of σ on $\mathrm{rec}^T_{F_v}(\pi_v)$. We write $P_{v,\sigma}(X) = \sum_{i=0}^n (-1)^i e_{v,i}(\sigma) X^{n-i}$.

If $\overline{v}$ is a place of F^+ unramified in F, and v is a place of F above $\overline{v}$, and $1 \le i \le 2n$ is an integer, then we write $\widetilde{T}_{v,i} \in \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \otimes_{\mathbf{Z}} \mathbf{Z}[q_{\overline{v}}^{-1}]$ for the operator denoted $T_{G,v,i}$ in [NT16, Prop.-Def. 5.2]. We define a polynomial

$$\begin{aligned} \widetilde{P}_v(X) = X^{2n} - \widetilde{T}_{v,1}X^{2n-1} + \cdots + (-1)^j q_v^{j(j-1)/2} \widetilde{T}_{v,j} + \cdots \\ \cdots + q_v^{n(2n-1)} \widetilde{T}_{v,2n} \in \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \otimes_{\mathbf{Z}} \mathbf{Z}[q_{\overline{v}}^{-1}][X]. \end{aligned} \tag{2.2.7}$$

It corresponds to the characteristic polynomial of a Frobenius element on $\mathrm{rec}^T_{F_v}(\pi_v)$, where π_v is the base change of an unramified representation $\sigma_{\overline{v}}$ of the group $\widetilde{G}(F^+_{\overline{v}})$. Again, if $\sigma \in W_{F_v}$, then we write

$$\widetilde{P}_{v,\sigma}(X) = \sum_{i=0}^{2n} (-1)^i \widetilde{e}_{v,i}(\sigma) X^{n-i} \in \widetilde{\mathbf{T}}_v[X] = (\mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \otimes_{\mathbf{Z}} \mathcal{O})[X]$$

for the polynomial corresponding to the characteristic polynomial of σ on $\mathrm{rec}^T_{F_v}(\pi_v)$.

We next define notation for some ramified Hecke operators. If v is a finite place of F, and $c \ge b \ge 0$ are integers, then we write $\mathrm{Iw}_v(b,c)$ for the subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$ consisting of those matrices that reduce to an upper-triangular matrix modulo ϖ_v^c and to a unipotent upper-triangular matrix modulo ϖ_v^b. We define $\mathrm{Iw}_v = \mathrm{Iw}(0,1)$ and $\mathrm{Iw}_{v,1} = \mathrm{Iw}_v(1,1)$; thus Iw_v is the standard Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$. If $1 \le i \le n$ is an integer and $c \ge 1$, then we will write $U_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{Iw}_v(b,c))$ for the double coset operator

$$U_{v,i} = [\mathrm{Iw}_v(b,c)\,\mathrm{diag}(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1)\mathrm{Iw}_v(b,c)],$$

where ϖ_v appears i times on the diagonal. Note that this depends both on the uniformizer ϖ_v and on the chosen level. We hope that this abuse of notation will not cause confusion. We also define

$$U_v = [\mathrm{Iw}_v(b,c)\,\mathrm{diag}(\varpi_v^{n-1}, \varpi_v^{n-2}, \ldots, \varpi_v, 1)\mathrm{Iw}_v(b,c)].$$

If $u \in T_n(\mathcal{O}_{F_v})$, then we define

$$\langle u \rangle = [\mathrm{Iw}_v(b,c) u \mathrm{Iw}_v(b,c)].$$

Note that the subgroups $\mathrm{Iw}_v(b,c)$ all admit Iwahori decompositions with respect to the standard upper-triangular Borel subgroup of GL_n. We write $\Delta_v \subset \mathrm{GL}_n(F_v)$ for the submonoid defined by

$$\Delta_v = \sqcup_{\mu \in \mathbf{Z}_+^n} \mathrm{Iw}_v \operatorname{diag}(\varpi_v^{\mu_1}, \dots, \varpi_v^{\mu_n}) \mathrm{Iw}_v.$$

We now assume that each p-adic place of F^+ splits in F. In this case we set $\Delta_p = \prod_{v \in S_p} \Delta_v$. If $\lambda \in (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$, then we define a homomorphism (of monoids) $\alpha_\lambda : \Delta_p \to E^\times$ by the formula

$$\begin{aligned}\alpha_\lambda((k_{v,1} \operatorname{diag}(\varpi_v^{a_{v,1}}, \dots, \varpi_v^{a_{v,n}}) k_{v,2})_{v \in S_p}) \\ = \prod_{v \in S_p} \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)} \prod_{i=1}^n \tau(\varpi_v)^{a_{v,i}(w_0^G \lambda)_{\tau,i}},\end{aligned}$$

where w_0^G is the longest element in the Weyl group

$$W((\mathrm{Res}_{F^+/\mathbf{Q}} G)_E, (\mathrm{Res}_{F^+/\mathbf{Q}} T)_E).$$

If $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ is dominant, then $\mathrm{GL}_n(F_p)$ acts on $\mathcal{V}_\lambda \otimes_{\mathcal{O}} E = V_\lambda$; we write $(g,x) \mapsto g \cdot x$ for this action. We endow $\mathcal{V}_\lambda$ with the structure of $\mathcal{O}[\Delta_p]$-module via the formula

$$g \cdot_p x = \alpha_\lambda(g)^{-1} g \cdot x.$$

This is well defined: the fact that the lattice $\mathcal{V}_\lambda$ is preserved under this twisted action follows as in [Ger19, Def. 2.8] from Lemma 2.2 of loc. cit.; the point is that $w_0^G \lambda$ is the lowest weight vector in $\mathcal{V}_\lambda$, so $g \cdot x$ is divisible by $\alpha_\lambda(g)$ when $g \in \Delta_p$. Using the construction in Section 2.1.2, we see that if $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ is a good subgroup, and for each place $v|p$ of F we have $K_v = \mathrm{Iw}_v(b,c)$, then there is a canonical homomorphism

$$\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{H}(\Delta_p, K_p) \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda))$$

and, in particular, all the Hecke operators $U_{v,i}$ and U_v act as endomorphisms of $R\Gamma(X_K, \mathcal{V}_\lambda)$. Note that the action of these operators depends on the choice of uniformizer ϖ_v, because the twisted action $\cdot_p$ does.

Now suppose that $\overline{v}$ is a finite place of F^+ that splits in F, and let v be a place of F above it. Then $\iota_v^{-1}(\mathrm{Iw}_v(b,c)) = \iota_{v^c}^{-1}(\mathrm{Iw}_{v^c}(b,c))$ (where here the subgroup $\mathrm{Iw}_v(b,c)$ is inside $\mathrm{GL}_{2n}(F_v)$), and we write $\widetilde{\mathrm{Iw}}_{\overline{v}}(b,c)$ for this subgroup of $\widetilde{G}(F_{\overline{v}}^+)$. We define a Hecke operator in $\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathrm{Iw}}_{\overline{v}}(b,c))$ for each $i = 1, \dots, 2n$ by the formula

$$\widetilde{U}_{v,i} = \iota_v^{-1}[\mathrm{Iw}_v(b,c) \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1) \mathrm{Iw}_v(b,c)],$$

where ϖ_v appears i times on the diagonal. We also define

$$\widetilde{U}_v = \iota_v^{-1}[\mathrm{Iw}_v(b,c) \operatorname{diag}(\varpi_v^{2n-1}, \varpi_v^{2n-2}, \dots, \varpi_v, 1) \mathrm{Iw}_v(b,c)].$$

If $u \in T(\mathcal{O}_{F^+,\overline{v}})$, then we define

$$\langle u \rangle = [\widetilde{\mathrm{Iw}}_{\overline{v}}(b,c) u \widetilde{\mathrm{Iw}}_{\overline{v}}(b,c)].$$

If $\varpi_{v^c} = \varpi_v^c$, then $\widetilde{U}_{v^c,i} = \widetilde{U}_{v,2n-i}\widetilde{U}_{v,2n}^{-1}$ and $\widetilde{U}_{v^c} = \widetilde{U}_v \widetilde{U}_{v,2n}^{1-2n}$.

We write $\widetilde{\Delta}_{\overline{v}} \subset \widetilde{G}(F_{\overline{v}}^+)$ for the submonoid defined by

$$\widetilde{\Delta}_{\overline{v}} = \iota_v^{-1}\left(\sqcup_{\mu \in \mathbf{Z}_+^{2n}} \mathrm{Iw}_v \operatorname{diag}(\varpi_v^{\mu_1}, \dots, \varpi_v^{\mu_{2n}}) \mathrm{Iw}_v\right)$$

(which is independent of the choice of v). Now suppose that each p-adic place of F^+ splits in F. In this case we set $\widetilde{\Delta}_p = \prod_{v \in \overline{S}_p} \widetilde{\Delta}_{\overline{v}}$. If $\widetilde{\lambda} \in (\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$, then we define a homomorphism $\widetilde{\alpha}_{\widetilde{\lambda}} : \widetilde{\Delta}_p \to E^\times$ by the formula

$$\begin{aligned}\widetilde{\alpha}_{\widetilde{\lambda}}((k_{\overline{v},1}\iota_{\widetilde{v}}^{-1}(\operatorname{diag}(\varpi_{\widetilde{v}}^{a_{\overline{v},1}}, \dots, \varpi_{\widetilde{v}}^{a_{\overline{v},2n}}))k_{\overline{v},2})_{\overline{v} \in \overline{S}_p}) \\ = \prod_{\overline{v} \in \overline{S}_p} \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}},E)} \prod_{i=1}^{2n} \tau(\varpi_{\widetilde{v}})^{a_{\overline{v},i}(w_0^{\widetilde{G}}\widetilde{\lambda})_{\tau,i}},\end{aligned}$$

where $w_0^{\widetilde{G}}$ is the longest element in the Weyl group

$$W((\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E, (\mathrm{Res}_{F^+/\mathbf{Q}} T)_E).$$

Here we recall that $\widetilde{v} \in \widetilde{S}_p$ is a fixed choice of place of F lying above $\overline{v}$, and that it appears also in the definition of $\mathcal{V}_{\widetilde{\lambda}}$. If $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ is dominant, then $\widetilde{G}_p$ acts on $\mathcal{V}_{\widetilde{\lambda}} \otimes_{\mathcal{O}} E = V_{\widetilde{\lambda}}$; we write $(g,x) \mapsto g \cdot x$ for this action. We endow $\mathcal{V}_{\widetilde{\lambda}}$ with the structure of $\mathcal{O}[\widetilde{\Delta}_p]$-module via the formula

$$g \cdot_p x = \widetilde{\alpha}_{\widetilde{\lambda}}(g)^{-1} g \cdot x.$$

Using the construction in Section 2.1.2, we see that if $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup, and for each place $\overline{v}|p$ of F^+ we have $\widetilde{K}_{\overline{v}} = \widetilde{\mathrm{Iw}}_{\overline{v}}(b,c)$, then there is a canonical homomorphism

$$\mathcal{H}(\widetilde{G}^S, \widetilde{K}^S) \otimes_{\mathbf{Z}} \mathcal{H}(\widetilde{\Delta}_p, \widetilde{K}_p) \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}))$$

and, in particular, all the Hecke operators $\widetilde{U}_{v,i}$ and $\widetilde{U}_v$ act as endomorphisms of $R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$.

If v is a finite place of F, prime to p, and I_v is an open compact subgroup of $\mathrm{GL}_n(F_v)$ satisfying $\mathrm{Iw}_v(1,1) \subset I_v \subset \mathrm{Iw}_v(0,1)$, then $\mathrm{Iw}_v(0,1)/I_v$ can be identified with a quotient of $(k(v)^\times)^n$, and we define

$$\Xi_v = (F_v^\times)^n / (\ker(\mathcal{O}_{F_v}^\times)^n \to (k(v)^\times)^n \to \mathrm{Iw}_v(0,1)/I_v).$$

The group I_v admits an Iwahori decomposition with respect to the parabolic subgroup $B_n = T_n N_n$ of GL_n, so we may apply the theory of Section 2.1.9. Moreover, [Fli11, Cor. 3.4] shows that for any $g \in \Xi_v$, the element $[I_v g I_v] \in \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}$ is invertible. (This uses our assumption that q_v is a

unit in $\mathcal{O}$.) Lemma 2.1.13 thus implies that there is an injective $\mathcal{O}$-algebra homomorphism

$$t : \Xi_v \to \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O},$$

which sends any positive $g \in \Xi_v$ to the double coset $[I_v g I_v]$. Given any $\alpha \in F_v^\times$, we write $t_{v,i}(\alpha) \in \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}$ for the image under t of the element $q_v^{-\langle \lambda, \rho + (1-n)/2 \det \rangle}(1, \dots, 1, \alpha, 1, \dots, 1)$ of $\mathcal{O}[\Xi_v]$, where $\lambda \in X_*(T_n)$ denotes the image of α under the natural projection $\Xi_v \to (F_v^\times / \mathcal{O}_{F_v}^\times)^n = X_*(T_n)$ and $\rho \in X^*(T_n)$ is the usual half-sum of positive roots, and where α sits in the ith position. We write $e_{v,i}(\alpha) \in \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}$ for the coefficient of $(-1)^i X^{n-i}$ in the polynomial $\prod_{i=1}^n (X - t_{v,i}(\alpha))$. If $\sigma \in W_{F_v}$, then we set $t_{v,i}(\sigma) = t_{v,i}(\alpha)$ and $e_{v,i}(\sigma) = e_{v,i}(\alpha)$, where $\alpha \in F_v^\times$ is such that the restriction of σ to the maximal abelian extension of F_v equals $\mathrm{Art}_{F_v}(\alpha)$. We define the polynomial

(2.2.8)

$$P_{v,\sigma}(X) = \prod_{i=1}^n (X - t_{v,i}(\sigma)) = \sum_{i=0}^n (-1)^i e_{v,i}(\sigma) X^{n-i} \in \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}[X].$$

Proposition 2.2.9. *Let π_v be an irreducible admissible $\overline{\mathbf{Q}}_p[\mathrm{GL}_n(F_v)]$-module.*

(1) *We have $\pi_v^{I_v} \neq 0$ if and only if π_v is isomorphic to an irreducible subquotient of a representation $\mathrm{Ind}_{B_n(F_v)}^{\mathrm{GL}_n(F_v)} \chi$, where $\chi = \chi_1 \otimes \cdots \otimes \chi_n : (F_v^\times)^n \to \overline{\mathbf{Q}}_p^\times$ is a smooth character that factors through the quotient $(F_v^\times)^n \to \Xi_v$.*
(2) *Suppose that $\pi_v^{I_v} \neq 0$. Then for any $\alpha \in F_v^\times$, $e_{v,i}(\alpha)$ acts on $\pi_v^{I_v}$ as a scalar $e_{v,i}(\alpha, \pi_v) \in \overline{\mathbf{Q}}_p^\times$.*
(3) *Suppose that $\pi_v^{I_v} \neq 0$, and let $(r_v, N_v) = \mathrm{rec}_{F_v}^T(\pi_v)$. Then for any $\sigma \in W_{F_v}$, the characteristic polynomial of $r_v(\sigma)$ is equal to $\sum_{i=0}^n (-1)^i e_{v,i}(\alpha, \pi_v) X^{n-i}$, where $\alpha = \mathrm{Art}_{F_v}^{-1}(\sigma|_{F_v^{ab}})$.*

Proof. The first part follows from [Fli11, Th. 2.1]. The second part is a consequence of the fact that the elements $e_{v,i}(\alpha)$ lie in the center of $\mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}$, which in turn follows from the explicit description of the center in [Fli11, Prop. 4.11]. The final part follows from the description in [Fli11, §4] of the action of this center on the I_v-invariants in the induced representation $\text{n-Ind}_{B_n(F_v)}^{\mathrm{GL}_n(F_v)} \chi$. □

Now suppose that v is a finite place of F, prime to p and split over F^+, and write $\mathfrak{p}_v \subset \mathrm{GL}_{2n}(\mathcal{O}_{F_v})$ for the parahoric subgroup consisting of matrices whose reduction modulo ϖ_v is block upper-triangular, with blocks of sizes $n, 1, 1, \dots, 1$. Projection to the lower right-hand block determines a homomorphism $\mathfrak{p}_v \to B_n(k(v))$. We write $\mathfrak{p}_{v,1} \subset \mathfrak{p}_v$ for the kernel of the composite homomorphism $\mathfrak{p}_v \to B_n(k(v)) \to T_n(k(v))$.

Let $\mathfrak{q}_v \subset \mathrm{GL}_{2n}(F_v)$ be an open compact subgroup such that $\mathfrak{p}_{v,1} \subset \mathfrak{q}_v \subset \mathfrak{p}_v$, and set $\widetilde{\mathfrak{p}}_v = \iota_v^{-1}(\mathfrak{p}_v)$, $\widetilde{\mathfrak{p}}_{v,1} = \iota_v^{-1}(\mathfrak{p}_{v,1})$, and $\widetilde{\mathfrak{q}}_v = \iota_v^{-1}(\mathfrak{q}_v)$. These are open compact subgroups of $\widetilde{G}(F_{\overline{v}}^+)$ and we can identify $\widetilde{\mathfrak{p}}_v \cap G(F_{\overline{v}}^+) = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times \mathrm{Iw}_v$ and $\widetilde{\mathfrak{p}}_{v,1} \cap G(F_{\overline{v}}^+) = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times \mathrm{Iw}_{v,1}$. In particular, we may identify the quotient $\widetilde{\mathfrak{p}}_v/\widetilde{\mathfrak{p}}_{v,1}$ with $(k(v)^\times)^n$. The group $\widetilde{\mathfrak{q}}_v$ admits an Iwahori decomposition with respect to the parabolic subgroup $P = GU$, so we may use the theory developed in Section 2.1.9.

Lemma 2.2.10. *The element $g = (\varpi_v^{-c} \cdot 1_n, 1_n) \in \mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v) = G(F_{\overline{v}}^+)$ is strongly positive, and the element $[\widetilde{\mathfrak{q}}_v g \widetilde{\mathfrak{q}}_v] \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ is invertible.*

Proof. After applying ι_v, we see that to prove the lemma it is enough to explain why $[\mathfrak{q}_v \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\mathfrak{q}_v]$ is an invertible element of $\mathcal{H}(\mathrm{GL}_{2n}(F_v), \mathfrak{q}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ (where ϖ_v, 1 each occur n times). It follows from [Fli11, Cor. 3.4] that $[\mathrm{Iw}_v(1) \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\mathrm{Iw}_v(1)]$ is invertible in $\mathcal{H}(\mathrm{GL}_{2n}(F_v), \mathrm{Iw}_v(1)) \otimes_{\mathbf{Z}} \mathcal{O}$, while it follows from [Fli11, Th. 4.5] that $[\mathrm{Iw}_v(1) \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\mathrm{Iw}_v(1)]$ and $[\mathfrak{q}_v]$ commute. We deduce that $[\mathfrak{q}_v \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\mathfrak{q}_v] = [\mathfrak{q}_v] \cdot [\mathrm{Iw}_v(1) \operatorname{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\mathrm{Iw}_v(1)]$ is invertible, as required. □

Lemma 2.1.13 implies that there is an injective $\mathcal{O}$-algebra homomorphism

$$t : \mathcal{H}(\mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v) \otimes_{\mathbf{Z}} \mathcal{O} \to \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}.$$

We write Ξ_v for the quotient of $(F_v^\times)^n$ associated to the group I_v. If $\sigma \in W_{F_v}$, we write, with apologies for the abuse of notation, $t_{v,i}(\sigma) \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ for the image under t of the element $\|\sigma\|_v^{-n} t_{v,i}(\sigma) \in \mathcal{H}(\mathrm{GL}_n(F_v), I_v) \otimes_{\mathbf{Z}} \mathcal{O}$ defined previously. We write $e_{v,i}(\sigma) \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ for the coefficient of $(-1)^i X^{n-i}$ in the polynomial

$$P_{v,\sigma}(X) = \prod_{i=1}^{n} (X - t_{v,i}(\sigma)) \in (\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O})[X]. \tag{2.2.11}$$

If $\sigma \in W_{F_{v^c}}$, then we define $e_{v^c,i}(\sigma) \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ to be the image under t of the element $\|\sigma\|_v^{i(n-1)} e_{v^c,i}(\sigma) \in \mathcal{H}(\mathrm{GL}_n(F_{v^c}), \mathrm{GL}_n(\mathcal{O}_{F_{v^c}})) \otimes_{\mathbf{Z}} \mathcal{O}$. We define the polynomial

$$P_{v^c,\sigma}(X) = \sum_{i=0}^{n} (-1)^i e_{v,i}(\sigma) X^{n-i} \in (\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O})[X]. \tag{2.2.12}$$

We finally define for any $\sigma \in W_{F_v}$ the polynomial

$$\widetilde{P}_{v,\sigma}(X) = P_{v^c,\sigma^{-c}}(X) P_{v,\sigma}(X) \in (\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O})[X],$$

and we define elements $\widetilde{e}_{v,i}(\sigma) \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O}$ by the formula $\widetilde{P}_{v,\sigma}(X) = \sum_{i=0}^{2n} (-1)^i \widetilde{e}_{v,i}(\sigma) X^{n-i}$.

LEMMA 2.2.13. *Suppose there is an irreducible admissible* $\overline{\mathbf{Q}}_p[\widetilde{G}(F_{\overline{v}}^+)]$-*module* $\widetilde{\pi}_{\overline{v}}$ *such that* $\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v} \neq 0$, *and let* $\sigma \in W_{F_v}$. *Then*

(1) *Each operator* $\widetilde{e}_{v,i}(\sigma)$ *acts by a scalar* $\widetilde{e}_{v,i}(\sigma, \widetilde{\pi}_{\overline{v}}) \in \overline{\mathbf{Q}}_p$ *on* $\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v}$.

(2) *Let* $(r_v, N_v) = \mathrm{rec}_{F_v}^T(\widetilde{\pi}_{\overline{v}} \circ \iota_v^{-1})$. *Then the characteristic polynomial of* $r_v(\sigma)$ *is equals* $\sum_{i=0}^{2n}(-1)^i \widetilde{e}_{v,i}(\sigma, \widetilde{\pi}_{\overline{v}}) X^{n-i}$.

Proof. We fix a choice of isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, so that normalized induction and normalized restriction (i.e., Jacquet module) may be defined over $\overline{\mathbf{Q}}_p$. The proof uses well-known principles (cf. [Ber84, Lem. 1.17]). First, there exists a tamely ramified character $\chi : T(F_{\overline{v}}^+) \to \overline{\mathbf{Q}}_p^\times$ such that $\widetilde{\pi}_{\overline{v}}$ is a subquotient of an induced representation $\widetilde{\Pi} = \text{n-Ind}_{B(F_{\overline{v}}^+)}^{\widetilde{G}(F_{\overline{v}}^+)} \chi$. Identifying $T(F_{\overline{v}}^+) = T_{2n}(F_v)$, we may identify χ with a tuple of tamely ramified characters $\psi_1, \ldots, \psi_{2n} : F_v^\times \to \overline{\mathbf{Q}}_p$. To prove the lemma, it suffices to show that $\widetilde{e}_{v,i}(\sigma)$ acts as a scalar on the subspace of $\widetilde{\mathfrak{q}}_v$-invariants of $\widetilde{\Pi}$, this scalar being equal to the degree i symmetric polynomial in $\psi_1(\mathrm{Art}_{F_v}^{-1}(\sigma)), \ldots, \psi_{2n}(\mathrm{Art}_{F_v}^{-1}(\sigma))$.

Let $R = \overline{\mathbf{Q}}_p[T(F_{\overline{v}}^+)/T(\mathcal{O}_{F_{\overline{v}}^+})]$, and let $\chi_u : T(F_{\overline{v}}^+) \to R^\times$ be the universal unramified character. We consider the induced representation $\widetilde{\Pi}_u = \text{n-Ind}_{B(F_{\overline{v}}^+)}^{\widetilde{G}(F_{\overline{v}}^+)}(\chi \otimes \chi_u)$, a smooth $R[\widetilde{G}(F_{\overline{v}}^+)]$-module. Then $\widetilde{\Pi}_u^{\widetilde{\mathfrak{q}}_v}$ is a finite free R-module, and for any homomorphism $x : R \to \overline{\mathbf{Q}}_p$, corresponding to an unramified character $\chi_x : T(F_{\overline{v}}^+) \to \overline{\mathbf{Q}}_p^\times$ with induced representation $\widetilde{\Pi}_x = \text{n-Ind}_{B(F_{\overline{v}}^+)}^{\widetilde{G}(F_{\overline{v}}^+)}(\chi \otimes \chi_x)$, the induced map

$$(\widetilde{\Pi}_u^{\widetilde{\mathfrak{q}}_v}) \otimes_{R,x} \overline{\mathbf{Q}}_p \to \widetilde{\Pi}_x^{\widetilde{\mathfrak{q}}_v}$$

is an isomorphism. We may identify $\chi \otimes \chi_x$ with a tuple $\psi_{x,1}, \ldots, \psi_{x,2n} : F_v^\times \to \overline{\mathbf{Q}}_p$ of tamely ramified characters. To prove the lemma, it in fact suffices to show for a Zariski dense set of points $x \in \operatorname{Spec} R(\overline{\mathbf{Q}}_p)$ that the Hecke operator $\widetilde{e}_{v,i}(\sigma)$ acts by a scalar on $\widetilde{\Pi}_x^{\widetilde{\mathfrak{q}}_v}$ that is equal to the degree i symmetric polynomial in $\psi_{x,1}(\mathrm{Art}_{F_v}^{-1}(\sigma)), \ldots, \psi_{x,2n}(\mathrm{Art}_{F_v}^{-1}(\sigma))$.

Consider the Jacquet module $r_P(\widetilde{\Pi}_x) = (\widetilde{\Pi}_x)_{U(F_{\overline{v}}^+)} \otimes \delta_P^{-1/2}$, an admissible $\overline{\mathbf{Q}}_p[G(F_{\overline{v}}^+)]$-module. Then [BK98, Th. 7.9] asserts that the natural map

$$q : (\widetilde{\Pi}_x)^{\widetilde{\mathfrak{q}}_v} \to r_P(\widetilde{\Pi}_x)^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v}$$

is an isomorphism that satisfies the formula

$$hq(x) = \delta_P^{1/2}(g) q(t(h)x)$$

for any $x \in (\widetilde{\Pi}_x)^{\widetilde{\mathfrak{q}}_v}$ and Hecke operator $h = [(\mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v) g (\mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v)] \in \mathcal{H}(\mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v) \otimes_{\mathbf{Z}} \mathcal{O}$. The geometrical lemma ([Zel80, 1.2, Theorem]) asserts that $r_P(\widetilde{\Pi}_x)$ admits a filtration by induced

representations σ_{x,Y_{v^c},Y_v}, indexed by partitions $\{1,\dots,2n\} = Y_{v^c} \sqcup Y_v$, which may be described as follows:

$$\sigma_{x,Y_{v^c},Y_v} = \left(\text{n-Ind}_{B_n(F_{v^c})}^{\mathrm{GL}_n(F_{v^c})} \otimes_{i\in Y_{v^c}} \psi_{x,i}^{-c}\right) \otimes \left(\text{n-Ind}_{B_n(F_v)}^{\mathrm{GL}_n(F_v)} \otimes_{i\in Y_v} \psi_{x,i}\right).$$

For a Zariski dense set of points x (including those for which the central element $(\varpi_{v^c}\cdot 1_n, 1_n) \in \mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v)$ acts by a distinct scalar on each induced representation σ_{x,Y_{v^c},Y_v}), this filtration splits and $r_P(\widetilde{\Pi}_x)$ is isomorphic to a sum of induced representations. The Hecke operators $e_{v,i}(\sigma)$ and $e_{v^c,i}(\sigma^{-c})$ act as a scalar in the subspace of $\mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v$-invariants in each summand, and a calculation shows that the scalar by which $\sum_{i+j=k} e_{v,i}(\sigma)e_{v^c,j}(\sigma^{-c})$ acts in each summand is the degree k elementary symmetric polynomial in $\psi_{x,1}(\mathrm{Art}_{F_v}^{-1}(\sigma)),\dots,\psi_{x,2n}(\mathrm{Art}_{F_v}^{-1}(\sigma))$ — in particular, independent of the choice of summand.

Transferring this information back along the map q shows that at such points x, the element $\widetilde{e}_{v,k}(\sigma) = \sum_{i+j=k} e_{v,i}(\sigma)e_{v^c,j}(\sigma^{-c})$ acts by a scalar on $(\widetilde{\Pi}_x)^{\widetilde{\mathfrak{q}}_v}$, which equals the degree k elementary symmetric polynomial in $\psi_{x,1}(\mathrm{Art}_{F_v}^{-1}(\sigma)),\dots,\psi_{x,2n}(\mathrm{Art}_{F_v}^{-1}(\sigma))$. This completes the proof. □

Fix a choice of Frobenius lift $\phi_v \in W_{F_v}$. We define $\mathrm{Res}_v \in (\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{\mathfrak{q}}_v) \otimes_{\mathbf{Z}} \mathcal{O})$ to be the resultant of the polynomials $P_{v^c,\phi_v^{-c}}(X)$ and $P_{v,\phi_v}(X)$.

Proposition 2.2.14. *Let $\widetilde{\pi}_{\overline{v}}$ be an irreducible admissible $\overline{\mathbf{Q}}_p[\widetilde{G}(F_{\overline{v}}^+)]$-module, suppose that $\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v} \neq 0$, and let $(r_v, N_v) = \mathrm{rec}_{F_v}^T(\widetilde{\pi}_{\overline{v}} \circ \iota_v^{-1})$. Let $T_{\overline{v}}$ denote the $\overline{\mathbf{Q}}_p$-subalgebra of $\mathrm{End}_{\overline{\mathbf{Q}}_p}(\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v})$ generated by the images of the elements $e_{v^c,i}(\phi_v^{-c})$ and $e_{v,i}(\phi_v)$. Then for each maximal ideal $\mathfrak{m} \subset T_{\overline{v}}$, either $\mathrm{Res}_v \in \mathfrak{m}$ or $\mathrm{Res}_v \notin \mathfrak{m}$, $T_{\overline{v},\mathfrak{m}} = T_{\overline{v}}/\mathfrak{m} = \overline{\mathbf{Q}}_p$, and for all $\tau_v \in I_{F_v}$, $N_v P_{v,\phi_v}(r_v(\phi_v)) = 0$ and $(r_v(\tau_v) - 1)P_{v,\phi_v}(r_v(\phi_v)) = 0$ in $M_{2n}(T_{\overline{v}}/\mathfrak{m}) = M_{2n}(\overline{\mathbf{Q}}_p)$.*

Proof. We again use some of the ideas in the proof of Lemma 2.2.13. Choose an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. For $m \geq 1$, let St_m denote the Steinberg representation of $\mathrm{GL}_m(F_v)$ (i.e., the square-integrable quotient of $\mathrm{Ind}_{B_m(F_v)}^{\mathrm{GL}_m(F_v)} \overline{\mathbf{Q}}_p$). Then there is an isomorphism $\mathrm{rec}_{F_v}^T(\mathrm{St}_m) = \mathrm{Sp}_m$, where Sp_m is the Weil–Deligne representation on $\overline{\mathbf{Q}}_p^m = \oplus_{i=1}^m \overline{\mathbf{Q}}_p \cdot e_i$, where W_{F_v} acts on e_i by the character $|\cdot|^{1-i} \circ \mathrm{Art}_{F_v}^{-1}$ and N_v acts by $N_v e_1 = 0$, $N_v e_i = e_{i-1}$ $(i = 2,\dots,m)$.

Since $\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v} \neq 0$, we can find an isomorphism

$$(r_v, N_v) \cong \oplus_{i=1}^s \mathrm{Sp}_{\alpha_i}(\psi_i |\cdot|^{(1-2n)/2})$$

for some tamely ramified characters $\psi_i : F_v^\times \to \overline{\mathbf{Q}}_p^\times$; then $\widetilde{\pi} \circ \iota_v^{-1}$ is isomorphic to a subquotient of the induced representation

$$\Pi = \text{n-Ind}_{P_\alpha(F_v)}^{\mathrm{GL}_{2n}(F_v)} \otimes_{i=1}^s \mathrm{St}_{\alpha_i}(\psi_i \circ \mathrm{Art}_{F_v}),$$

where P_α is the standard parabolic subgroup of GL_{2n} corresponding to the partition $2n = \alpha_1 + \cdots + \alpha_s$. Let $\widetilde{\Pi} = \Pi \circ \iota_v$. Let $T'_{\overline{v}}$ denote the $\overline{\mathbf{Q}}_p$-subalgebra of $\mathrm{End}_{\overline{\mathbf{Q}}_p}(\widetilde{\Pi}^{\widetilde{\mathfrak{q}}_v})$ generated by the images of the elements $e_{v^c,i}(\phi_v^{-c})$ and $e_{v,i}(\phi_v)$. Then $T_{\overline{v}}$ is a quotient of $T'_{\overline{v}}$, and it suffices to show that the conclusion of the lemma holds with $T_{\overline{v}}$ replaced by $T'_{\overline{v}}$.

By the geometrical lemma, we can find a filtration of $r_P(\widetilde{\Pi})$ with graded pieces indexed by tuples $\mu = (\mu_{ij})_{i=1,2,j=1,\ldots,s}$, where the μ_{ij} are non-negative integers such that for each $j = 1, \ldots, s$, $\mu_{1j} + \mu_{2j} = \alpha_j$ and for each $i = 1, 2$, $\mu_{i1} + \cdots + \mu_{is} = n$. The representation of $G(F_{\overline{v}}^+) = \mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v)$ indexed by the tuple μ is

$$\sigma_\mu = \left(\text{n-Ind}_{P_1(F_{v^c})}^{\mathrm{GL}_n(F_{v^c})} \mathrm{St}_{\mu_{1s}}(\theta_{1s}^{-c}) \otimes \cdots \otimes \mathrm{St}_{\mu_{11}}(\theta_{11}^{-c})\right) \otimes \left(\text{n-Ind}_{P_2(F_v)}^{\mathrm{GL}_n(F_v)} \mathrm{St}_{\mu_{21}}(\theta_{21}) \otimes \cdots \otimes \mathrm{St}_{\mu_{2s}}(\theta_{2s})\right),$$

where $\theta_{ij} : F_v^\times \to \overline{\mathbf{Q}}_p^\times$ is the character given by the formulae

$$\theta_{1j} = \psi_j|\cdot|^{\mu_{2j}/2}, \theta_{2j} = \psi_j|\cdot|^{-\mu_{1j}/2}$$

for $j = 1, \ldots, s$.

We recall that the natural projection $\widetilde{\Pi}^{\widetilde{\mathfrak{q}}_v} \to r_P(\widetilde{\Pi})^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}})\times I_v}$ is an isomorphism. The maximal ideals $\mathfrak{m} \subset T'_{\overline{v}}$ correspond to the different factorizations $\widetilde{P}_{v,\phi_v}(X) = P_{v^c,\phi_v^{-c}}(X)P_{v,\phi_v}(X)$ that occur in $\widetilde{\Pi}^{\widetilde{\mathfrak{q}}_v}$. Each factorization arises from (at least one) μ such that $(\sigma_\mu)^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}})\times I_v} \neq 0$; the corresponding factorization is given by

$$\begin{aligned} P_{v^c,\phi_v^{-c}}(X) &= \prod_{j=1}^{s}\prod_{k=1}^{\mu_{1j}} (X - (\theta_{1j}^{-c}|\cdot|^{(1-2n-\alpha_j+2k-1)/2})(\phi_v^{-c})) \\ &= \prod_{j=1}^{s}\prod_{k=1}^{\mu_{1j}} (X - (\theta^{1j}|\cdot|^{(1-2n+\alpha_j+1-2k)/2})(\phi_v)), \\ P_{v,\phi_v}(X) &= \prod_{j=1}^{s}\prod_{k=1}^{\mu_{2j}} (X - \theta_{2j}|\cdot|^{(1-2n+\mu_{2j}-\mu_{1j}+1-2k)/2})(\phi_v)). \end{aligned}$$

If $(\sigma_\mu)^{\mathrm{GL}_n(\mathcal{O}_{F_{v^c}})\times I_v} \neq 0$, then we must have $\mu_{1j} \in \{0, 1\}$ for all $j = 1, \ldots, s$. Let us choose therefore a maximal ideal $\mathfrak{m}$ such that $\mathrm{Res}_v \notin \mathfrak{m}$ and a tuple μ giving rise to $\mathfrak{m}$. Combining [Tho21, Prop. 2.2] and Lemma 2.2.13, we find that $T'_{\overline{v},\mathfrak{m}} = T'_{\overline{v}}/\mathfrak{m} = \overline{\mathbf{Q}}_p$. Let $Q(X)$ denote the image of $P_{v,\phi_v}(X)$ modulo $\mathfrak{m}$. Examining the action of $Q(r_v(\phi_v))$ in the summand $\mathrm{Sp}_{\alpha_j}(\psi_j|\cdot|^{(1-2n)/2})$ of (r_v, N_v), we see that $Q(r_v(\phi_v))$ either annihilates this summand (if $\mu_{1j} = 0$) or at least has image contained in the span of the vector e_1. In either case we find that I_{F_v} acts trivially on the image of $Q(r_v(\phi_v))$ and N_v annihilates this image. This is what we needed to show. $\square$

Corollary 2.2.15. *Let $\widetilde{\pi}_{\overline{v}}$ be an irreducible admissible $\overline{\mathbf{Q}}_p[\widetilde{G}(F^+_{\overline{v}})]$-module, and suppose that $\widetilde{\pi}_{\overline{v}}^{\widetilde{\mathfrak{q}}_v} \neq 0$. Let $\rho : G_{F_v} \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ be a continuous representation such that $\mathrm{WD}(\rho)^{F-ss} \cong \mathrm{rec}^T_{F_v}(\widetilde{\pi}_{\overline{v}} \circ \iota_v^{-1})$. Let $T_{\overline{v}}$ be defined as in the proposition. Then for all $\tau_v \in I_{F_v}$, we have the equality*

$$\mathrm{Res}_v^{(2n)!}(\rho(\tau_v) - 1)P_{v,\phi_v}(\rho(\phi_v)) = 0$$

in $M_{2n}(\overline{\mathbf{Q}}_p) \otimes_{\overline{\mathbf{Q}}_p} T_{\overline{v}} = M_{2n}(T_{\overline{v}})$.

Proof. We can again take this statement "one maximal ideal of $T_{\overline{v}}$ at a time." The number $(2n)!$ is a crude upper bound for the $\overline{\mathbf{Q}}_p$-dimension of $T_{\overline{v}}$. In particular, if $\mathrm{Res}_v \in \mathfrak{m}$, then $\mathrm{Res}_v^{(2n)!} T_{\overline{v},\mathfrak{m}} = 0$. We therefore need only show that for each maximal ideal $\mathfrak{m}$ such that $\mathrm{Res}_v \notin \mathfrak{m}$, we have the equality

$$(\rho(\tau_v) - 1)Q(\rho(\phi_v)) = 0$$

in $M_{2n}(\overline{\mathbf{Q}}_p)$ for every $\tau_v \in I_{F_v}$, where $Q(X)$ denotes the image of $P_{v,\phi_v}(X)$ modulo $\mathfrak{m}$. Let $\rho(\phi_v) = su$ be the multiplicative Jordan decomposition (so that s is semisimple, u is unipotent, and s, u commute). Then $r_v(\phi_v) = s$, by definition of Frobenius semi-simplification. Since $\mathrm{Res}_v \mod \mathfrak{m}$ is non-zero, $Q(\rho(\phi_v))$ and $Q(r_v(\phi_v))$ have the same image, which is the span of the eigenspaces of $r_v(\phi_v)$ corresponding to eigenvalues that are not roots of $Q(X)$. Since $N_v Q(r_v(\phi_v)) = 0$, we find that for each $\tau_v \in I_{F_v}$, $\rho(\tau_v)$ and $r(\tau_v)$ agree on the image of $Q(r_v(\phi_v))$. We finally conclude that

$$(\rho(\tau_v) - 1)Q(\rho(\phi_v)) = (r_v(\tau_v) - 1)Q(r_v(\phi_v)) = 0,$$

as required. □

We now describe the behavior of some of the above Hecke operators under parabolic restriction, with respect to the Siegel parabolic. We first give the statements in the unramified case. In order to ease notation, we use the following convention: if $f(X)$ is a polynomial of degree d, with unit constant term a_0, then $f^\vee(X) = a_0^{-1} X^d f(X^{-1})$.

Proposition 2.2.16. *Let v be a place of F, unramified over the place $\overline{v}$ of F^+. Let*

$$\mathcal{S} : \mathcal{H}(\widetilde{G}(F^+_{\overline{v}}), \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})) \to \mathcal{H}(G(F^+_{\overline{v}}), G(\mathcal{O}_{F^+_{\overline{v}}}))$$

denote the homomorphism defined by (2.1.8). Then we have

$$\mathcal{S}(\widetilde{P}_v(X)) = P_v(X) q_v^{n(2n-1)} P_{v^c}^\vee(q_v^{1-2n} X).$$

Proof. See [NT16, Prop.-Def. 5.3]. □

We now discuss the ramified split case. Suppose first that v is a place of F that is split over the place $\overline{v}$ of F^+. Let $\widetilde{I}_{\overline{v}}$ be a subgroup of $\widetilde{G}(F^+_{\overline{v}})$ satisfying $\widetilde{\mathrm{Iw}}_{\overline{v}}(1,1) \subset \widetilde{I}_{\overline{v}} \subset \widetilde{\mathrm{Iw}}_{\overline{v}}(0,1)$. Then $\widetilde{I}_{\overline{v}} \cap G(F^+_{\overline{v}})$ may be identified with a product

$I_{v^c} \times I_v$ of open compact subgroups of $\mathrm{GL}_n(F_{v^c})$ and $\mathrm{GL}_n(F_v)$, respectively. If $\sigma \in W_{F_v}$, then we define

$$\begin{aligned}\widetilde{P}_{v,\sigma}(X) &= \prod_{i=1}^{2n}(X - \iota_v^{-1}(t_{v,i}(\sigma))) \\ &= \sum_{i=0}^{2n}(-1)^i \iota_v^{-1}(e_{v,i}(\sigma))X^{2n-i} \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{I}_{\overline{v}}) \otimes_{\mathbf{Z}} \mathcal{O}[X].\end{aligned} \tag{2.2.17}$$

The group $\widetilde{I}_{\overline{v}}$ admits an Iwahori decomposition with respect to the parabolic subgroup P, and the element $(\varpi_{v^c}^{-1} \cdot 1_n, 1_n) \in G(F_{\overline{v}}^+) = \mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v)$ is strongly positive and defines a Hecke operator $[\widetilde{I}_{\overline{v}}(\varpi_{v^c}^{-1} \cdot 1_n, 1_n)\widetilde{I}_{\overline{v}}]$ that is invertible in $\mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{I}_{\overline{v}}) \otimes_{\mathbf{Z}} \mathcal{O}$. We can therefore apply Lemma 2.1.13, allowing us to state the following result:

Proposition 2.2.18. *For any $\sigma \in W_{F_v}$, we have*

$$\mathcal{S}(\widetilde{P}_{v,\sigma}(X)) = P_{v,\sigma}(X)\|\sigma\|_v^{n(1-2n)} P_{v^c,\sigma^{-c}}(\|\sigma\|_v^{2n-1}X).$$

Proof. This results from a calculation using the definition of $\widetilde{P}_{v,\sigma}(X)$ and the formula for the composite $\mathcal{S} \circ t$ given in Section 2.1.9. (Let $\alpha \in F_{v^c}^\times$ be such that $\mathrm{Art}_{F_{v^c}}(\alpha)$ agrees with the restriction of σ^c to the maximal abelian extension of F_{v^c}. For the element $((1, \dots \alpha^{-1}, \dots 1), 1_n)$ of $\mathrm{GL}_n(F_{v^c}) \times \mathrm{GL}_n(F_v)$, the action of $\mathcal{S} \circ t$ on the corresponding Hecke operator is by multiplication by $\|\sigma\|_v^{-n}$.) □

Suppose next that v is a place of F that is split over the place $\overline{v}$ of F^+ and that $\widetilde{\mathfrak{q}}_v \subset \widetilde{G}(F_{\overline{v}}^+)$ is an open compact subgroup such that $\widetilde{\mathfrak{p}}_{v,1} \subset \widetilde{\mathfrak{q}}_v \subset \widetilde{\mathfrak{p}}_v$. Write $\widetilde{\mathfrak{q}}_v \cap G(F_{\overline{v}}^+) = \mathrm{GL}_n(\mathcal{O}_{F_{v^c}}) \times I_v$. We have already observed that Lemma 2.1.13 applies in this situation, and we have the following analogue of Proposition 2.2.18, which is proved in the same way:

Proposition 2.2.19. *For any $\sigma \in W_{F_v}$, we have $\mathcal{S}(P_{v,\sigma}(X)) = P_{v,\sigma}(X)$ and $\mathcal{S}(P_{v^c,\sigma^{-c}}(X)) = \|\sigma\|_v^{n(1-2n)} P_{v^c,\sigma^{-c}}(\|\sigma\|_v^{2n-1}X)$.*

2.2.20. *Duality and twisting.* In this section we record some operations that relate different cohomology groups and the actions of the corresponding Hecke operators. We deal with duality first. Let S be a finite set of finite places of F such that $S = S^c$. There are anti-involutions

$$\iota : \mathcal{H}(G^S, K^S) \to \mathcal{H}(G^S, K^S)$$

and (if $S = S^c$)

$$\widetilde{\iota} : \mathcal{H}(\widetilde{G}^S, \widetilde{K}^S) \to \mathcal{H}(\widetilde{G}^S, \widetilde{K}^S)$$

given on double cosets by $\widetilde{\iota}([\widetilde{K}^S g \widetilde{K}^S]) = [\widetilde{K}^S g^{-1} \widetilde{K}^S]$ (resp. $\iota([K^S g K^S]) = [K^S g^{-1} K^S]$). In particular, we have anti-involutions $\widetilde{\iota} : \widetilde{\mathbf{T}}^S \to \widetilde{\mathbf{T}}^S$, $\iota : \mathbf{T}^S \to \mathbf{T}^S$

(which are actually involutions, since these Hecke algebras are commutative). If $v \notin S$, then we have the formulae

$$\widetilde{\iota}(\widetilde{P}_v(X)) = q_v^{2n(2n-1)}\widetilde{P}_v^\vee(q_v^{1-2n}X) = \widetilde{P}_{v^c}(X),$$
$$\iota(P_v(X)) = q_v^{n(n-1)}P_v^\vee(q_v^{1-n}X).$$

If $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S$ (resp. $\mathfrak{m} \subset \mathbf{T}^S$) is a maximal ideal with residue field a finite extension of k, then we define $\widetilde{\mathfrak{m}}^\vee = \widetilde{\iota}(\mathfrak{m})$ (resp. $\mathfrak{m}^\vee = \iota(\mathfrak{m})$). The interaction between these involutions and Poincaré duality is described by the following proposition. We write $\widetilde{D} = \dim_{\mathbf{R}} \widetilde{X}$ (resp. $D = \dim_{\mathbf{R}} X$).

Proposition 2.2.21. *Let $R = \mathcal{O}$ or $\mathcal{O}/\varpi^m$ for some $m \geq 1$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ (resp. $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$) be a good subgroup, and let $\mathcal{V}$ be an $R[\widetilde{K}_S]$-module (resp. $R[K_S]$-module), which is finite free as an R-module. Let $\mathcal{V}^\vee = \mathrm{Hom}(\mathcal{V}, R)$. Then there is an isomorphism*

$$R\,\mathrm{Hom}_R(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{V}), R) \cong R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee)[\widetilde{D}]$$

(*resp.*

$$R\,\mathrm{Hom}_R(R\Gamma_c(X_K, \mathcal{V}), R) \cong R\Gamma(X_K, \mathcal{V}^\vee)[D])$$

in $\mathbf{D}(R)$ that is equivariant for the action of $\mathcal{H}(\widetilde{G}^S, \widetilde{K}^S)$ (resp. $\mathcal{H}(G^S, K^S)$) when this Hecke algebra acts by $\widetilde{\iota}^t$ (resp. ι^t) on the left-hand side and in the usual way on the right-hand side.

Proof. See [NT16, Prop. 3.7]. □

Corollary 2.2.22. *Let $R = \mathcal{O}$ or $\mathcal{O}/\varpi^m$ for some $m \geq 1$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ (resp. $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$) be a good subgroup, and let $\mathcal{V}$ be an $R[\widetilde{K}_S]$-module (resp. $R[K_S]$-module), which is finite free as an R-module. Let $\mathcal{V}^\vee = \mathrm{Hom}(\mathcal{V}, R)$. Then $\widetilde{\iota}$ (resp. ι) descends to an isomorphism*

$$\widetilde{\mathbf{T}}^S(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{V})) \cong \widetilde{\mathbf{T}}^S(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee))$$

(*resp. an isomorphism*

$$\mathbf{T}^S(R\Gamma_c(X_K, \mathcal{V})) \cong \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}^\vee))$$

of R-algebras. In particular, if $\widetilde{\mathfrak{m}}$ (resp. $\mathfrak{m}$) is a maximal ideal of $\widetilde{\mathbf{T}}^S$ (resp. $\mathbf{T}^S$) in the support of $H_c^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V})$ (resp. $H_c^*(X_K, \mathcal{V})$), then $\widetilde{\mathfrak{m}}^\vee$ (resp. $\mathfrak{m}^\vee$) is in the support of $H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}^\vee)$ (resp. $H^*(X_K, \mathcal{V}^\vee)$).

Proof. We justify the statements for GL_n. The proposition implies that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{T}^S & \longrightarrow & \mathrm{End}_{\mathbf{D}(R)}(R\Gamma_c(X_K, \mathcal{V})) \\ \iota \big\downarrow & & \big\downarrow \\ \mathbf{T}^S & \longrightarrow & \mathrm{End}_{\mathbf{D}(R)}(R\Gamma(X_K, \mathcal{V}^\vee)), \end{array}$$

where the horizontal arrows are the canonical ones and the right vertical arrow is the one induced by the Poincaré duality isomorphism. The statement of the corollary is equivalent to the assertion that image under ι of the kernel of the top horizontal arrow is equal to the kernel of the lower horizontal arrow. This follows from the commutativity of the diagram. $\square$

We next deal with twisting for the group G. Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and let $\psi : G_F \to \mathcal{O}^\times$ be a continuous character such that $\psi \circ \mathrm{Art}_{F_v}$ is trivial on $\det(K_v)$ for each place $v \notin S$ of F. We define an isomorphism of $\mathcal{O}$-algebras $f_\psi : \mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O} \to \mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ by the formula $f_\psi(f)(g) = \psi(\mathrm{Art}_F(\det(g)))^{-1} f(g)$. (It is an isomorphism because it has an inverse, given by the formula $f_\psi^{-1} = f_{\psi^{-1}}$.) If $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for each $v \notin S$ (so that ψ is unramified outside S and $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O} = \mathbf{T}^S$), then we have the formula $f_\psi(T_{v,i}) = \psi(\mathrm{Frob}_v)^{-i} T_{v,i}$ for each finite place $v \notin S$ of F. If $\mathfrak{m} \subset \mathbf{T}^S$ is a maximal ideal with residue field a finite extension of k, then we define $\mathfrak{m}(\psi) = f_\psi(\mathfrak{m})$.

Proposition 2.2.23. *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and suppose that S contains the p-adic places of F. Let $\psi : G_F \to \mathcal{O}^\times$ be a continuous character satisfying the following conditions*:

(1) *For each finite place $v \nmid p$ of F, $\psi \circ \mathrm{Art}_{F_v}$ is trivial on $\det(K_v)$.*
(2) *There is $m = (m_\tau)_\tau \in \mathbf{Z}^{\mathrm{Hom}(F,E)}$ such that for each place $v|p$ of F, and for each $k \in \det(K_v)$, we have*

$$\psi(\mathrm{Art}_{F_v}(k)) = \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)} \tau(k)^{-m_\tau}.$$

Let $\mu \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ be the dominant weight defined by the formula $\mu_\tau = (m_\tau, \ldots, m_\tau)$ for each $\tau \in \mathrm{Hom}(F, E)$. Then for any $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, there is an isomorphism

$$R\Gamma(X_K, \mathcal{V}_\lambda) \cong R\Gamma(X_K, \mathcal{V}_{\lambda+\mu})$$

in $\mathbf{D}(\mathcal{O})$, which is equivariant for the action of $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ when $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ acts in the usual way on the left-hand side and by f_ψ on the right-hand side.

Proof. The character ψ defines a class in $H^0(X_K, \mathcal{V}_\mu) = \mathrm{Hom}_{\mathrm{Sh}(X_K)}(\underline{\mathcal{O}}, \mathcal{V}_\mu)$. By tensor product this determines a morphism $\mathcal{V}_\lambda \to \mathcal{V}_\lambda \otimes_{\mathcal{O}} \mathcal{V}_\mu \cong \mathcal{V}_{\lambda+\mu}$ of sheaves on X_K, hence a morphism $R\Gamma(X_K, \mathcal{V}_\lambda) \to R\Gamma(X_K, \mathcal{V}_{\lambda+\mu})$ in $\mathbf{D}(\mathcal{O})$. In order to determine how this morphism behaves with respect to the action of Hecke operators, we will repeat this calculation in $\mathbf{D}(\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O})$.

Let $\mathcal{A} = \mathrm{Ind}_{\mathrm{GL}_n(F)}^{\mathrm{GL}_n(\mathbf{A}_F^\infty)} \mathcal{O} = H^0(\mathfrak{X}_G, \mathcal{O})$. There is an isomorphism

$$H^0(\mathfrak{X}_G, \mathcal{V}_\lambda) \cong \mathcal{A} \otimes_{\mathcal{O}} \mathcal{V}_\lambda$$

of $\mathcal{O}[\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}) \times K_S)]$-modules, and hence a canonical isomorphism in $\mathbf{D}(\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O})$:

$$R\Gamma(X_K, \mathcal{V}_\lambda) \cong R\Gamma(K, \mathcal{A} \otimes_{\mathcal{O}} \mathcal{V}_\lambda).$$

The same applies when λ is replaced by any dominant weight in $(\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. The class ψ in $H^0(X_K, \mathcal{V}_\mu)$ corresponds to the K-equivariant map $g_\psi : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}} \mathcal{V}_\mu$ that sends a function $f : \mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbf{A}_F^\infty) \to \mathcal{O}$ to $g_\psi(f)(g) = \psi(\det(g))f(g)$. The map g_ψ becomes $G^S \times K_S$-equivariant when we twist the action on the source, giving

$$g_\psi : \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{O}} \mathcal{V}_\mu(\psi^{-1,S}).$$

By definition, the twist $(\psi^{-1,S})$ means that the action of an element $g \in G^S$ is twisted by $\psi(\det(g))^{-1}$. Taking the tensor product by $\mathcal{V}_\lambda$ and then taking derived K-invariants gives a morphism

$$R\Gamma(X_K, \mathcal{V}_\lambda) \to R\Gamma(X_K, \mathcal{V}_{\lambda+\mu}(\psi^{-1,S}))$$

in $\mathbf{D}(\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O})$, hence a $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$-equivariant isomorphism

$$R\Gamma(X_K, \mathcal{V}_\lambda) \to R\Gamma(X_K, \mathcal{V}_{\lambda+\mu}(\psi^{-1,S}))$$

in $\mathbf{D}(\mathcal{O})$. The proof of the proposition is complete on noting that there is a canonical isomorphism

$$R\Gamma(X_K, \mathcal{V}_{\lambda+\mu}(\psi^{-1,S})) \cong R\Gamma(X_K, \mathcal{V}_{\lambda+\mu})$$

in $\mathbf{D}(\mathcal{O})$, which is equivariant for the action of $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ when $\mathcal{H}(G^S, K^S) \otimes_{\mathbf{Z}} \mathcal{O}$ acts in the natural way on the source and by f_ψ on the target. $\square$

Corollary 2.2.24. *Suppose that S contains the p-adic places of F, and let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup such that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for each place $v \notin S$ of F. Let $\psi : G_F \to \mathcal{O}^\times$ be a continuous character satisfying the following conditions*:

(1) *For each finite place $v \nmid p$ of F, $\psi \circ \mathrm{Art}_{F_v}$ is trivial on* $\det(K_v)$.

(2) *There is* $m = (m_\tau)_\tau \in \mathbf{Z}^{\mathrm{Hom}(F,E)}$ *such that for each place* $v|p$ *of* F, *and for each* $k \in \det(K_v)$, *we have*

$$\psi(\mathrm{Art}_{F_v}(k)) = \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v,E)} \tau(k)^{-m_\tau}.$$

Let $\mu \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ *be the dominant weight defined for each* $\tau \in \mathrm{Hom}(F,E)$ *by the formula* $\mu_\tau = (m_\tau, \dots, m_\tau)$. *Then for any* $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, f_ψ *descends to an isomorphism*

$$\mathbf{T}^S(K,\lambda) \cong \mathbf{T}^S(K,\lambda+\mu).$$

In particular, if $\mathfrak{m}$ *is a maximal ideal of* $\mathbf{T}^S$ *that is in the support of* $H^*(X_K, \mathcal{V}_\lambda)$, *then* $\mathfrak{m}(\psi)$ *is in the support of* $H^*(X_K, \mathcal{V}_{\lambda+\mu})$.

Proof. This is an immediate consequence of Proposition 2.2.23. □

2.3. *Some automorphic Galois representations.* In the next two sections of this chapter, we state some results asserting the existence of Galois representations associated to automorphic forms. Although the main results of this paper concern the relation between classical automorphic representations and Galois representations, we must also consider the Galois representations associated to torsion classes, and therefore valued in (possibly p-torsion) Hecke algebras. This goes some way towards explaining the need to state so many closely related results here. A large part of this paper will be taken up with the problem of studying the local properties of the Hecke–algebra valued Galois representations whose existence is asserted in the statement of Theorem 2.3.7.

2.3.1. *Existence of Galois representations attached to automorphic forms.* If π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\mathbf{C})}$, we say that π is of weight λ if the infinitesimal character of π_∞ is the same as that of $V_\lambda^\vee$.

THEOREM 2.3.2. *Let* $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\mathbf{C})}$, *and let* π *be a cuspidal automorphic representation of* $\mathrm{GL}_n(\mathbf{A}_F)$ *of weight* λ. *Then for any isomorphism* $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, *there exists a continuous semisimple representation* $r_\iota(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ *satisfying the following condition*: *for each prime* $l \neq p$ *above that both* F *and* π *are unramified, and for each place* $v|l$ *of* F, $r_\iota(\pi)|_{G_{F_v}}$ *is unramified and the characteristic polynomial of* $r_\iota(\pi)(\mathrm{Frob}_v)$ *is equal to the image of* $P_v(X)$ *in* $\overline{\mathbf{Q}}_p[X]$ *under the homomorphism* $\mathbf{T}_v \to \overline{\mathbf{Q}}_p$ *associated to* $\iota^{-1}\pi_v$.

Proof. This is the main theorem of [HLTT16]. □

THEOREM 2.3.3. *Suppose that* F *contains an imaginary quadratic field. Let* π *be a cuspidal automorphic representation of* $\widetilde{G}(\mathbf{A}_{F^+})$, *and let* ξ *be an irreducible algebraic representation of* $\widetilde{G}_{\mathbf{C}}$ *such that* π *is* ξ*-cohomological. Then*

there exist a partition $2n = n_1 + \cdots + n_r$ *and discrete, conjugate self-dual automorphic representations* $\Pi_1, \ldots, \Pi_r$ *of* $\mathrm{GL}_{n_1}(\mathbf{A}_F), \ldots, \mathrm{GL}_{n_r}(\mathbf{A}_F)$, *satisfying the following conditions*:

(1) *Let* $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$. *If* l *is a prime unramified in* F *and above that* π *is unramified, then* Π *is unramified above* l *and for each place* $v|l$ *of* F *lying above a place* $\overline{v}$ *of* F^+, Π_v *and* $\pi_{\overline{v}}$ *are related by unramified base change.*

(2) *If* $F_0 \subset F$ *is an imaginary quadratic field and* l' *is a prime that splits in* F_0, *then for each place* $v|l'$ *of* F *lying above a place* $\overline{v}$ *of* F^+, Π_v *and* $\pi_{\overline{v}}$ *are identified under the induced isomorphism* $\iota_v : \widetilde{G}(F^+_{\overline{v}}) \cong \mathrm{GL}_{2n}(F_v)$.

(3) *The infinitesimal character of* Π *is the same as that of the representation* $(\xi \otimes \xi)^\vee$ *of* $\mathrm{GL}_{2n}(F \otimes_{\mathbf{Q}} \mathbf{C})$.

Consequently,[1] *there exists for any isomorphism* $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ *a continuous semisimple representation* $r_\iota(\pi) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ *satisfying the following conditions*:

(a) *For each prime* $l \neq p$ *which is unramified in* F *and above which* π *is unramified, and for each place* $v|l$ *of* F, $r_\iota(\pi)|_{G_{F_v}}$ *is unramified and the characteristic polynomial of* $r_\iota(\pi)(\mathrm{Frob}_v)$ *is equal to the image of* $\widetilde{P}_v(X)$ *in* $\overline{\mathbf{Q}}_p[X]$.

(b) *For each place* $v|p$ *of* F, $r_\iota(\pi)$ *is de Rham, and for each embedding* $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$, *we have*

$$\mathrm{HT}_\tau(r_\iota(\pi)) = \{\widetilde{\lambda}_{\tau,1} + 2n - 1, \widetilde{\lambda}_{\tau,2} + 2n - 2, \ldots, \widetilde{\lambda}_{\tau,2n}\},$$

where $\widetilde{\lambda} \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ *is the highest weight of the representation given by* $\iota^{-1}(\xi \otimes \xi)^\vee$ *of* GL_{2n} *over* $\overline{\mathbf{Q}}_p$.

(c) *If* $F_0 \subset F$ *is an imaginary quadratic field and* l *is a prime which splits in* F_0, *then for each place* $v|l$ *of* F *lying above a place* $\overline{v}$ *of* F^+, *there is an isomorphism* $\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{F\text{-}ss} \cong \mathrm{rec}^T_{F_v}(\pi_{\overline{v}} \circ \iota_v)$.

Proof. We will deduce this from [Shi14]. The main wrinkle is that this reference gives a case of base change for unitary similitude groups (while our group $\widetilde{G}$ is a unitary group, with trivial similitude factor). Let l^{aux} be an auxiliary prime at which both F and π are unramified. In order to prove the proposition, it suffices to prove the existence of an automorphic representation Π of $\mathrm{GL}_{2n}(\mathbf{A}_F)$ satisfying the second and third requirements, and satisfying the first requirement at almost all rational primes, including l^{aux}. We can then use strong multiplicity 1 and our freedom to vary l^{aux} in order to recover the proposition as stated. The existence and local properties of the Galois

[1]The fact that the Π_i are not mentioned in these consequences is not an oversight. We use the Galois representations associated to the Π_i in order to construct $r_\iota(\pi)$ and verify that it has the expected properties.

representation are then a consequence of the existence of Π (a result due to many people, but see, e.g., [Car14]).

Let $\widetilde{G}'$ denote the similitude group associated to $\widetilde{G}$; thus there is a short exact sequence

$$1 \longrightarrow \mathrm{Res}_{F^+/\mathbf{Q}}\, \widetilde{G} \longrightarrow \widetilde{G}' \longrightarrow \mathbf{G}_m \longrightarrow 1$$

of reductive groups over $\mathbf{Q}$. By the main result of [Shi14], it suffices to find an irreducible algebraic representation ξ' of $\widetilde{G}'_{\mathbf{C}}$ and a cuspidal automorphic representation π' of $\widetilde{G}'(\mathbf{A}_{\mathbf{Q}})$ satisfying the following conditions:

- The restriction $\pi'|_{\widetilde{G}(\mathbf{A}_{F^+})}$ contains π.
- π' is ξ'-cohomological.
- π' is unramified at l^{aux}.

Arguing as in the proof of [HT01, Th. VI.2.9], we see that it is enough to show the existence of a continuous character $\psi : \mathbf{A}_{F_0}^{\times}/{F_0}^{\times} \to \mathbf{C}^{\times}$ satisfying the following conditions:

- The restriction $\psi|_{(\mathbf{A}_{F_0}^{\times})^{c=1}}$ is equal to the restriction of the central character $\omega_\pi : (\mathbf{A}_F^{\times})^{c=1} \to \mathbf{C}^{\times}$ of π to $(\mathbf{A}_{F_0}^{\times})^{c=1}$.
- ψ is of type A_0; i.e., its restriction to $F_{0,\infty}^{\times}$ arises from a character of the torus $(\mathrm{Res}_{F_0/\mathbf{Q}}\, \mathbf{G}_m)_{\mathbf{C}}$.
- $\psi|_{\mathcal{O}_{F_0,l^{\mathrm{aux}}}^{\times}}$ is trivial.

The existence of such a character follows from the algebraicity of $\omega_\pi|_{(\mathbf{A}_{F_0}^{\times})^{c=1}}$, itself a consequence of the fact that π is ξ-cohomological. $\square$

2.3.4. *Existence of Hecke algebra-valued Galois representations.* Let S be a finite set of finite places of F, containing the p-adic places.

THEOREM 2.3.5. *Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a maximal ideal. Suppose that S satisfies the following conditions*:

- $S = S^c$.
- *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

Then there exists a continuous, semi-simple representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)/\mathfrak{m})$$

satisfying the following condition: for each finite place $v \notin S$ of F, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}(K,\lambda)/\mathfrak{m})[X]$.

We note that our condition on S can always be achieved after possibly enlarging S.

Proof. Fix an embedding $\mathbf{T}^S(K,\lambda)/\mathfrak{m} \hookrightarrow \overline{\mathbf{F}}_p$. From [Sch15, Cor. 5.4.3], we deduce there is an n-dimensional continuous semisimple Galois representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ such that for each finite place $v \notin S$ of F, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n}$$

in $\overline{\mathbf{F}}_p[X]$. (Our condition on S ensures that we can appeal to the results of [Sch15] in a case where they are unconditional; cf. Theorem 2.3.3 and the discussion in [Sch15, Rem. 5.4.6].) Combining the Chebotarev density theorem, the Brauer–Nesbitt Theorem and the vanishing of the Brauer group of a finite field [DS74, Lem. 6.13], we see that $\overline{\rho}_{\mathfrak{m}}$ can in fact be realized over $\mathbf{T}^S(K,\lambda)/\mathfrak{m}$. □

Definition 2.3.6. We say that a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S$ is of Galois type if its residue field is a finite extension of k, and there exists a continuous, semi-simple representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S/\mathfrak{m})$ such that for each finite place $v \notin S$ of F, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}^S/\mathfrak{m})[X]$.

We say that a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S$ is non-Eisenstein if it is of Galois type and $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

Note that Theorem 2.3.5 can be viewed as asserting that, under a suitable condition on S, any maximal ideal of $\mathbf{T}^S$ in the support of $H^*(X_K, \mathcal{V}_\lambda)$ is of Galois type. We observe that if $\mathfrak{m} \subset \mathbf{T}^S$ is of Galois type, then so is $\mathfrak{m}^\vee$, and in fact $\overline{\rho}_{\mathfrak{m}^\vee} \cong \overline{\rho}_{\mathfrak{m}}^\vee \otimes \epsilon^{1-n}$. In particular, if $\mathfrak{m}$ is non-Eisenstein, then so is $\mathfrak{m}^\vee$. Similarly, if $\psi : G_{F,S} \to \mathcal{O}^\times$ is a continuous character, and $\mathfrak{m} \subset \mathbf{T}^S$ is a maximal ideal of Galois type, then so is $\mathfrak{m}(\psi)$, and in fact $\overline{\rho}_{\mathfrak{m}(\psi)} \cong \overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi}$. In particular, if $\mathfrak{m}$ is non-Eisenstein, then so is $\mathfrak{m}(\psi)$.

THEOREM 2.3.7. *Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a maximal ideal. Suppose that S satisfies the following conditions*:

- $S = S^c$.
- *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

Suppose, moreover, that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then there exist an integer $N \geq 1$, which depends only on n and $[F : \mathbf{Q}]$, an ideal $I \subset \mathbf{T}^S(K,\lambda)$ satisfying $I^N = 0$, and a continuous homomorphism

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/I)$$

satisfying the following condition: for each finite place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}(K,\lambda)_{\mathfrak{m}}/I)[X]$.

Proof. This follows from [Sch15, Cor. 5.4.4]. □

THEOREM 2.3.8. *Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$ be a maximal ideal. Suppose that S satisfies the following condition*:

- *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

(*Note that the condition $S = S^c$ is implicit in the use of the notation $\widetilde{\mathbf{T}}^S$ here.*) *Then there is a continuous, semi-simple representation*

$$\overline{\rho}_{\widetilde{\mathfrak{m}}} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/\widetilde{\mathfrak{m}})$$

satisfying the following condition: *for each finite place $v \notin S$ of F, the characteristic polynomial of $\overline{\rho}_{\widetilde{\mathfrak{m}}}(\mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/\widetilde{\mathfrak{m}})[X]$.*

Proof. The existence of a $2n$-dimensional group determinant $\widetilde{D}_{\widetilde{\mathfrak{m}}}$ valued in $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/\widetilde{\mathfrak{m}}$ and with the given characteristic polynomials on Frobenius elements at places $v \notin S$ is implicit in [Sch15] and also follows from [NT16, Th. 5.7], as we now explain. The result [NT16, Th. 5.7] shows that if the group $\widetilde{K}$ is small, in the sense that there is a rational prime $q \neq p$ such that $\widetilde{K}_q$ is contained in the principal congruence subgroup at level q (if q is odd) or $2q$ (if $q = 2$), then there is even a $2n$-dimensional group determinant valued in $\widetilde{\mathbf{T}}^S(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/(\varpi)))$ such that for each finite place $v \notin S$ of F, the characteristic polynomial of Frob_v is equal to the image of $\widetilde{P}_v(X)$. The surjection

$$\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda}) \to \widetilde{\mathbf{T}}^S(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/(\varpi)))$$

is bijective at the level of maximal ideals, so this implies the existence of the desired group determinant when $\widetilde{K}$ is small. When $\widetilde{K}$ is not small, we choose an odd rational prime q_1 which is prime to S, and we let $\widetilde{K}_1$ denote the intersection of $\widetilde{K}$ with the principal congruence subgroup of $\widetilde{G}(\widehat{\mathcal{O}}_{F^+})$ of level q_1. Let S_1 denote the union of S with the set of q_1-adic places of F. Then there is a diagram of Hecke algebras

$$\widetilde{\mathbf{T}}^{S_1}(\widetilde{K}_1, \widetilde{\lambda}) \leftarrow \widetilde{\mathbf{T}}^{S_1}(\widetilde{K}/\widetilde{K}_1, \widetilde{\lambda}) \twoheadrightarrow \widetilde{\mathbf{T}}^{S_1}(\widetilde{K}, \widetilde{\lambda}) \hookrightarrow \widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda}),$$

where the left-facing arrow has nilpotent kernel and so induces a bijection at the level of maximal ideals. Let $\widetilde{\mathfrak{m}}_1 \subset \widetilde{\mathbf{T}}^{S_1}(\widetilde{K}, \widetilde{\lambda})$ denote the pullback of $\widetilde{\mathfrak{m}}$ along the right-hand inclusion. Since $\widetilde{K}_1$ is small, there exists a group determinant $\widetilde{D}_{\widetilde{\mathfrak{m}}_1}$ valued in $\widetilde{\mathbf{T}}^{S_1}(\widetilde{K}, \widetilde{\lambda})/\widetilde{\mathfrak{m}}_1$ and with the correct characteristic polynomials at places outside S_1. Let $\widetilde{D}_{\widetilde{\mathfrak{m}},1}$ denote the pushforward of $\widetilde{D}_{\widetilde{\mathfrak{m}}_1}$ to $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/\widetilde{\mathfrak{m}}$. Thus $\widetilde{D}_{\widetilde{\mathfrak{m}},1}$ is a $2n$-dimensional group determinant of G_{F,S_1} with the property that for any finite place $v \notin S_1$ of F, $\widetilde{D}_{\widetilde{\mathfrak{m}},1}(X - \mathrm{Frob}_v)$ equals the image of $\widetilde{P}_v(X)$.

Choose another odd rational prime $q_2 \neq q_1$ which is prime to S, and repeat this construction to obtain a group determinant $\widetilde{D}_{\widetilde{\mathfrak{m}},2}$ of G_{F,S_2} valued in $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/\widetilde{\mathfrak{m}}$ with the property that for any finite place $v \notin S_2$ of F, $\widetilde{D}_{\widetilde{\mathfrak{m}},2}(X - \mathrm{Frob}_v)$ equals the image of $\widetilde{P}_v(X)$. Since the Frobenius elements at places $v \notin S_1 \cup S_2$ are dense in $G_{F,S_1\cup S_2}$, the group determinants $\widetilde{D}_{\widetilde{\mathfrak{m}},1}$ and $\widetilde{D}_{\widetilde{\mathfrak{m}},2}$ have the same characteristic polynomials on all elements of $G_{F,S_1\cup S_2}$. By [Che14, Lem. 1.12], these group determinants are equal and we can take $\widetilde{D}_{\widetilde{\mathfrak{m}}} = \widetilde{D}_{\widetilde{\mathfrak{m}},1} = \widetilde{D}_{\widetilde{\mathfrak{m}},2}$.

To obtain a true representation from this group determinant, we first fix an embedding $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/\widetilde{\mathfrak{m}} \hookrightarrow \overline{\mathbf{F}}_p$. The group determinant determines a representation over $\overline{\mathbf{F}}_p$, by [Che14, Th. A]. It follows by the same argument as in the proof of Theorem 2.3.5 that this representation can in fact be realized over $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/\widetilde{\mathfrak{m}}$. □

A similar argument shows that [NT16, Th. 5.7] implies the following result:

Proposition 2.3.9. *Suppose that S satisfies the following conditions*:

- $S = S^c$.
- *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

Then there exists an ideal $I \subset \widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})$ satisfying $I^{2\dim_{\mathbf{R}}\widetilde{X}} = 0$ and a $2n$-dimensional group determinant $\widetilde{D}$ of $G_{F,S}$ valued in $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/I$ such that for each finite place $v \notin S$ of F, the characteristic polynomial $\widetilde{D}(X - \mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/I)[X]$.

Proof. When $\widetilde{K}$ is small, this is an immediate consequence of [NT16, Th. 5.7], together with the observation that the natural map

$$\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda}) \to \varprojlim_{m\geq 1} \widetilde{\mathbf{T}}^S(R\Gamma(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/(\varpi^m)))$$

is an isomorphism. Moreover, in this case we can take $I = 0$. In general, we introduce an odd rational prime q_1 as in the proof of Theorem 2.3.8 and consider again the diagram

$$\widetilde{\mathbf{T}}^{S_1}(\widetilde{K}_1,\widetilde{\lambda}) \leftarrow \widetilde{\mathbf{T}}^{S_1}(\widetilde{K}/\widetilde{K}_1,\widetilde{\lambda}) \twoheadrightarrow \widetilde{\mathbf{T}}^{S_1}(\widetilde{K},\widetilde{\lambda}) \hookrightarrow \widetilde{\mathbf{T}}^{S}(\widetilde{K},\widetilde{\lambda}).$$

The map $\widetilde{\mathbf{T}}^{S_1}(\widetilde{K}/\widetilde{K}_1,\widetilde{\lambda}) \to \widetilde{\mathbf{T}}^{S_1}(\widetilde{K}_1,\widetilde{\lambda})$ has kernel J_1 satisfying $J_1^{\dim_{\mathbf{R}}\widetilde{X}} = 0$ (since the cohomology of $R\Gamma(\widetilde{X}_{\widetilde{K}_1}, \mathcal{V}_{\widetilde{\lambda}})$ is 0 for degrees not lying in $[0, \dim_{\mathbf{R}}\widetilde{X} - 1]$). Taking I_1 to be the ideal of $\widetilde{\mathbf{T}}^{S_1}(\widetilde{K},\widetilde{\lambda})$ generated by the image of J_1, we obtain a $2n$-dimensional group determinant $\widetilde{D}_1$ of G_{F,S_1} valued in $\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})/I_1$ such that for each finite place $v \notin S_1$ of F, $\widetilde{D}_1(X - \mathrm{Frob}_v)$ equals the image of $\widetilde{P}_v(X)$, and moreover $I_1^{\dim_{\mathbf{R}}\widetilde{X}} = 0$.

Introducing an odd rational prime $q_2 \neq q_1$ which is prime to S, we obtain similarly an ideal $I_2 \subset \widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$ satisfying $I_2^{\dim_{\mathbf{R}} \widetilde{X}} = 0$ and a group determinant $\widetilde{D}_2$ valued in $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/I_2$ and having properties analogous to $\widetilde{D}_1$. We take $I = (I_1, I_2)$ and $\widetilde{D}$ to be the projection of $\widetilde{D}_1$ to $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/I$. Consideration of characteristic polynomials at places $v \notin S_1 \cup S_2$, as in the proof of Theorem 2.3.8, shows that $\widetilde{D}$ equals the projection of $\widetilde{D}_2$ to $\widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})/I$. It follows that $\widetilde{D}$ has the property required by the proposition. □

2.4. *Boundary cohomology.* In the remaining section of this chapter we prove some results about the boundary cohomology of the arithmetic locally symmetric spaces of G and $\widetilde{G}$. This is made possible by the existence of Galois representations attached to Hecke eigenclasses in the cohomology of these groups and of their Levi subgroups. The important observation is usually that the cohomology of a certain stratum in the boundary can be observed to vanish after localization at a sufficiently nice (e.g., non-Eisenstein) maximal ideal of a suitable Hecke algebra.

2.4.1. *The Siegel parabolic.* Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup which is decomposed with respect to the Levi decomposition $P = GU$ (cf. Section 2.1.2). We set $K = \widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$ and $K_U = \widetilde{K} \cap U(\mathbf{A}_{F^+}^\infty)$.

Let $\mathfrak{m} \subset \mathbf{T}^S(K, \lambda)$ be a non-Eisenstein maximal ideal, and suppose that $S = S^c$. Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S$ denote the pullback of $\mathfrak{m}$ under the homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$ defined in (2.2.3). In order to state the first main result of this subsection, we recall that the boundary $\partial \widetilde{X}_{\widetilde{K}} = \overline{\widetilde{X}}_{\widetilde{K}} - \widetilde{X}_{\widetilde{K}}$ of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$ has a $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$-equivariant stratification indexed by the parabolic subgroups of $\widetilde{G}$ which contain B. See [NT16, §3.1.2], especially [NT16, Lem. 3.10] for more details. For such a standard parabolic subgroup Q, we denote by $\widetilde{X}^Q_{\widetilde{K}}$ the stratum labelled by Q. The stratum $\widetilde{X}^Q_{\widetilde{K}}$ can be written as

$$\widetilde{X}^Q_{\widetilde{K}} = Q(F^+) \backslash \left(X^Q \times \widetilde{G}(\mathbf{A}_{F^+}^\infty)/\widetilde{K} \right).$$

As discussed in Section 2.1.2, there is, for any $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+, E)}$, a homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})).$$

Therefore, we can define the localization $R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$. (This complex will be non-zero in $\mathbf{D}(\mathcal{O})$ if and only if the maximal ideal $\widetilde{\mathfrak{m}}$ of $\widetilde{\mathbf{T}}^S$ occurs in the support of the cohomology groups $H^*(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$.)

THEOREM 2.4.2. *Let $\mathfrak{m} \subset \mathbf{T}^S(K, \lambda)$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$. Let $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+, E)}$. Then there is a natural*

$\widetilde{\mathbf{T}}^S$*-equivariant isomorphism*

$$R\Gamma(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \xrightarrow{\sim} R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$$

in $\mathbf{D}(\mathcal{O})$.

Proof. There is no harm in enlarging S, so we first add finitely many places to S, ensuring that it satisfies the condition of Theorem 2.3.5. The proof is similar to the proof of [NT16, Th. 4.2], which applies to the case of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ and which shows that the cohomology of the stratum labelled by any proper parabolic subgroup of $\mathrm{Res}_{F/\mathbf{Q}}\mathrm{GL}_n$ is Eisenstein. Since P is a maximal parabolic of $\widetilde{G}$, the inclusion $\widetilde{X}^P_{\widetilde{K}} \subset \partial\widetilde{X}_{\widetilde{K}}$ is an open embedding, which induces a natural, $\widetilde{\mathbf{T}}^S$-equivariant map

$$R\Gamma_c(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}},$$

and which fits into an excision distinguished triangle

$$R\Gamma_c(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to R\Gamma(\partial\widetilde{X}_{\widetilde{K}} \setminus \widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \xrightarrow{[1]}.$$

We will show that $R\Gamma(\partial\widetilde{X}_{\widetilde{K}} \setminus \widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} = 0$ in $\mathbf{D}(\mathcal{O})$, by showing that for each standard proper parabolic subgroup $Q \subset \widetilde{G}$ with $Q \neq P$, we have $R\Gamma_c(\widetilde{X}^Q_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} = 0$ in $\mathbf{D}(\mathcal{O})$. Applying the same argument to the excision triangle for the inclusion from $\widetilde{X}^P_{\widetilde{K}}$ to its closure, this will also show that the natural map

$$R\Gamma_c(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to R\Gamma(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$$

is an isomorphism.

In order to show this vanishing, it suffices (after possibly shrinking $\widetilde{K}$ at the p-adic places of F^+) to show that if $Q \neq P$ is a standard proper parabolic subgroup of $\widetilde{G}$, then $R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, k)_{\widetilde{\mathfrak{m}}} = 0$. (We are using here that if $C^\bullet$ is a perfect complex in $\mathbf{D}(\mathcal{O})$, then $C^\bullet = 0$ in $\mathbf{D}(\mathcal{O})$ if and only if $C^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} k = 0$ in $\mathbf{D}(k)$. We are also using Poincaré duality to exchange cohomology with compact support for usual cohomology, as in [NT16, Prop. 3.7].)

We will, in fact, show that for any maximal ideal $\widetilde{\mathfrak{m}}' \subset \widetilde{\mathbf{T}}^S$ in the support of $R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, k)$, there exists a semisimple residual Galois representation

$$\bar{\rho}_{\widetilde{\mathfrak{m}}'} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^S/\widetilde{\mathfrak{m}}')$$

such that for each place $v \notin S$ of F, the characteristic polynomial of $\bar{\rho}_{\widetilde{\mathfrak{m}}'}$ equals the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}^S/\widetilde{\mathfrak{m}}')[X]$. Moreover, assume that the Levi component M of Q is of the form

$$\mathrm{Res}_{F/F^+}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{F/F^+}\mathrm{GL}_{n_r} \times \widetilde{G}_{n-s}$$

for integers $r \geq 1, n_i \geq 1, s \in \{1, \ldots, n\}$ satisfying $\sum_{i=1}^r n_i = s$. (More precisely, assume that it is the block diagonal subgroup of $\widetilde{G}$ associated to the partition

$2n = n_1 + \cdots + n_r + 2(n-s) + n_r + \cdots + n_1$. These describe all the standard F^+-rational Levi subgroups of $\widetilde{G}$.) Then we have

$$\bar{\rho}_{\widetilde{\mathfrak{m}}'} = \oplus_{i=1}^r \bar{\rho}_i' \oplus \bar{\rho}'(n-s) \oplus_{i=1}^r (\bar{\rho}_i')^{c,\vee}, \tag{2.4.3}$$

where $\bar{\rho}_i'$ is n_i-dimensional and $\bar{\rho}'(n-s)$ is $(2n-2s)$-dimensional. The non-Eisenstein condition on $\mathfrak{m}$ implies that

$$\bar{\rho}_{\widetilde{\mathfrak{m}}} = \bar{\rho}_1 \oplus \bar{\rho}_2,$$

where both $\bar{\rho}_1, \bar{\rho}_2$ are (absolutely) irreducible n-dimensional representations. This shows that, unless $r = 1$ and $s = n$, $R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, k)_{\widetilde{\mathfrak{m}}} = 0$. The case $r = 1, s = n$ corresponds precisely to the Siegel parabolic P.

Let us define $\mathbf{T}^S_Q = \mathcal{H}(Q^S, \widetilde{K}^S_Q) \otimes_{\mathbf{Z}} \mathcal{O}$ and $\mathbf{T}^S_M = \mathcal{H}(M^S, \widetilde{K}^S_M) \otimes_{\mathbf{Z}} \mathcal{O}$. We recall (cf. Section 2.1.2) that there are homomorphisms $r_Q : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S_Q$ and $r_M : \mathbf{T}^S_Q \to \mathbf{T}^S_M$, and that we set $\mathcal{S}^{\widetilde{G}}_M = r_M \circ r_Q$. We first claim that for any maximal ideal $\widetilde{\mathfrak{m}}'$ of $\widetilde{\mathbf{T}}^S$ in the support of $H^*(\widetilde{X}^Q_{\widetilde{K}}, k)$, there exist a good subgroup $\widetilde{K}'_M \subset \widetilde{K}_M$ with $(\widetilde{K}'_M)^S = \widetilde{K}^S_M$ and a maximal ideal $\mathfrak{m}'$ of $\mathbf{T}^S_M$ in the support of $H^*(X^M_{\widetilde{K}'_M}, k)$ such that $\widetilde{\mathfrak{m}}' = \mathcal{S}^{\widetilde{G},*}_M(\mathfrak{m}')$. This follows the same steps as the proof of [NT16, Th. 4.2], which we outline here.

Firstly, one can describe the cohomology $R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, k)$ together with its $\widetilde{\mathbf{T}}^S$-action in terms of the pullback under $r_Q : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S_Q$ of the cohomology of finitely many locally symmetric spaces for Q. More precisely, using the Iwasawa decomposition away from S, we can write $\widetilde{G}(\mathbf{A}^\infty_{F^+}) = \bigsqcup_{i=1}^r Q(\mathbf{A}^\infty_{F^+}) g_i \widetilde{K}$ and obtain r locally symmetric spaces $X^Q_{K_{Q,i}}$, with $K_{Q,i} := Q(\mathbf{A}^\infty_{F^+}) \cap g_i \widetilde{K} (g_i)^{-1}$, together with an isomorphism

$$R\Gamma(\widetilde{X}^Q_{\widetilde{K}}, k) \simeq \bigoplus_{i=1}^r r_Q^* \left(R\Gamma(X^Q_{K_{Q,i}}, k) \right)$$

in $\mathbf{D}(\widetilde{\mathbf{T}}^S)$. The proof in [NT16], which identifies equations (4.2) and (4.3) of loc. cit., applies verbatim to our situation, so we do not repeat it here.

Secondly, fix a neat compact open subgroup $K_Q \subset Q(\mathbf{A}^\infty_{F^+})$, which can be any of the $K_{Q,i}$ considered above. Let $Q = M \ltimes N$ be a Levi decomposition of Q. Let K_M be the image of K_Q in $M(\mathbf{A}^\infty_{F^+})$ and $K_N := K_Q \cap N(\mathbf{A}^\infty_{F^+})$. Let $\mathcal{W}$ be the object in the derived category of sheaves on $X^M_{K_M}$ corresponding to the object $R\Gamma(K_{N,S}, k)$ in $D(k[K_{M,S}])$ under the formalism in Section 2.1.2. Then, using an argument that is formally identical to that on pages 56–58 of [NT16], we obtain an isomorphism

$$R\Gamma(X^Q_{K_Q}, k) \simeq r_M^* \left(R\Gamma(X^M_{K_M}, \mathcal{W}) \right)$$

in $D(\mathbf{T}_Q^S)$. The corresponding statement in loc. cit. is obtained by combining the second and fourth displayed equations on page 58.

Finally, we consider the spectral sequence that computes the total cohomology of $R\Gamma(X_{K_M}^M, \mathcal{W})$. Let $\mathcal{W}^i$ be the local systems on $X_{K_M}^M$ corresponding to the cohomology groups $H^i(K_{N,S}, k)$. The first two steps above show that there exists a maximal ideal $\mathfrak{m}'$ of $\mathbf{T}_M^S$ in the support of some $H^*(X_{\widetilde{K}'_M}^M, \mathcal{W}^i)$ such that $\widetilde{\mathfrak{m}}' = \mathcal{S}_M^{\widetilde{G},*}(\mathfrak{m}')$. It remains to shrink the level $K_{M,S}$ in order to trivialize all the $\mathcal{W}^i$. This could, a priori, cause a problem because the map on cohomology groups need not be injective. However, we are only interested in keeping track of a maximal ideal $\mathfrak{m}'$ of $\mathbf{T}_M^S$. The Hochschild–Serre spectral sequence shows that shrinking the level does not cause problems, as in the proof of [NT16, Lem. 4.3]. (See also the example below Lemma 2.2.4 for an illustration of the same phenomenon in the derived category.)

In order to complete the proof of the theorem, it therefore suffices to show that for any good subgroup $K_M \subset M(\mathbf{A}_F^\infty)$ with $K_M^S = \widetilde{K}_M^S$ and for any maximal ideal $\mathfrak{m}'$ of $\mathbf{T}_M^S$ in the support of $H^*(X_{K_M}^M, k)$, there exists a semisimple residual Galois representation

$$\bar{\rho}_{\mathcal{S}^*(\mathfrak{m}')} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^S/\mathcal{S}^*(\mathfrak{m}'))$$

such that for each place $v \notin S$ of F, the characteristic polynomial of $\bar{\rho}_{\mathcal{S}^*(\mathfrak{m}')}$ equals the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}^S/\widetilde{\mathfrak{m}}')[X]$ and, moreover, that this Galois representation admits a decomposition of the form (2.4.3).

After possibly shrinking K_M once more, we can assume that it admits a decomposition $K_M = K_1 \times \cdots \times K_r \times \widetilde{K}_s$, where $K_i \subset \mathrm{GL}_{n_i}(\mathbf{A}_F^\infty)$ and $\widetilde{K}_s \subset \widetilde{G}_{n-s}(\mathbf{A}_{F^+}^\infty)$. After possibly enlarging k we can, moreover, assume, in the obvious notation, (by the Künneth formula) that there exist maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r, \widetilde{\mathfrak{m}}_s$ of the Hecke algebras $\mathbf{T}_{\mathrm{GL}_{n_1}}^S, \ldots, \mathbf{T}_{\mathrm{GL}_{n_r}}^S, \mathbf{T}_{\widetilde{G}_{n-s}}^S$, respectively, which are in the supports of the groups

$$H^*(X_{K_1}^{\mathrm{GL}_{n_1}}, k), \ldots, H^*(X_{K_r}^{\mathrm{GL}_{n_r}}, k), H^*(X_{\widetilde{K}_s}^{\widetilde{G}_{n-s}}, k),$$

respectively, and such that $\mathfrak{m}'$ is identified with $(\mathfrak{m}_1, \ldots, \mathfrak{m}_r, \widetilde{\mathfrak{m}}_s)$ under the isomorphism

$$\mathbf{T}_M^S \to \mathbf{T}_{\mathrm{GL}_{n_1}}^S \otimes_{\mathcal{O}} \otimes \cdots \otimes_{\mathcal{O}} \mathbf{T}_{\mathrm{GL}_{n_r}}^S \otimes_{\mathcal{O}} \mathbf{T}_{\widetilde{G}_{n-s}}^S.$$

We can, moreover, assume that all of the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r, \widetilde{\mathfrak{m}}_s$ and $\mathfrak{m}'$ have residue field k.

Let us write $P_v^i(X) \in \mathbf{T}_{\mathrm{GL}_{n_i}}^S[X]$ and $\widetilde{P}_v^s(X) \in \mathbf{T}_{\widetilde{G}_{n-s}}^S[X]$ for the analogues for the groups GL_{n_i} and $\widetilde{G}_{n-s}$ of the Hecke polynomials defined in Section 2.2.5.

By Theorems 2.3.5 and 2.3.8, there exist continuous, semi-simple representations

$$\overline{\rho}_{\mathfrak{m}_i} : G_{F,S} \to \mathrm{GL}_{n_i}(k) \ (i = 1, \dots, r)$$

and

$$\overline{\rho}_{\widetilde{\mathfrak{m}}_s} : G_{F,S} \to \mathrm{GL}_{2(n-s)}(k)$$

such that for each finite place $v \notin S$ of F and for each $i = 1, \dots, r$, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}_i}(\mathrm{Frob}_v)$ is equal to $P_v^i(X)$ mod $\mathfrak{m}_i$, and the characteristic polynomial of $\overline{\rho}_{\widetilde{\mathfrak{m}}_s}(\mathrm{Frob}_v)$ is equal to $\widetilde{P}_v^s(X)$ mod $\widetilde{\mathfrak{m}}_s$.

The proof of the theorem is complete on noting that we can take

$$\overline{\rho}_{\mathcal{S}^*(\mathfrak{m}')} = \bigoplus_{i=1}^{r} \left(\overline{\rho}_{\mathfrak{m}_i} \otimes \epsilon^{n_1+\cdots+n_i-2n} \oplus \overline{\rho}_{\mathfrak{m}_i}^{c,\vee} \otimes \epsilon^{1-(n_1+\cdots+n_i)} \right) \oplus \overline{\rho}_{\widetilde{\mathfrak{m}}_s} \otimes \epsilon^{-s}.$$

That this choice is valid rests on the computation of the image of $\widetilde{P}_v(X)$ under the map $\mathcal{S}_M^{\widetilde{G}}$. The details are very similar to the proof of [NT16, Prop.-Def. 5.3], and they are omitted. □

We can now state the second main result of this subsection, which takes Theorem 2.4.2 as its starting point.

THEOREM 2.4.4. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be as at the start of Section 2.4.1, and let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ be a dominant weight whose image in $(\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$ is $\widetilde{G}$-dominant. Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S$ denote its pullback under the homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$. Then the homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$ descends to a homomorphism*

$$\widetilde{\mathbf{T}}^S(R\Gamma(\partial \widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}) \to \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}).$$

Proof. The first step in the proof is to note that it suffices to show that $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^S(R\Gamma(\partial \widetilde{X}_{\widetilde{K}}^P, \mathcal{V}_{\widetilde{\lambda}})) \to \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda)),$$

by Theorem 2.4.2. On the other hand, the discussion at the end of Section 2.1.9 shows that $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^S(R\Gamma(\partial \widetilde{X}_{\widetilde{K}}^P, \mathcal{V}_{\widetilde{\lambda}})) \to \widetilde{\mathbf{T}}^S(R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{V}_{\widetilde{\lambda}})), \tag{2.4.5}$$

where $\widetilde{\mathbf{T}}^S$ acts on the latter complex via r_P. It therefore suffices to show that $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^S(R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{V}_{\widetilde{\lambda}})) \to \widetilde{\mathbf{T}}^S(R\Gamma(X_K, \mathcal{V}_\lambda)), \tag{2.4.6}$$

where $\widetilde{\mathbf{T}}^S$ acts on the latter cohomology groups via $\mathcal{S} = r_G \circ r_P$. In fact, it even suffices to show that for each $m \geq 1$, $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^S(R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)) \to \widetilde{\mathbf{T}}^S(R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)); \tag{2.4.7}$$

cf. [NT16, Lem. 3.12].

However, arguing in the same way as on [NT16, p. 58], we see that there is an isomorphism

$$R\Gamma(X^P_{\widetilde{K}_P}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \cong R\Gamma(\widetilde{K}^S_P \times K_S, R\Gamma(\mathrm{Inf}^{P^S\times K_S}_{G^S\times K_S}\, \mathfrak{X}_G, R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m)),$$

where the derived pushforward sends (complexes of) $P^S \times \widetilde{K}_{P,S}$-equivariant sheaves on $\mathfrak{X}_G$ to $P^S \times K_S$-equivariant sheaves on $\mathfrak{X}_G$. Suppose we knew that $\mathcal{V}_\lambda/\varpi^m$ was a direct summand of $R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m$; then we could conclude, by arguing in the same way as at the top of [NT16, p. 59], that $r_G^* R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)$ is isomorphic to a direct summand of $R\Gamma(X^P_{\widetilde{K}_P}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)$ in the category

$$\mathbf{D}(\mathcal{H}(P^S \times \widetilde{K}_{P,S}, \widetilde{K}_P) \otimes_{\mathbf{Z}} \mathcal{O}/\varpi^m),$$

implying the existence of the homomorphism (2.4.7).

It remains to construct the desired splitting of $\mathcal{V}_\lambda/\varpi^m$ as a direct summand of $R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m$. To do this, we recall the following two facts:

- $\widetilde{K}_P$ is a semidirect product $\widetilde{K}_P = \widetilde{K}_U \rtimes K$. (By assumption, $\widetilde{K}$ is decomposed with respect to the Levi decomposition $P = GU$.)
- There is a $\widetilde{K}_P$-equivariant embedding $\mathcal{V}_\lambda \to \mathcal{V}_{\widetilde{\lambda}}$, which splits after restriction to K. (This follows from [NT16, Cor. 2.11].)

The morphism $\mathcal{V}_\lambda/\varpi^m \to R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m$ is the composite of the reduction modulo ϖ^m of the given map $\mathcal{V}_\lambda \to \mathcal{V}_{\widetilde{\lambda}}^{\widetilde{K}_{U,S}}$, together with the morphism $(\mathcal{V}_{\widetilde{\lambda}}/\varpi^m)^{\widetilde{K}_{U,S}} \to R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m$ whose existence is assured by the universal property of the derived functor.

The morphism $R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m \to \mathcal{V}_\lambda/\varpi^m$ is the composite of the morphism $R1_*^{\widetilde{K}_{U,S}}\mathcal{V}_{\widetilde{\lambda}}/\varpi^m \to \mathcal{V}_{\widetilde{\lambda}}/\varpi^m$ (given by restriction to the trivial subgroup) and the reduction modulo ϖ^m of the K-equivariant splitting $\mathcal{V}_{\widetilde{\lambda}} \to \mathcal{V}_\lambda$. This completes the proof. □

Here is a variant of Theorem 2.4.4 where we now take trivial coefficients but consider additional Hecke operators at some ramified places.

THEOREM 2.4.8. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ be as at the start of Section 2.4.1, let $\mathfrak{m} \subset \mathbf{T}^S(K,0)$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S$ denote its pullback under the homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$. Suppose, moreover, that there is a subset $R \subset S$ satisfying the following conditions:*

- *Each place $v \in R$ is prime to p and is split over F^+.*
- *For each place $v \in R - R^c$ lying over a place $\overline{v}$ of F^+, $\widetilde{K}_{\overline{v}} = \widetilde{\mathfrak{q}}_v$, where $\widetilde{\mathfrak{q}}_v$ contains $\widetilde{\mathfrak{p}}_{v,1}$ and is contained in $\widetilde{\mathfrak{p}}_v$. For each place $v \in R \cap R^c$ lying over a place $\overline{v}$ of F^+, $\widetilde{K}_{\overline{v}} = \widetilde{I}_{\overline{v}}$, where $\widetilde{I}_{\overline{v}}$ contains $\widetilde{\mathrm{Iw}}_{\overline{v},1}$ and is contained in $\widetilde{\mathrm{Iw}}_{\overline{v}}$.*

Let $T = S - (R^c - R)$. *Let* $\widetilde{\mathbf{T}}_R^T \subset \mathcal{H}(\widetilde{G}(\mathbf{A}_{F+}^\infty), \widetilde{K}) \otimes_{\mathbf{Z}} \mathcal{O}$ *denote the (commutative)* $\mathcal{O}$*-subalgebra generated by* $\widetilde{\mathbf{T}}^S$ *and all the elements* $t_{v,i}(\sigma)$ $(v \in R, \sigma \in W_{F_v})$ *and* $e_{v,i}(\sigma)$ $(v \in R^c - R, \sigma \in W_{F_v})$. *Let* $\mathbf{T}_R^T \subset \mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^\infty), K) \otimes_{\mathbf{Z}} \mathcal{O}$ *denote the (commutative)* $\mathcal{O}$*-subalgebra generated by* $\mathbf{T}^T$ *and all the elements* $t_{v,i}(\sigma)$ $(v \in R, \sigma \in W_{F_v})$. *Then there is a map* $\mathcal{S} : \widetilde{\mathbf{T}}_R^T \to \mathbf{T}_R^T$, *which descends to an* $\mathcal{O}$*-algebra homomorphism*

$$\widetilde{\mathbf{T}}_R^T(R\Gamma(\partial \widetilde{X}_{\widetilde{K}}, \mathcal{O})_{\widetilde{\mathfrak{m}}}) \to \mathbf{T}_R^T(R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}).$$

Proof. Let $R_0 = R \cup R^c$, $S_0 = S - R_0$. The map $\mathcal{S}$ is the one described in Section 2.1.2 (at unramified places) and Section 2.1.9. (See, in particular, Lemma 2.1.13, which applies at the ramified places we consider here; cf. the discussion at the end of Section 2.2.5.) Once again, by Theorem 2.4.2, it will be enough for us to show that $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}}^P, \mathcal{O})) \to \mathbf{T}_R^T(R\Gamma(X_K, \mathcal{O})).$$

In order to show the existence of this homomorphism we first recall, following the discussion at the end of Section 2.1.9, that $\mathcal{S}$ arises by localization from the composite of homomorphisms

$$r_P : \mathcal{H}(\widetilde{G}^S \times \Delta_{\widetilde{G},R_0}, \widetilde{K}^{S_0}) \to \mathcal{H}(P^S \times \Delta_{P,R_0}, \widetilde{K}_P^{S_0})$$

and

$$r_G : \mathcal{H}(P^S \times \Delta_{P,R_0}, \widetilde{K}_P^{S_0}) \to \mathcal{H}(G^S \times \Delta_{G,R_0}, K^{S_0}).$$

Moreover, there are morphisms of complexes

$$R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{O}) \xrightarrow{\alpha} R\Gamma(\widetilde{X}_{\widetilde{K}}^P, \mathcal{O}) \xrightarrow{\beta} R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{O})$$

and

$$R\Gamma(X_K, \mathcal{O}) \xrightarrow{\gamma} R\Gamma(X_{\widetilde{K}_P}^P, \mathcal{O}) \xrightarrow{\delta} R\Gamma(X_K, \mathcal{O})$$

satisfying the following conditions:

- β respects the action of $\mathcal{H}(\widetilde{G}^S \times \Delta_{\widetilde{G},R_0}, \widetilde{K}^{S_0})$ (when this algebra acts by r_P on the target of β).
- γ respects the action of $\mathcal{H}(P^S \times \Delta_{P,R_0}, \widetilde{K}_P^{S_0})$ (when this algebra acts by r_G on the source of γ).
- $\beta\alpha$ and $\delta\gamma$ both equal the identity.

Let $\widetilde{\mathbf{T}}_{R,+}^T$ denote the intersection of $\widetilde{\mathbf{T}}_R^T$ with $\mathcal{H}(\widetilde{G}^S \times \Delta_{\widetilde{G},R_0}, \widetilde{K}^{S_0}) \otimes_{\mathbf{Z}} \mathcal{O}$ (intersection taken inside $\mathcal{H}(\widetilde{G}^{S_0}, \widetilde{K}^{S_0}) \otimes_{\mathbf{Z}} \mathcal{O}$). Define $\mathbf{T}_{R,+}^T$ similarly. Then $\mathcal{S}(\widetilde{\mathbf{T}}_{R,+}^T) \subset \mathbf{T}_{R,+}^T$, and the above listed properties immediately imply that $\mathcal{S}_+ = \mathcal{S}|_{\widetilde{\mathbf{T}}_{R,+}^T}$ descends to a morphism as in the top horizontal arrow of the

following diagram:

$$\begin{array}{ccc} \widetilde{\mathbf{T}}^T_{R,+}(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O})) & \longrightarrow & \mathbf{T}^T_{R,+}(R\Gamma(X_K,\mathcal{O})) \\ \downarrow \subset & & \downarrow \subset \\ \widetilde{\mathbf{T}}^T_R(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O})) & & \mathbf{T}^T_R(R\Gamma(X_K,\mathcal{O})). \end{array}$$

By construction, we can find an element $z \in \widetilde{\mathbf{T}}^T_{R,+}(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O}))$ with the following properties:

- z is a unit in $\widetilde{\mathbf{T}}^T_R(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O}))$ and
$$\widetilde{\mathbf{T}}^T_R(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O})) = \widetilde{\mathbf{T}}^T_{R,+}(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O}))[z^{-1}].$$
- $\mathcal{S}_+(z)$ is a unit in $\mathbf{T}^T_R(R\Gamma(X_K,\mathcal{O}))$ and
$$\mathbf{T}^T_R(R\Gamma(X_K,\mathcal{O})) = \mathbf{T}^T_{R,+}(R\Gamma(X_K,\mathcal{O}))[\mathcal{S}_+(z)^{-1}].$$

Indeed, we can take z to be a product (over places of $\overline{R}_0$) of strongly positive Hecke operators, as in the statement of Lemma 2.1.13. We deduce that in fact $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}^T_R(R\Gamma(\widetilde{X}^P_{\widetilde{K}},\mathcal{O})) \to \mathbf{T}^T_R(R\Gamma(X_K,\mathcal{O})),$$

as required. □

2.4.9. *Some results on rational cohomology.*

THEOREM 2.4.10. *Fix a choice of isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$.*

(1) *Let π be a cuspidal, regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$. Suppose that there exists a good subgroup $K \subset \mathrm{GL}_n(\mathbf{A}_F)$ such that $(\pi^\infty)^K \neq 0$. Then the map $\mathbf{T}^S \to \overline{\mathbf{Q}}_p$ associated to the Hecke eigenvalues of $(\iota^{-1}\pi^\infty)^K$ factors through the quotient $\mathbf{T}^S \to \mathbf{T}^S(K,\lambda)$.*

(2) *Let $q_0 = [F^+ : \mathbf{Q}]n(n-1)/2$, $l_0 = [F^+ : \mathbf{Q}]n - 1$. Let $K \subset \mathrm{GL}_n(\mathbf{A}_F)$ be a good subgroup, and let $\mathfrak{m} \subset \mathbf{T}^S(K,\mathcal{V}_\lambda)$ be a maximal ideal such that $\overline{\rho}_\mathfrak{m}$ is absolutely irreducible. Then for each $j \in \mathbf{Z}$, the group*
$$H^j(X_K,\mathcal{V}_\lambda)_\mathfrak{m}[1/p]$$
is non-zero only if $j \in [q_0, q_0 + l_0]$; moreover if one of the groups in this range is non-zero, then they all are.

If $f : \mathbf{T}^S(K,\mathcal{V}_\lambda)_\mathfrak{m} \to \overline{\mathbf{Q}}_p$ is a homomorphism, then there exists a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$ such that f is associated to the Hecke eigenvalues of $(\iota^{-1}\pi^\infty)^K$. In particular, there is an isomorphism $\overline{r_\iota(\pi)} \cong \overline{\rho}_\mathfrak{m}$.

Proof. For the first part, it suffices to show that there is a non-zero eigenvector for $\mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S)$ in $H^*(X_K, V_{\iota\lambda})$ with the eigenvalue of $T_{v,i}$ equal to its eigenvalue of $T_{v,i}$ on $\pi_v^{K_v}$.

Likewise, for the second part it suffices to show that the group $H^j(X_K, V_{\iota\lambda})_{\mathfrak{m}} := H^j(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}} \otimes_{\mathcal{O},\iota} \mathbf{C}$ is non-zero only if $j \in [q_0, q_0 + l_0]$, that if one of the groups in this range is non-zero they all are, and that if $f : \mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S) \to \mathbf{C}$ is a system of Hecke eigenvalues appearing in $H^*(X_K, V_{\iota\lambda})_{\mathfrak{m}}$, then there is a cuspidal, regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$ giving rise to this system of Hecke eigenvalues.

As a consequence of Franke's theorem [Fra98, Th. 18], as in [FS98, §2.2], we have a canonical decomposition

$$H^*(X_K, V_{\iota\lambda}) = \left(\bigoplus_{\{Q\}\in\mathcal{C}} H^*(\mathfrak{m}_G, K_\infty; A_{V_{\iota\lambda},\{Q\}} \otimes_{\mathbf{C}} V_{\iota\lambda})(\chi_\lambda) \right)^K .$$

In this formula, $\mathcal{C}$ is the set of associate classes of parabolic $\mathbf{Q}$-subgroups of $\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n$. The cohomology on the right-hand side is relative Lie algebra cohomology, $\mathfrak{m}_G$ is the Lie algebra of the real points of the algebraic group given by the kernel of the map $N_{F/\mathbf{Q}} \circ \det : \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n \to \mathrm{GL}_1$, and $A_{V_{\iota\lambda},\{Q\}}$ is a certain space of automorphic forms (in particular, it is a $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-module). Finally, the (χ_λ) denotes a twist of the $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-module structure, determined by the central character of $V_{\iota\lambda}$, which appears because the automorphic forms considered in *loc. cit.* are by definition invariant under translation by $\mathbf{R}^{>0} \subset (\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)(\mathbf{R})$. We set $E_{\{Q\}} = H^*(\mathfrak{m}_G, K_\infty; A_{V_{\iota\lambda},\{Q\}} \otimes_{\mathbf{C}} V_{\iota\lambda})(\chi_\lambda)$. The summand $E^K_{\{G\}}$ is the cuspidal cohomology group

$$H^*_{cusp}(X_K, V_{\iota\lambda}) = \bigoplus_{\pi} (\pi^\infty)^K \otimes_{\mathbf{C}} H^*(\mathfrak{m}_G, K_\infty; \pi_\infty \otimes_{\mathbf{C}} V_{\iota\lambda}),$$

where the sum is over cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbf{A}_F)$ with central character ξ satisfying $\xi|_{\mathbf{R}_{>0}} = \xi_{\iota\lambda}^{-1}|_{\mathbf{R}_{>0}}$, where $\xi_{\iota\lambda}$ is the central character of $V_{\iota\lambda}$.

Let $\mathfrak{M}$ be a maximal ideal of $\mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S) \otimes_{\mathbf{Z}} \mathbf{C}$ in the support of $E^K_{\{Q\}}$. Suppose $Q \subset \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n$ is the standard (block upper triangular) parabolic subgroup corresponding to the partition $n = n_1 + \cdots + n_r$. We denote its standard (block diagonal) Levi factor by L_Q. In order to simplify notation, we set

$$W = W((\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)_{\mathbf{C}}, (\mathrm{Res}_{F/\mathbf{Q}}\,T_n)_{\mathbf{C}}),$$

$$W_Q = W_Q((\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)_{\mathbf{C}}, (\mathrm{Res}_{F/\mathbf{Q}}\,T_n)_{\mathbf{C}}),$$

and

$$W^Q = W^Q((\mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GL}_n)_{\mathbf{C}}, (\mathrm{Res}_{F/\mathbf{Q}}\,T_n)_{\mathbf{C}})$$

for the respective Weyl groups (notation as in Section 1.2). It follows from [FS98, Prop. 3.3] (see also the proof of [Fra98, Th. 20]) that $\mathfrak{M}$ corresponds to the system of Hecke eigenvalues for the (unnormalized) parabolic induction $\mathrm{Ind}_{Q(\mathbf{A}^\infty)}^{\mathrm{GL}_n(\mathbf{A}_F^\infty)} \sigma^\infty$, where $\sigma = \bigotimes_{i=1}^r \pi_i$ is a cuspidal automorphic representation of $L_Q(\mathbf{A_Q}) = \prod_{i=1}^r \mathrm{GL}_{n_i}(\mathbf{A}_F)$ whose infinitesimal character matches that of the dual of the $(L_Q)_{\mathbf{C}}$-representation with highest weight $w(\iota\lambda + \rho) - \rho$, for some w in the set W^Q. Here ρ denotes half the sum of the $(\mathrm{Res}_{F/\mathbf{Q}} B_n)_{\mathbf{C}}$-positive roots, and we note that each $w(\iota\lambda + \rho) - \rho$ is a dominant weight for $(L_Q)_{\mathbf{C}}$. In particular, the π_i are regular algebraic cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ (whose weight depends on w).

We sketch how this statement can be deduced from the proof of [FS98, Prop. 3.3]. The space $A_{V_{\iota\lambda},\{Q\}}$ decomposes, as a $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-module, into a direct sum $\oplus_\varphi A_{V_{\iota\lambda},\{Q\},\varphi}$. Each space of automorphic forms $A_{V_{\iota\lambda},\{Q\},\varphi}$ is the quotient of a space denoted $W_{Q,\widetilde{\pi}} \otimes S(\check{\mathfrak{a}}_Q^G)$ in *loc. cit.* It is also observed in the proof of [FS98, Prop. 3.3] that this space, as a $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-module, has a filtration whose quotients are isomorphic as $\mathrm{GL}_n(\mathbf{A}_F^\infty)$-modules to a normalized parabolic induction $\mathrm{Ind}_{Q(\mathbf{A_Q})}^{\mathrm{GL}_n(\mathbf{A}_F^\infty)}(\delta_Q \otimes \pi^\infty)$. Our notation differs from [FS98], as we are writing Ind for unnormalized parabolic induction. Here π is a cuspidal automorphic representation of $L_Q(\mathbf{A_Q})$ whose infinitesimal character corresponds under the normalized Harish-Chandra isomorphism to a weight in the W-orbit of the infinitesimal character of $V_{\iota\lambda}^\vee$ (by [FS98, 1.2(c)]). The normalization is given by the character δ_Q of $L_Q(\mathbf{A_Q})$ defined by $\delta_Q(l) = e^{\langle H_Q(l), \rho_Q\rangle}$, where H_Q is the standard height function defined in [FS98, p. 769] and ρ_Q is half the sum of the roots in the unipotent radical of Q. Although π will not always be regular algebraic, the twist $\sigma := \delta_Q \otimes \pi$ will be. More precisely, we show that the infinitesimal character of σ equals that of the dual of the $(L_Q)_{\mathbf{C}}$-representation with highest weight $\lambda_w := w(\iota\lambda + \rho) - \rho$ for some $w \in W^Q$. Indeed, we have $v \in W^Q$ such that the infinitesimal character $\chi_\sigma = \chi_\pi + \rho_Q$ is equal to $v(\iota\lambda^\vee + \rho) + \rho_Q$, where $\iota\lambda^\vee$ is the highest weight of $V_{\iota\lambda}^\vee$ (which has infinitesimal character $\iota\lambda^\vee + \rho$). A short calculation shows that $\chi_\sigma = \lambda^\vee_{w_{0,Q} v w_0} + \rho_{L_Q}$, where ρ_{L_Q} is half the sum of the positive roots for L_Q, w_0 is the longest element of W and $w_{0,Q}$ is the longest element of W_Q. Note that since W^Q is characterized by taking dominant weights for GL_n to dominant weights for L_Q (equivalently, taking anti-dominant weights to anti-dominant weights), $w_{0,Q} v w_0$ is an element of W^Q, so this gives the desired statement.

Returning to the proof of the theorem, it now follows from Theorem 2.3.2 that there is a Galois representation

$$r_\iota(\mathfrak{M}) : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

such that for all but finitely many $v \notin S$, the characteristic polynomial of $r_\iota(\mathfrak{M})(\mathrm{Frob}_v)$ equals $P_v(X)$ mod $\mathfrak{M}$. Indeed, (cf. the proof of [NT16, Th. 4.2])

we have

$$r_\iota(\mathfrak{M}) = \bigoplus_{i=1}^{r} r_\iota(\pi_i) \otimes \epsilon^{-(n_{i+1}+\cdots+n_r)}.$$

We can now deduce that if Q is a proper parabolic, then $(E^K_{\{Q\}})_{\mathfrak{m}} = E^K_{\{Q\}} \otimes_{\mathbf{T}^S(K,\mathcal{V}_\lambda)} \mathbf{T}^S(K,\mathcal{V}_\lambda)_{\mathfrak{m}}$ vanishes. Suppose $\mathfrak{M}$ is a maximal ideal of

$$\mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S) \otimes_{\mathbf{Z}} \mathbf{C}$$

in the support of $(E^K_{\{Q\}})_{\mathfrak{m}}$. On the one hand, the representation $r_\iota(\mathfrak{M})$ is reducible in this case, but we have an isomorphism $\overline{r_\iota(\pi)} \cong \overline{\rho}_{\mathfrak{m}}$. This contradicts the assumption that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, so we deduce that $(E^K_{\{Q\}})_{\mathfrak{m}} = 0$.

Finally, we show both parts of the theorem. It suffices to show that if π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ with central character matching $\xi_{\iota\lambda}^{-1}$ on $\mathbf{R}_{>0}$, then

(1) $H^*(\mathfrak{m}_G, K_\infty; \pi_\infty \otimes V_{\iota\lambda})$ is zero unless π is regular algebraic of weight $\iota\lambda$.
(2) If π is regular algebraic of weight $\iota\lambda$, then $H^j(\mathfrak{m}_G, K_\infty; \pi_\infty \otimes V_{\iota\lambda})$ vanishes for $j \notin [q_0, q_0 + l_0]$ and is non-zero for $j \in [q_0, q_0 + l_0]$.

The first claim follows from [BW00, Ch. II, Prop. 3.1]. The second claim follows from [Clo90, Lem. 3.14] and the Künneth formula for relative Lie algebra cohomology. (In the notation of *loc. cit.*, our Lie algebra $\mathfrak{m}_G$ is a direct sum of $\widetilde{\mathfrak{g}}_v$ for each infinite place v of F and an abelian Lie algebra of dimension $[F^+ : \mathbf{Q}] - 1$; the range of non-zero cohomological degrees is $n-1$ for $(\widetilde{\mathfrak{g}}_v, K_v)$-cohomology, so we get range $(n-1)[F^+ : \mathbf{Q}] + [F^+ : \mathbf{Q}] - 1 = l_0$ in total.) □

THEOREM 2.4.11. *Let $\rho \in X^*(\mathrm{Res}_{F^+/\mathbf{Q}} T_n)$ denote half the sum of the positive roots of $\mathrm{Res}_{F^+/\mathbf{Q}} \widetilde{G}$. Fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Let $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E))}$ be a highest weight with the following property: for any $w \in W^P((\mathrm{Res}_{F^+/\mathbf{Q}} \widetilde{G})_{\mathbf{C}}, (\mathrm{Res}_{F^+/\mathbf{Q}} T)_{\mathbf{C}})$, there are no (characteristic 0) cuspidal automorphic representations for G of weight $\iota\lambda_w$, where $\lambda_w = w(\widetilde{\lambda} + \rho) - \rho$.*

Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S$ be a maximal ideal that is in the support of $H^*(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})$ with the property that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is a direct sum of n-dimensional absolutely irreducible representations of G_F. Let $d = \frac{1}{2}\dim_{\mathbf{R}} X^{\widetilde{G}} = n^2[F^+ : \mathbf{Q}]$.

Then $H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}[1/p]$ is a semisimple $\widetilde{\mathbf{T}}^S[1/p]$-module, and for every homomorphism $f : \widetilde{\mathbf{T}}^S(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}) \to \overline{\mathbf{Q}}_p$, there exists a cuspidal, regular algebraic automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbf{A}_{F^+})$ of weight $\iota\widetilde{\lambda}$ such that f is associated to the Hecke eigenvalues of $(\iota^{-1}\widetilde{\pi}^\infty)^{\widetilde{K}}$.

Proof. The proof uses similar ingredients to the proof of Theorem 2.4.10 above. We must understand the systems of $\widetilde{\mathbf{T}}^S$-eigenvalues occurring in

$$H^d(\widetilde{X}_{\widetilde{K}}, V_{\iota\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} := H^d\left(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}\right)_{\widetilde{\mathfrak{m}}}[1/p] \otimes_{E,\iota} \mathbf{C}.$$

As a consequence of [Fra98, Th. 18], as in [FS98, §2.2], applied to the group $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$, we have a canonical decomposition

$$H^d(\widetilde{X}_{\widetilde{K}}, V_{\iota\widetilde{\lambda}}) = \left(\bigoplus_{\{\widetilde{P}\}\in\mathcal{C}} H^d\left(\widetilde{\mathfrak{g}}, \widetilde{K}_\infty; A_{V_{\iota\widetilde{\lambda}},\{\widetilde{P}\}} \otimes_{\mathbf{C}} V_{\iota\widetilde{\lambda}}\right)\right)^{\widetilde{K}}.$$

Here, $\mathcal{C}$ is the set of associate classes of parabolic $\mathbf{Q}$-subgroups of $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$. The cohomology on the right-hand side is relative Lie algebra cohomology, $\widetilde{\mathfrak{g}}$ is the Lie algebra of $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})(\mathbf{R})$, and $A_{V_{\iota\widetilde{\lambda}},\{\widetilde{P}\}}$ is a certain space of automorphic forms for $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$. (We note that in this case there is no additional character twist of the $\widetilde{G}(\mathbf{A}_F^\infty)$-module structure, because the maximal split torus in the center of $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$ is trivial; equivalently, the $\mathfrak{m}_{\widetilde{G}}$ of [FS98] is equal to $\widetilde{\mathfrak{g}}$ because $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$ has no rational characters.) Set $E_{\{\widetilde{P}\}} := H^d\big(\mathfrak{m}_{\widetilde{G}}, \widetilde{K}_\infty; A_{V_{\iota\widetilde{\lambda}},\{\widetilde{P}\}} \otimes_{\mathbf{C}} V_{\iota\widetilde{\lambda}}\big)$. The summand $E^{\widetilde{K}}_{\{\widetilde{G}\}}$ is the cuspidal cohomology group

$$H^d_{cusp}\left(\widetilde{X}_{\widetilde{K}}, V_{\iota\widetilde{\lambda}}\right) = \bigoplus_{\widetilde{\pi}} (\widetilde{\pi}^\infty)^{\widetilde{K}} \otimes_{\mathbf{C}} H^d\left(\widetilde{\mathfrak{g}}, \widetilde{K}_\infty; \widetilde{\pi}_\infty \otimes_{\mathbf{C}} V_{\iota\widetilde{\lambda}}\right),$$

where the sum runs over cuspidal automorphic representations of $\widetilde{G}(\mathbf{A}_{F^+})$. We see that the theorem will be proved if we can establish the following two claims:

(1) If $\widetilde{P}$ is a proper standard parabolic subgroup of $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$ different from the Siegel parabolic, then

$$\left(\iota^{-1}E^{\widetilde{K}}_{\{\widetilde{P}\}}\right)_{\widetilde{\mathfrak{m}}} := \iota^{-1}E^{\widetilde{K}}_{\{\widetilde{P}\}} \otimes_{\widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})} \widetilde{\mathbf{T}}^S(\widetilde{K},\widetilde{\lambda})_{\widetilde{\mathfrak{m}}} = 0.$$

(2) If $\widetilde{P} = P$ is the Siegel parabolic subgroup of $\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G}$, then we also have $\left(\iota^{-1}E^{\widetilde{K}}_{\{\widetilde{P}\}}\right)_{\widetilde{\mathfrak{m}}} = 0$.

The same argument as in the proof of Theorem 2.4.10 shows that if $\widetilde{\mathfrak{M}}$ is a maximal ideal of $\widetilde{\mathbf{T}}^S[1/p]$ that occurs in the support of $\iota^{-1}E^{\widetilde{K}}_{\{\widetilde{P}\}}$, then $\widetilde{\mathfrak{M}}$ corresponds to the system of Hecke eigenvalues appearing in

$$\left(\mathrm{Ind}_{\widetilde{P}(\mathbf{A}_{F^+}^\infty)}^{\widetilde{G}(\mathbf{A}_{F^+}^\infty)} \iota^{-1}\sigma^\infty\right)^{\widetilde{K}},$$

where σ is a cuspidal automorphic representation of $L_{\widetilde{P}}(\mathbf{A}_{F^+})$ whose infinitesimal character equals the dual of the infinitesimal character of the irreducible algebraic representation of $L_{\widetilde{P}}$ of highest weight $w(\iota\widetilde{\lambda}+\rho)-\rho$ for some

$w \in W^{\widetilde{P}}((\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\mathbf{C}}, (\mathrm{Res}_{F^+/\mathbf{Q}}T)_{\mathbf{C}})$. The second claim now follows immediately from our hypothesis that there are no such automorphic representations in the case $\widetilde{P} = P$.

As in the proof of Theorem 2.4.2, we note that the Levi subgroup $L_{\widetilde{P}}$ is isomorphic to a product $\mathrm{Res}_{F/F^+}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{F/F^+}\mathrm{GL}_{n_r} \times U(n-s, n-s)$ for some decomposition $2n = n_1 + \cdots + n_r + 2(n-s)$. We can now establish the first claim: using the existence of Galois representations attached to regular algebraic cuspidal automorphic representations of GL_m and $U(m,m)$ for $m \leq n$ (i.e., using Theorems 2.3.2 and 2.3.3), we see that there exists a Galois representation $r(\widetilde{\mathfrak{M}}) : G_F \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$ such that for all but finitely many places v of F, $r(\widetilde{\mathfrak{M}})$ is unramified at v and $r(\widetilde{\mathfrak{M}})$ has characteristic polynomial equal to $\widetilde{P}_v(X)$ mod $\widetilde{\mathfrak{M}}$. Moreover, this representation has at least three Jordan–Hölder factors as soon as $(r,s) \notin \{(1,n),(0,0)\}$ (by an argument identical to the one appearing at the end of the proof of Theorem 2.4.2). Since we are assuming that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ has two irreducible constituents, each of dimension n, this would lead to a contradiction, showing that we must in fact have $\left(\iota^{-1}E^{\widetilde{K}}_{\{\widetilde{P}\}}\right)_{\widetilde{\mathfrak{m}}} = 0$. This completes the proof. $\square$

3. Local-global compatibility, $l \neq p$

3.1. *Statements.* Let F be a CM field containing an imaginary quadratic field, and fix an integer $n \geq 1$. Let p be a prime, and let E be a finite extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ large enough to contain the images of all embeddings of F in $\overline{\mathbf{Q}}_p$. We assume that each p-adic place $\overline{v}$ of F^+ splits in F.

Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Let S be a finite set of finite places of F, containing the p-adic places, and satisfying the following conditions:

- $S = S^c$.
- Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.

We recall (Theorem 2.3.7) that under these hypotheses, if $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ is a non-Eisenstein maximal ideal, then there is a continuous homomorphism

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/I)$$

characterized, up to conjugation, by the characteristic polynomials of Frobenius elements at places $v \notin S$; here I is a nilpotent ideal whose exponent depends only on n and $[F:\mathbf{Q}]$. Our goal in this chapter is to describe the restriction of $\rho_{\mathfrak{m}}$ to decomposition groups at some prime-to-p places where ramification is allowed.

To this end, we suppose given as well a subset $R \subset S$ satisfying the following conditions:

- Each place $v \in R$ is prime to p.
- For each place $v \in R$, there exists an imaginary quadratic field $F_0 \subset F$ in which the residue characteristic of v splits. In particular, v is split over F^+.
- For each place $v \in R$, K_v contains $\mathrm{Iw}_{v,1}$ and is contained in Iw_v. For each place $v \in R^c - R$, $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$. (Note that $R^c \subset S$ since S is assumed stable under complex conjugation.)

Let $T = S - (R^c - R)$. We define $\mathbf{T}_R^T \subset \mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^\infty), K) \otimes_{\mathbf{Z}} \mathcal{O}$ to be the (commutative) $\mathcal{O}$-subalgebra generated by $\mathbf{T}^T$ and all the elements $t_{v,i}(\sigma)$ $(v \in R, \sigma \in W_{F_v})$, as in the statement of Theorem 2.4.8. We define

$$\mathbf{T}_R^T(K,\lambda) \subset \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda))$$

to be the image of $\mathbf{T}_R^T$. Thus there are inclusions

$$\mathbf{T}^S(K,\lambda) \subset \mathbf{T}^T(K,\lambda) \subset \mathbf{T}_R^T(K,\lambda).$$

THEOREM 3.1.1. *Let notation and assumptions be as above. Then we can find an integer $N \geq 1$ (depending only on n and $[F : \mathbf{Q}]$), an ideal $I_R \subset \mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}$ satisfying $I_R^N = 0$, and a continuous homomorphism*

$$\rho_{\mathfrak{m},R} : G_{F,T} \to \mathrm{GL}_n(\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R)$$

satisfying the following conditions:

(1) *For each place $v \notin T$ of F, the characteristic polynomial of $\rho_{\mathfrak{m},R}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R)[X]$.*
(2) *For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of $\rho_{\mathfrak{m},R}(\sigma)$ is equal to the image of $P_{v,\sigma}(X)$ in $(\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R)[X]$.*

In the statement of this theorem, $\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}$ is the localization of $\mathbf{T}_R^T(K,\lambda)$ as a $\mathbf{T}^S(K,\lambda)$-algebra; it is an $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}})$ that contains $\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}$. Instead of proving this theorem directly, we will in fact prove the following statement:

PROPOSITION 3.1.2. *Let notation and assumptions be as above. Then there exist an integer $N \geq 1$ (depending only on n and $[F : \mathbf{Q}]$), an ideal $I_R \subset \mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}$ satisfying $I_R^N = 0$, and a $\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R$-valued determinant $D_{\mathfrak{m},R}$ on $G_{F,T}$ of dimension n satisfying the following conditions*:

(1) *For each place $v \notin T$, the characteristic polynomial of Frob_v in $D_{\mathfrak{m},R}$ is equal to the image of $P_v(X)$ in $(\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R)[X]$.*
(2) *For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of σ is equal to the image of $P_{v,\sigma}(X)$ in $(\mathbf{T}_R^T(K,\lambda)_{\mathfrak{m}}/I_R)[X]$.*

Proposition 3.1.2 implies Theorem 3.1.1 by [Che14, Th. 2.22]. The remainder of Section 3 is devoted to the proof of Proposition 3.1.2. Although Proposition 3.1.2 is an assertion about determinants, not true representations, we will still use the assumption that $\mathfrak{m}$ is non-Eisenstein in the proof — in particular, as it simplifies our analysis of the boundary cohomology (using the results proved in Section 2.4.1).

3.2. *The proof of Proposition 3.1.2.* Let $R_1 = R \cap R^c$. Let $\overline{R}$ (resp. $\overline{R}_1$) denote the set of places of F^+ lying below a place of R (resp. R_1). We begin with a preliminary reduction.

LEMMA 3.2.1. *Fix for each $v \in R$ a choice of Frobenius lift $\phi_v \in W_{F_v}$. In order to prove Proposition 3.1.2, it is enough to prove it under the following additional assumptions*:

(1) *$K_v = \mathrm{Iw}_{v,1}$ for each place $v \in R$. There exists an odd prime q, prime to R and p, such that $K_q = \ker(\mathrm{GL}_n(\mathcal{O}_{F,q}) \to \mathrm{GL}_n(\mathcal{O}_F/(q)))$.*
(2) *For each place $v \in R$, the characteristic polynomials of $\overline{\rho}_{\mathfrak{m}}(\phi_v)$ and $(\overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\phi_v)$ are coprime.*
(3) *There exists a character $\psi : G_F \to \mathcal{O}^\times$ of finite prime-to-p order, unramified above $R \cup R^c \cup S_p$, such that the composite $\psi \circ \mathrm{Art}_F \circ \det : K \to \mathcal{O}^\times$ is trivial and for each $v \in R$, the characteristic polynomials of $\overline{\psi}(\mathrm{Frob}_v)\overline{\rho}_{\mathfrak{m}}(\phi_v)$ and $\overline{\psi}(\mathrm{Frob}_{v^c})^{-1}(\overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\phi_v)$ are coprime.*
(4) *$\lambda = 0$.*
(5) *There exists a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ satisfying the following conditions*:
   (a) *$\widetilde{K}$ is decomposed with respect to P, and $K = \widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty)$.*
   (a) *$\widetilde{K}_q = \ker(\widetilde{G}(\mathcal{O}_{F^+,q}) \to \widetilde{G}(\mathcal{O}_{F^+}/(q)))$.*
   (a) *If $\overline{v}$ is a finite place of F^+ that is prime to S, then $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+,v})$. If $\overline{v} \in \overline{R}_1$, then $\widetilde{K}_v = \widetilde{\mathrm{Iw}}_{\overline{v}}(1,1)$. If $\overline{v} \in \overline{R} - \overline{R}_1$ and v is the unique place of R lying above $\overline{v}$, then $\widetilde{K}_{\overline{v}} = \widetilde{\mathfrak{p}}_{v,1}$.*

Proof. We first show that if Proposition 3.1.2 holds under assumption (1), then it holds without this assumption. Let assumptions be as in Proposition 3.1.2, and let $q_1, q_2 \neq p$ be distinct odd primes not dividing any element of S. Let $K_i \subset K$ be the normal subgroup with $K_{i,v} = \mathrm{Iw}_v(1,1)$ if $v \in R$, $K_{i,q_i} = \ker(\mathrm{GL}_n(\mathcal{O}_{F,q_i}) \to \mathrm{GL}_n(\mathcal{O}_F/(q_i)))$, and $K_i^{R,q_i} = K$. Let S_i (resp. T_i) denote the union of S (resp. T) with the set of q_i-adic places of F. Let $\mathfrak{m}_i \subset \mathbf{T}^{T_i}$ denote the pullback of $\mathfrak{m}$ under the inclusion $\mathbf{T}^{T_i} \to \mathbf{T}^T$. For each $i = 1, 2$, there is a diagram of $\mathbf{T}_R^{T_i}$-algebras

$$\mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}_i} \leftarrow \mathbf{T}_R^{T_i}(K/K_i, \lambda)_{\mathfrak{m}_i} \twoheadrightarrow \mathbf{T}_R^{T_i}(K, \lambda)_{\mathfrak{m}_i} \to \mathbf{T}_R^{T}(K, \lambda)_{\mathfrak{m}}.$$

The left-hand arrow has nilpotent kernel of exponent d depending only on n and $[F : \mathbf{Q}]$, by Lemma 2.2.4. By hypothesis, there exists an integer $N \geq 1$, depending only on n and $[F : \mathbf{Q}]$, ideals $J_i \subset \mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}_i}$ satisfying $J_i^N = 0$, and n-dimensional group determinants D_i of G_{F,T_i} with coefficients in $\mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}_i}/J_i$ satisfying conditions (1) and (2) of Proposition 3.1.2.

Let I_i denote the image in $\mathbf{T}_R^{T_i}(K, \lambda)_{\mathfrak{m}_i}$ of the pre-image of J_i in the ring $\mathbf{T}_R^{T_i}(K/K_i, \lambda)_{\mathfrak{m}_i}$, and let $I \subset \mathbf{T}_R^T(K, \lambda)_{\mathfrak{m}}$ denote the ideal generated by the images of I_1 and I_2. Then $I^{2Nd} = 0$. Let $D_{\mathfrak{m}}$ denote the pushforward of the determinant D_1 to $\mathbf{T}_R^T(K, \lambda)_{\mathfrak{m}}/I$. Then by construction, $D_{\mathfrak{m}}$ is an n-dimensional determinant of G_{F,T_1} satisfying condition (1) of Proposition 3.1.2 at prime-to-T_1 places and condition (2) at each place of R. However, the Chebotarev density theorem and [Che14, Lem. 1.12] imply that $D_{\mathfrak{m}}$ is also equal to the pushforward of D_2 to $\mathbf{T}_R^T(K, \lambda)_{\mathfrak{m}}/I$. We therefore obtain the required local-global compatibility also at the q_1-adic places of F. The proof of this step is complete on noting that the exponent $2Nd$ of I indeed still depends only on n and the degree $[F : \mathbf{Q}]$.

We next show that if Proposition 3.1.2 holds under assumptions (1) and (2) in the statement of the lemma, then it holds under assumption (1). After possibly enlarging $\mathcal{O}$, we can find characters $\psi_1, \psi_2 : G_F \to \mathcal{O}^\times$ of finite, prime-to-p order satisfying the following conditions:

- Both ψ_1, ψ_2 are unramified at each place of S.
- There is no rational prime r such that ψ_1, ψ_2 are both ramified at r.
- For each $i = 1, 2$ and for each place $v \in R$, the characteristic polynomials of $(\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi_i})(\phi_v)$ and $((\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi}_i))^{c,\vee} \otimes \epsilon^{1-2n})(\phi_v)$ are coprime.

Let $K_i = \prod_v \ker(\psi_i \circ \mathrm{Art}_{F_v} \circ \det : K_v \to \mathcal{O}^\times)$, and let T_i denote the union of T with the set of places dividing a rational prime above which ψ_i is ramified. Let $\mathfrak{m}_i$ denote the pullback of $\mathfrak{m}$ to $\mathbf{T}^{T_i}$. Proposition 2.2.23 shows that the truth of Proposition 3.1.2 for $\mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}_i}$ is equivalent to the truth of Proposition 3.1.2 for $\mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}_i(\psi_i)}$, which we are assuming. On the other hand, an argument very similar to the one given in the first part of the proof shows that the truth of Proposition 3.1.2 for $\mathbf{T}_R^{T_i}(K_i, \lambda)_{\mathfrak{m}}$ $(i = 1, 2)$ implies the truth of this proposition for $\mathbf{T}_R^{T_i}(K, \lambda)_{\mathfrak{m}_i}$ $(i = 1, 2)$ and then for $\mathbf{T}_R^T(K, \lambda)_{\mathfrak{m}}$. A very similar argument shows that if the proposition holds under assumptions (1)–(3) in the statement of the lemma, then it holds under (1) and (2).

We next show that if Proposition 3.1.2 holds under assumptions (1)–(4) in the statement of the lemma, then it holds under assumptions (1)–(3). Let K be a good subgroup satisfying assumptions (1)–(3). The natural map

$$\mathbf{T}_R^T(K, \lambda) \to \varprojlim_{m \geq 1} \mathbf{T}_R^T(R\Gamma(X_K, \mathcal{V}_\lambda/(\varpi^m)))$$

is an isomorphism. For each $m \geq 1$, let $K(p^m) = \ker(K \to \mathrm{GL}_n(\mathcal{O}_{F,p}/(p^m)))$. Then $K(p^m)$ also satisfies assumption (1). The local system $\mathcal{V}_\lambda/(\varpi^m)$ on

$X_{K(p^m)}$ is constant, so there is a canonical isomorphism of Hecke algebras

$$\mathbf{T}_R^T(K(p^m), \mathcal{V}_\lambda/(\varpi^m)) \cong \mathbf{T}_R^T(K(p^m), \mathcal{O}/(\varpi^m)).$$

There is also a canonical surjection

$$\mathbf{T}_R^T(K(p^m), \mathcal{O}) \to \mathbf{T}_R^T(K(p^m), \mathcal{O}/(\varpi^m)).$$

We consider the diagram of Hecke algebras

$$\mathbf{T}_R^T(K(p^m), \mathcal{O}/(\varpi^m)) \leftarrow \mathbf{T}_R^T(K/K(p^m), \mathcal{V}_\lambda/(\varpi^m)) \to \mathbf{T}_R^T(K, \mathcal{V}_\lambda/(\varpi^m)),$$

where by Lemma 2.2.4, there is an integer $d \geq 1$ depending only on n and $[F : \mathbf{Q}]$ such that the kernel of the left-hand arrow is nilpotent of exponent d. By assumption, therefore, we can find ideals $I_m \subset \mathbf{T}_R^T(K, \mathcal{V}_\lambda/(\varpi^m))_\mathfrak{m}$ satisfying $I_m^{Nd} = 0$ and n-dimensional group determinants D_m of $G_{F,T}$ valued in $\mathbf{T}_R^T(K, \mathcal{V}_\lambda/(\varpi^m))_\mathfrak{m}/I_m$ and satisfying the conditions (1) and (2) in the statement of Proposition 3.1.2. Let

$$I = \ker\left(\mathbf{T}_R^T(K, \lambda)_\mathfrak{m} \to \prod_{m \geq 1} \mathbf{T}_R^T(K, \mathcal{V}_\lambda/(\varpi^m))_\mathfrak{m}/I_m\right).$$

Then $I^{Nd} = 0$ and, by [Che14, Ex. 2.32], there is a unique n-dimensional group determinant $D_\mathfrak{m}$ valued in $\mathbf{T}_R^T(K, \lambda)_\mathfrak{m}/I$ whose pushforward to each ring $\mathbf{T}_R^T(K, \mathcal{V}_\lambda/(\varpi^m))_\mathfrak{m}/I_m$ equals D_m. This determinant $D_\mathfrak{m}$ necessarily has the required properties.

We finally show that if Proposition 3.1.2 holds under assumptions (1)–(5) in the statement of the lemma, then it holds under assumptions (1)–(4). Assume (1)–(4). We define $\widetilde{K} = \prod_{\overline{v}} \widetilde{K}_{\overline{v}}$ as follows:

- If $\overline{v} \notin \overline{S}$, then $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$.
- If $\overline{v} \in \overline{R}_1$, then $\widetilde{K}_{\overline{v}} = \widetilde{\mathrm{Iw}}_{\overline{v}}(1,1)$. If $\overline{v} \in \overline{R} - \overline{R}_1$ and v is the unique place of R lying above $\overline{v}$, then $\widetilde{K}_{\overline{v}} = \widetilde{\mathfrak{p}}_{v,1}$.
- $\widetilde{K}_q = \ker(\widetilde{G}(\mathcal{O}_{F^+,q}) \to \widetilde{G}(\mathcal{O}_{F^+}/(q)))$.
- If $\overline{v}$ is any other finite place of F^+, then fix $m \geq 1$ such that $\ker(\widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}}) \to \widetilde{G}(\mathcal{O}_{F^+}/(\varpi_{\overline{v}}^m))) \cap G(\mathcal{O}_{F^+_{\overline{v}}}) \subset K \cap G(\mathcal{O}_{F^+_{\overline{v}}})$, and set $\widetilde{K}_{\overline{v}} = \ker(\widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}}) \to \widetilde{G}(\mathcal{O}_{F^+}/(\varpi_{\overline{v}}^m))) \cdot (K \cap G(\mathcal{O}_{F^+_{\overline{v}}}))$.

It is easy to check that $\widetilde{K}$ is a good open subgroup of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ that is decomposed with respect to P and that satisfies $\widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty) = K$. The group K therefore satisfies condition (5) of the lemma, and the proof of the lemma is complete. □

We henceforth fix a choice of Frobenius lift $\phi_v \in W_{F_v}$ for each place $v \in R$ and assume that K and $\mathfrak{m}$ satisfy assumptions (1)–(5) of Lemma 3.2.1, and we prove Proposition 3.1.2 for the Hecke algebra $\mathbf{T}_R^T(K, 0)$. Let $\widetilde{K}$ be the good subgroup of $\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ as in the statement of the lemma, and let $\widetilde{\mathbf{T}}_R^T \subset$

$\mathcal{H}(\widetilde{G}(\mathbf{A}_{F^+}^\infty), \widetilde{K}) \otimes_{\mathbf{Z}} \mathcal{O}$ denote the (commutative) $\mathcal{O}$-subalgebra generated by $\widetilde{\mathbf{T}}^S$ and all the elements $t_{v,i}(\sigma)$ ($v \in R, \sigma \in W_{F_v}$) and $e_{v,i}(\sigma)$ ($v \in R^c - R, \sigma \in W_{F_v}$), as in the statement of Theorem 2.4.8. Thus we have constructed an extension of the homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$ to a homomorphism $\mathcal{S} : \widetilde{\mathbf{T}}_R^T \to \mathbf{T}_R^T$. These homomorphisms, together with the analogue of Proposition 3.1.2 for the group $\widetilde{G}$, will be the key to the proof. This analogue is as follows; it makes use of the resultant $\mathrm{Res}_v \in \mathcal{H}(\widetilde{G}(F_{\overline{v}}^+), \widetilde{K}_{\overline{v}}) \otimes_{\mathbf{Z}} \mathcal{O}$ of the polynomials $P_{v^c,\phi_v^{-c}}(X)$ and $P_{v,\phi_v}(X)$ for a place $v \in R - R^c$, which was introduced before Proposition 2.2.14.

Proposition 3.2.2. *There exist an integer $N \geq 1$, depending only on $[F : \mathbf{Q}]$ and n, an ideal $\widetilde{I}_{c,R} \subset \widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))$ satisfying $\widetilde{I}_{c,R}^N = 0$, and a $\widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_{c,R}$-valued determinant $\widetilde{D}_{c,R}$ on $G_{F,S}$ of dimension $2n$ satisfying the following conditions*:

(1) *For each place $v \notin S$ of F, the characteristic polynomial of* Frob_v *is equal to the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_R)[X]$.*
(2) *For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of σ is equal to the image of $\widetilde{P}_{v,\sigma}(X)$ in $(\widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_R)[X]$.*
(3) *Let $\widetilde{\mathrm{tr}}_{c,R} : \widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))[G_{F,S}] \to \widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_{c,R}$ be the trace associated to $\widetilde{D}_{c,R}$ (cf.* [Che14, §1.10]). *Then for each place $v \in R - R^c$, each $\sigma \in G_{F,S}$, and each $\tau_v \in I_{F_v}$, we have* $\mathrm{Res}_v^{(2n)!}\, \widetilde{\mathrm{tr}}_{c,R}(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$.

We note that this result, in the case where R is empty, is Proposition 2.3.9. The result in this case is also contained implicitly in the proof of [Sch15, Cor. 5.2.6].

Proof. The proposition can be proved by re-doing the proof of [Sch15, Cor. 5.2.6] to keep track of the action of the additional Hecke operators at R. For the reader's benefit, we single out the following essential statement (cf. [Sch15, Th. 4.3.1, Cor. 5.1.11]): let $C = \widehat{\overline{\mathbf{Q}}}_p$, let $m \geq 1$ be an integer, and let $\mathbf{T}_{\mathrm{cl}}$ denote $\widetilde{\mathbf{T}}_R^T$, endowed with the weakest topology for which all of the maps

$$\widetilde{\mathbf{T}}_R^T \to \mathrm{End}_C(H^0(\mathcal{X}_{\widetilde{K}^p\widetilde{K}_p}, \omega_{\widetilde{K}^p\widetilde{K}_p}^{mk} \otimes \mathcal{I}))$$

are continuous. (Here the right-hand side, defined as in the statement of [Sch15, Th. 4.3.1], is endowed with its natural (finite-dimensional C-vector space) topology, and we are varying over all $k \geq 1$ and open compact subgroups $\widetilde{K}_p \subset \widetilde{G}(F_p^+)$ such that $\widetilde{K}^p\widetilde{K}_p$ is a good subgroup.) Then for any continuous quotient $\mathbf{T}_{\mathrm{cl}} \to A$, where A is a ring with the discrete topology, there is a unique A-valued determinant D_A of $G_{F,S}$ of dimension $2n$ satisfying the following conditions:

- For each place $v \notin S$ of F, the characteristic polynomial of Frob_v equals the image of $\widetilde{P}_v(X)$ in $A[X]$.
- For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of σ equals the image of $\widetilde{P}_{v,\sigma}(X)$ in $A[X]$.
- Let $\mathrm{tr}_A : A[G_{F,S}] \to A$ denote the trace associated to D_A. Then for each place $v \in R - R^c$, for each $\sigma \in G_{F,S}$, and for each $\tau_v \in I_{F_v}$, we have $\mathrm{Res}_v^{(2n)!}\,\mathrm{tr}_A(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$.

This statement can be proved in exactly the same way as [Sch15, Cor. 5.1.11], by combining [Che14, Ex. 2.32] with the following observation: take a cuspidal, cohomological automorphic representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbf{A}_{F^+})$ such that $\widetilde{\pi}^{\infty,\widetilde{K}} \neq 0$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, and let $\widetilde{\mathbf{T}}_R^T(\widetilde{\pi}) = \mathrm{im}(\widetilde{\mathbf{T}}_R^T \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \to \mathrm{End}_{\overline{\mathbf{Q}}_p}(\iota^{-1}\widetilde{\pi}^{\infty,\widetilde{K}}))$. Consider the associated Galois representation (whose existence and local properties are described by Theorem 2.3.3):

$$r_\iota(\widetilde{\pi}) : G_{F,S} \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p).$$

Let $\rho : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}_R^T(\widetilde{\pi}))$ denote the composite of $r_\iota(\widetilde{\pi})$ with the inclusion $\mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p) \subset \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}_R^T(\widetilde{\pi}))$. Then we have the following properties:

- For each place $v \notin S$ of F, the characteristic polynomial of $\rho(\mathrm{Frob}_v)$ equals the image of $\widetilde{P}_v(X)$ in $\widetilde{\mathbf{T}}_R^T(\widetilde{\pi})[X]$.
- For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of $\rho(\sigma)$ equals the image of $\widetilde{P}_{v,\sigma}(X)$ in $\widetilde{\mathbf{T}}_R^T(\widetilde{\pi})[X]$.
- For each place $v \in R^c - R$ and for each $\tau_v \in I_{F_v}$, we have the equality $\mathrm{Res}_v^{(2n)!}\,\rho((\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$ in $M_{2n}(\widetilde{\mathbf{T}}_R^T(\widetilde{\pi}))$.

The first two points follow from Theorem 2.3.3 and Proposition 2.2.9. (Note that the images of $\widetilde{P}_v(X)$ and $\widetilde{P}_{v,\sigma}(X)$ in $\widetilde{\mathbf{T}}_R^T(\widetilde{\pi})[X]$ in fact lie in $\overline{\mathbf{Q}}_p[X]$.) The third point follows from the same theorem and Corollary 2.2.15. (Our appeals to Theorem 2.3.3 here are the source of our assumption, at the beginning of this section, that each place of R has residue characteristic that splits in an imaginary quadratic subfield of F.) □

Corollary 3.2.3. *There exist an integer $N \geq 1$, depending only on $[F : \mathbf{Q}]$ and n, an ideal $\widetilde{I}_{\partial,R} \subset \widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{O}))$ satisfying $\widetilde{I}_{\partial,R}^N = 0$, and a $\widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_{\partial,R}$-valued determinant $\widetilde{D}_{\partial,R}$ on $G_{F,S}$ of dimension $2n$ satisfying the following conditions*:

(1) *For each place $v \notin S$ of F, the characteristic polynomial of* Frob_v *is equal to the image of $\widetilde{P}_v(X)$ in $(\widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_{\partial,R})[X]$.*
(2) *For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of σ is equal to the image of $\widetilde{P}_{v,\sigma}(X)$ in $(\widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{O}))/\widetilde{I}_{\partial,R})[X]$.*

(3) *Let* $\widetilde{\mathrm{tr}}_{\partial,R} : \widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}},\mathcal{O}))[G_{F,S}] \to \widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{I}_{\partial,R}$ *be the trace associated to* $\widetilde{D}_{\partial,R}$. *Then for each place* $v \in R - R^c$, *for each* $\sigma \in G_{F,S}$, *and for each* $\tau_v \in I_{F_v}$, *we have* $\mathrm{Res}_v^{(2n)!}\,\mathrm{tr}_{\partial,R}(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$.

Proof. There is a $\widetilde{\mathbf{T}}_R^T$-equivariant exact triangle in $\mathbf{D}(\mathcal{O})$,

$$R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O}) \longrightarrow R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}) \longrightarrow R\Gamma(\partial\widetilde{X}_{\widetilde{K}},\mathcal{O}) \longrightarrow,$$

and consequently a natural homomorphism

$$\widetilde{\mathbf{T}}_R^T(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O}) \oplus R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O})) \to \widetilde{\mathbf{T}}_R^T(R\Gamma(\partial\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{J},$$

where $\widetilde{J}$ is an ideal of square 0. To prove the corollary, it is therefore enough to show that there are an integer $N \geq 1$, depending only on $[F:\mathbf{Q}]$ and n, an ideal $\widetilde{I} \subset \widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))$ satisfying $\widetilde{I}^N = 0$, and a $\widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))$-valued determinant $\widetilde{D}$ on $G_{F,S}$ of dimension $2n$ satisfying the following conditions:

(1) For each place $v \notin S$ of F, the characteristic polynomial $\widetilde{D}(X - \mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$ in $\widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{I}[X]$.
(2) For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of σ is equal to the image of $\widetilde{P}_{v,\sigma}(X)$ in $\widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{I}[X]$.
(3) Let $\widetilde{\mathrm{tr}} : \widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))[G_{F,S}] \to \widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{I}$ be the trace associated to $\widetilde{D}$. Then for each place $v \in R - R^c$, for each $\sigma \in G_{F,S}$, and for each $\tau_v \in I_{F_v}$, we have $\mathrm{Res}_v^{(2n)!}\,\widetilde{\mathrm{tr}}(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$.

By Proposition 2.2.21, there is a commutative diagram (determined by Verdier duality)

$$\begin{array}{ccc} \mathcal{H}(\widetilde{G}^\infty,\widetilde{K}) & \longrightarrow & \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O})) \\ \downarrow{\scriptstyle \widetilde{\iota}} & & \downarrow \\ \mathcal{H}(\widetilde{G}^\infty,\widetilde{K}) & \longrightarrow & \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O})). \end{array}$$

Let $\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O}))$ denote the image of the composite map

$$\widetilde{\mathbf{T}}_R^T \to \mathcal{H}(\widetilde{G}^\infty,\widetilde{K}) \otimes_{\mathbf{Z}} \mathcal{O} \xrightarrow{\widetilde{\iota}} \mathcal{H}(\widetilde{G}^\infty,\widetilde{K}) \otimes_{\mathbf{Z}} \mathcal{O} \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O})),$$

where the first and last maps are the canonical ones. The existence of the above commutative diagram shows that $\widetilde{\iota}$ descends to an isomorphism

$$\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O})) \to \widetilde{\mathbf{T}}_R^T(R\Gamma(\widetilde{X}_{\widetilde{K}},\mathcal{O})).$$

To complete the proof of the corollary, it is therefore enough to show that there is a determinant $\widetilde{D}_{c,R,\vee}$ of $G_{F,S}$ of dimension $2n$ with coefficients in a quotient $\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(R\Gamma_c(\widetilde{X}_{\widetilde{K}},\mathcal{O}))/\widetilde{I}_{c,R,\vee}$ by some nilpotent ideal $\widetilde{I}_{c,R,\vee}$ of exponent bounded solely in terms of $[F:\mathbf{Q}]$ and n, and satisfying conditions analogous to those required of $\widetilde{D}$. Using the same argument as in the statement of

Proposition 3.2.2, it is enough to show the following: Let $\widetilde{\pi}$ be a cuspidal, cohomological automorphic representation of $\widetilde{G}(\mathbf{A}_{F^+})$ such that $\widetilde{\pi}^{\infty,\widetilde{K}} \neq 0$, and let $\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi})$ denote the image of the composite

$$\widetilde{\mathbf{T}}_R^T \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \to \mathcal{H}(\widetilde{G}^\infty, \widetilde{K}^\infty) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p \xrightarrow{\widetilde{\iota}} \mathcal{H}(\widetilde{G}^\infty, \widetilde{K}^\infty) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p \to \mathrm{End}_{\overline{\mathbf{Q}}_p}(\iota^{-1}\widetilde{\pi}^{\infty,\widetilde{K}}).$$

Consider the associated Galois representation $r_\iota(\widetilde{\pi}) : G_{F,S} \to \mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p)$, and let $\rho : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi}))$ denote the composite of $r_\iota(\widetilde{\pi})^\vee \otimes \epsilon^{1-2n}$ with the inclusion $\mathrm{GL}_{2n}(\overline{\mathbf{Q}}_p) \subset \mathrm{GL}_{2n}(\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi}))$. Then we have the following properties:

- For each place $v \notin S$ of F, the characteristic polynomial of $\rho(\mathrm{Frob}_v)$ equals the image of $\widetilde{P}_v(X)$ in $\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi})[X]$.
- For each place $v \in R$, and for each element $\sigma \in W_{F_v}$, the characteristic polynomial of $\rho(\sigma)$ equals the image of $\widetilde{P}_{v,\sigma}(X)$ in $\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi})[X]$.
- For each place $v \in R - R^c$ and for each $\tau_v \in I_{F_v}$, we have the equality $\mathrm{Res}_v^{(2n)!}\, \rho(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$ in $M_{2n}(\widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi}))$.

To see why these properties hold, we note that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\widetilde{G}^\infty, \widetilde{K}) & \longrightarrow & \mathrm{End}_{\overline{\mathbf{Q}}_p}(\iota^{-1}\widetilde{\pi}^{\vee,\infty,\widetilde{K}}) \\ \Big\downarrow{\scriptstyle \widetilde{\iota}} & & \Big\downarrow \\ \mathcal{H}(\widetilde{G}^\infty, \widetilde{K}) & \longrightarrow & \mathrm{End}_{\overline{\mathbf{Q}}_p}(\iota^{-1}\widetilde{\pi}^{\infty,\widetilde{K}}), \end{array}$$

where the horizontal arrows are the canonical ones and the right vertical arrow is transpose with respect to the natural duality between $\iota^{-1}\widetilde{\pi}^{\infty,\widetilde{K}}$ and $\iota^{-1}\widetilde{\pi}^{\vee,\infty,\widetilde{K}}$. In particular, $\widetilde{\iota}$ determines an isomorphism $\widetilde{\mathbf{T}}_R^T(\widetilde{\pi}^\vee) \to \widetilde{\iota}(\widetilde{\mathbf{T}}_R^T)(\widetilde{\pi})$. The above points therefore follow from the analogous points for the cuspidal, cohomological automorphic representation $\widetilde{\pi}^\vee$ of $\widetilde{G}(\mathbf{A}_{F^+})$, already established in the proof of Proposition 3.2.2, together with the observation that there is an isomorphism $r_\iota(\widetilde{\pi}^\vee) \cong r_\iota(\widetilde{\pi})^\vee \otimes \epsilon^{1-2n}$. □

We need one more lemma, which is an analogue of Hensel's lemma for group determinants.

Lemma 3.2.4. *Let A be a complete Noetherian local $\mathcal{O}$-algebra with residue field k, and let Γ be a group. Fix natural numbers n_1, n_2, and set $n = n_1 + n_2$. Suppose given group determinants D_1, D_2 of Γ of dimensions n_1, n_2 with coefficients in A, and let $D = D_1 D_2$. Suppose moreover that, if $\overline{D}_i = D_i$ mod $\mathfrak{m}_A$, then the semisimple representations $\overline{\rho}_i : \Gamma \to \mathrm{GL}_{n_i}(k)$ with $\det \overline{\rho}_i = \overline{D}_i$ for $i = 1, 2$ have no common Jordan–Hölder factors.*

Then

(1) *For any other group determinants E_1, E_2 of Γ of dimensions n_1, n_2 with E_i mod $\mathfrak{m}_A = \overline{D}_i$ for $i = 1, 2$ and $E_1 E_2 = D$, we have $E_1 = D_1$ and $E_2 = D_2$.*
(2) *We have* $\ker D = \ker(D_1) \cap \ker(D_2)$.

Proof. We will give an expression for D_1 that depends only on D, $\overline{D}_1$, and $\overline{D}_2$. This will establish the first part of the lemma. Let $R = A[\Gamma]$, and let $S = R/\operatorname{CH}(D)$, where the Cayley–Hamilton ideal $\operatorname{CH}(D)$ is defined in [Che14, §1.17]. By [Che14, Lem. 1.21], the homomorphism $R \to M_{n_1}(k) \times M_{n_2}(k)$ determined by $\overline{D}_1$, $\overline{D}_2$ factors through S. Let $\overline{e}_1, \overline{e}_2 \in M_{n_1}(k) \times M_{n_2}(k)$ be the central idempotents that are the identity in one factor and zero in the other. Following [BC09, p. 32, footnote], we may lift $\overline{e}_1, \overline{e}_2$ to idempotents $e_1, e_2 \in S$ such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

We now consider the polynomial law D_{1,e_1} on $e_1 S e_1$ given by the formula $D_{1,e_1}(x) = D_1(x + e_2)$. According to [Che14, Lem. 2.4], D_{1,e_1} is a determinant $e_1 S e_1 \to A$ of some dimension $d_1 \leq n_1$. Reducing modulo $\mathfrak{m}_A$, we see that $d_1 = n_1$. It follows that the polynomial law D_{1,e_2} on $e_2 S e_2$ given by the formula $D_{1,e_2}(x) = D_1(x + e_1)$ is of dimension 0, therefore constant and equal to 1. Working over $A[X]$, and invoking [Che14, Lem. 2.4(2)], we have

$$D_1(X - e_2) = D_{1,e_1}(X) = X^{n_1},$$

hence $e_2^{n_1} = e_2 \in \operatorname{CH}(D_1) \subset \ker(D_1)$. Similarly we deduce that $e_1 \in \ker(D_2)$. We find that for any A-algebra B and any $x \in S \otimes_A B$, we have

$$D_1(x) = D_1(e_1 x + e_2 x) = D_1(e_1 x),$$

and so

$$D(e_1 x + e_2) = D_1(x e_1) = D_1(x).$$

Since the expression $D_1(x) = D(e_1 x + e_2)$ only depends on D, $\overline{D}_1$, and $\overline{D}_2$, this proves the first part of the lemma. For the second, we note that the inclusion $\ker(D_1) \cap \ker(D_2) \subset \ker(D)$ follows immediately from the definition. For the other inclusion, take $x \in \ker(D)$, an A-algebra B, and $y \in R \otimes_A B$. By symmetry, it is enough to show that $D_1(1 + xy) = 1$. We have

$$D_1(1 + xy) = D_1(e_1(1 + xy)) = D(1 + e_1 xy) = 1,$$

since $e_1 x \in \ker(D)$. This concludes the proof. □

We can now complete the proof of Proposition 3.1.2.

Proof of Proposition 3.1.2. Let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$. By Theorem 2.4.8, the map $\mathcal{S}$ descends to a homomorphism

$$\widetilde{\mathbf{T}}_R^T(R\Gamma(\partial \widetilde{X}_{\widetilde{K}}, \mathcal{O})_{\widetilde{\mathfrak{m}}}) \to \mathbf{T}_R^T(R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}).$$

By Propositions 2.2.16, 2.2.18 and 2.2.19 and Corollary 3.2.3, we see that we can find an integer $N \geq 1$, depending only on $[F : \mathbf{Q}]$ and n, an ideal $I_R \subset \mathbf{T}_R^T(R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}})$ satisfying $I_R^N = 0$, and a $\mathbf{T}_R^T(K, 0)_{\mathfrak{m}}/I_R$-valued determinant D' on $G_{F,S}$ of dimension $2n$ satisfying the following conditions:

(1) For each place $v \notin S$ of F, the characteristic polynomial of Frob_v under D' is equal to the image of $P_v(X) q_v^{n(2n-1)} P_{v^c}^{\vee}(q_v^{1-2n} X)$ in $(\mathbf{T}_R^T(K, 0)_{\mathfrak{m}}/I_R)[X]$.

(2) For each $v \in R$, and for each $\sigma \in W_{F_v}$, the characteristic polynomial of σ under D' is equal to the image of $P_{v,\sigma}(X)\|\sigma\|_v^{n(1-2n)} P^\vee_{v^c,\sigma^c}(\|\sigma\|_v^{2n-1}X)$ in $(\mathbf{T}_R^T(K,0)_\mathfrak{m}/I_R)[X]$.

(3) Let $\mathrm{tr}' : \mathbf{T}_R^T(R\Gamma(X_K,\mathcal{O})_\mathfrak{m})[G_{F,S}] \to \mathbf{T}_R^T(R\Gamma(X_K,\mathcal{O})_\mathfrak{m})/I_R$ be the trace associated to D'. Then for each place $v \in R - R^c$, for each $\sigma \in G_{F,S}$, and for each $\tau_v \in I_{F_v}$, we have $\mathrm{Res}_v^{(2n)!}\,\mathrm{tr}'(\sigma(\tau_v - 1)P_{v,\phi_v}(\phi_v)) = 0$.

By Theorem 2.3.7, we can assume (after possibly enlarging I_R and increasing N in a way still depending only on $[F:\mathbf{Q}]$ and n) that there exists a continuous representation $\rho_\mathfrak{m} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}_R^T(K,0)_\mathfrak{m}/I_R)$ such that for each finite place $v \notin S$ of F, $\det(X - \rho_\mathfrak{m}(\mathrm{Frob}_v))$ equals the image of $P_v(X)$ in $\mathbf{T}_R^T(K,0)_\mathfrak{m}/I_R$. Let $D = \det \rho_\mathfrak{m}$. Looking at characteristic polynomials of Frobenius elements for places $v \notin S$, we conclude that $D' = \det(\rho_\mathfrak{m} \oplus \rho_\mathfrak{m}^{c,\vee} \otimes \epsilon^{1-2n}) = D(D^{c,\vee} \otimes \epsilon^{1-2n})$. (Note that our notation for twisted determinants is chosen so that it matches the twisted representation; the polynomial law underlying $D^{c,\vee} \otimes \epsilon^{1-2n}$ is given by twisting with $\det(\epsilon^{1-2n}) = \epsilon^{n(1-2n)}$.) To complete the proof of Proposition 3.1.2, we need to show that D satisfies the following conditions:

- For each place $v \in R$ and for each $\sigma \in W_{F_v}$, the characteristic polynomial of σ under D is the image of $P_{v,\sigma}(X)$ in $\mathbf{T}_R^T(K,0)_\mathfrak{m}/I_R$.
- D factors through $G_{F,T}$ and for each $v \in S - T$, the characteristic polynomial of Frob_v under D is the image of $P_v(X)$ in $\mathbf{T}_R^T(K,0)_\mathfrak{m}/I_R$.

We will then (in the notation of Proposition 3.1.2) be able to take $D_{\mathfrak{m},R} = D$. We take points these in turn. If $v \in R$, then there is a unique n-dimensional group determinant E_v of W_{F_v} with coefficients in $\mathbf{T}_R^T(K,0)_\mathfrak{m}$ such that for each $\sigma \in W_{F_v}$, the characteristic polynomial of σ under E_v equals the image of $P_{v,\sigma}(X)$. Similarly, if $v \in R^c - R$, there is a unique n-dimensional group determinant E_v of W_{F_v} with coefficients in $\mathbf{T}_R^T(K,0)_\mathfrak{m}$ that is unramified and such that the characteristic polynomial of Frob_v equals $P_v(X)$. Our assumptions imply that for each $v \in R$, we have $D|_{W_{F_v}}(D^{c,\vee} \otimes \epsilon^{1-2n})|_{W_{F_v}} = E_v(E_{v^c}^{c,\vee} \otimes \epsilon^{1-2n})$. We would like to deduce that $D|_{W_{F_v}} = E_v$.

We first show that this holds in any quotient of $\mathbf{T}_R^T(K,0)_\mathfrak{m}$ by a maximal ideal. (Recall that $\mathfrak{m}$ is, by assumption, a maximal ideal of $\mathbf{T}^S(K,0)$, so that the ring $\mathbf{T}_R^T(K,0)_\mathfrak{m}$ is not necessarily local.) By assumption (i.e., by Lemma 3.2.1(3)), there is a character $\psi : G_{F,S} \to \mathcal{O}^\times$, unramified above $R \cup R^c \cup S_p$, such that for each place $v \in R$, the characteristic polynomials of $\overline{\psi}(\mathrm{Frob}_v)\overline{\rho}_\mathfrak{m}(\phi_v)$ and $\overline{\psi}(\mathrm{Frob}_{v^c})^{-1}(\overline{\rho}_\mathfrak{m} \oplus \overline{\rho}_\mathfrak{m}^{c,\vee} \otimes \epsilon^{1-2n})(\phi_v)$ are coprime. On the other hand, Proposition 2.2.23 implies that for any $v \in R$ equalities, we have

$$\det(\rho_\mathfrak{m} \otimes \psi \oplus (\rho_\mathfrak{m} \otimes \psi)^{c,\vee} \otimes \epsilon^{1-2n})|_{W_{F_v}} = (E_v \otimes \psi|_{W_{F_v}})((E_{v^c} \otimes \psi|_{W_{F_{v^c}}})^{c,\vee} \otimes \epsilon^{1-2n}).$$

Looking at the roots of the characteristic polynomial of ϕ_v in each determinant and using the bijection between group determinants over a finite field and isomorphism classes of semisimple representations [Che14, Th. 2.12], we conclude that

we must have $\det \overline{\rho}_{\mathfrak{m}}|_{W_{F_v}} = E_v \bmod \mathfrak{n}$ for every maximal ideal $\mathfrak{n} \subset \mathbf{T}_R^T(K,0)_{\mathfrak{m}}$. Lemma 3.2.4 then immediately implies that we have $D|_{W_{F_v}} = \det \rho_{\mathfrak{m}}|_{W_{F_v}} = E_v$.

It remains to check that for each place $v \in R - R^c$ (hence $v^c \in S - T$), $\rho_{\mathfrak{m}}|_{W_{F_{v^c}}}$ is unramified and $\det(X - \rho_{\mathfrak{m}}(\mathrm{Frob}_{v^c}))$ equals the image of $P_{v^c}(X)$ in $\mathbf{T}_R^T(K,0)_{\mathfrak{m}}/I_R[X]$. Equivalently, we must check that for each place $v \in R - R^c$, $\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}|_{W_{F_v}}$ is unramified and $\det(X - (\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\mathrm{Frob}_v))$ equals the image of $q_v^{n(2n-1)} P_{v^c}^{\vee}(q_v^{1-2n} X)$ in $\mathbf{T}_R^T(K,0)_{\mathfrak{m}}/I_R[X]$. The computation of $\det(X - (\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\mathrm{Frob}_v))$ follows from what we have done already, so we just need to show that $\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}|_{W_{F_v}}$ is unramified. (Note that this is stronger, in general, than the assertion that the associated group determinant of $\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}|_{W_{F_v}}$ is unramified.) To show this, we use the following set of relations, which follow on applying $\mathcal{S}$ to the corresponding set of relations for the determinant D':

- For each place $v \in R - R^c$, for each $\sigma \in G_{F,S}$, and for each $\tau_v \in I_{F_v}$, we have

$$\mathcal{S}(\mathrm{Res}_v)^{(2n)!} \operatorname{tr}(\rho_{\mathfrak{m}}(\sigma(\tau_v - 1) P_{v,\phi_v}(\phi_v))) \\ + \mathcal{S}(\mathrm{Res}_v)^{(2n)!} \operatorname{tr}((\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\sigma(\tau_v - 1) P_{v,\phi_v}(\phi_v))) = 0.$$

We have already seen that if $v \in R - R^c$, then $P_{v,\phi_v}(\rho_{\mathfrak{m}}(\phi_v)) = 0$, so we deduce that for each $v \in R - R^c$, each $\sigma \in G_{F,S}$, and each $\tau_v \in I_{F_v}$, we have

$$\mathcal{S}(\mathrm{Res}_v)^{(2n)!} \operatorname{tr}((\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\sigma(\tau_v - 1) P_{v,\phi_v}(\phi_v))) = 0.$$

By definition, $\mathrm{Res}_v \in \widetilde{\mathbf{T}}_R^T$ is the resultant of the polynomials $P_{v,\phi_v}(X)$ and $P_{v^c,\phi_{v^c}^{-1}}(X)$ in $\widetilde{\mathbf{T}}_R^T[X]$. The images of these polynomials in $\mathbf{T}_R^T[X]$ under the map $\mathcal{S}$ are computed by Proposition 2.2.19; they are (respectively) $P_{v,\phi_v}(X)$ and $q_v^{n(2n-1)} P_{v^c,\phi_{v^c}^{-1}}(q_v^{1-2n} X) = q_v^{n(2n-1)} P_{v^c}^{\vee}(q_v^{1-2n} X)$. Thus $\mathcal{S}(\mathrm{Res}_v) \in \mathbf{T}_R^T$ is the resultant of these two polynomials, and the image of $\mathcal{S}(\mathrm{Res}_v)$ modulo any maximal ideal of $\mathbf{T}_R^T(K,0)_{\mathfrak{m}}$ coincides with the resultant of $\det(X - \overline{\rho}_{\mathfrak{m}}(\phi_v))$ and $\det(X - \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}(\phi_v))$. These polynomials in $k[X]$ are coprime by assumption (cf. Lemma 3.2.1), so we find that $\mathcal{S}(\mathrm{Res}_v)$ is a unit in $\mathbf{T}_R^T(K,0)_{\mathfrak{m}}$ and therefore that we have the stronger identity

$$\operatorname{tr}((\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\sigma(\tau_v - 1) P_{v,\phi_v}(\phi_v))) = 0.$$

The matrix $(\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(P_{v,\phi_v}(\phi_v)) = P_{v,\phi_v}((\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\phi_v))$ has unit determinant. Since $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible and $\sigma \in G_{F,S}$ is arbitrary, we conclude that we must have $(\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})(\tau_v - 1) = 0$ for all $\tau_v \in I_{F_v}$ or, equivalently, that $\rho_{\mathfrak{m}}|_{W_{F_{v^c}}}$ is unramified. This is what we needed to show. □

4. Local-global compatibility, $l = p$ (Fontaine–Laffaille case)

4.1. *Statements.* Let F be a CM field containing an imaginary quadratic field, and fix an integer $n \geq 1$. Let p be a prime, and let E be a finite extension

of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ large enough to contain the images of all embeddings of F in $\overline{\mathbf{Q}}_p$. We assume throughout this chapter that F satisfies the following standing hypothesis:

- The prime p is unramified in F. Moreover, F contains an imaginary quadratic field in which p splits.

Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Let S be a finite set of finite places of F, containing the p-adic places, stable under complex conjugation, and satisfying the following condition:

- Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic subfield of F in which l splits.

We recall (Theorem 2.3.7) that under these hypotheses, if $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ is a non-Eisenstein maximal ideal, then there is a continuous homomorphism

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/J)$$

characterized, up to conjugation, by the characteristic polynomials of Frobenius elements at places $v \notin S$; here J is a nilpotent ideal whose exponent depends only on n and $[F:\mathbf{Q}]$. Our goal in this chapter is to show that under certain conditions, we can show that the restrictions of $\rho_{\mathfrak{m}}$ to decomposition groups at the p-adic places of F satisfy conditions coming from p-adic Hodge theory. More precisely, we can show, after perhaps enlarging the nilpotent ideal J, that they are Fontaine–Laffaille with the expected Hodge–Tate weights.

Before stating the main theorem of this chapter we first briefly recall some of the properties of the Fontaine–Laffaille functor [FL82], with normalizations as in [CHT08, §2.4.1].

Let v be a p-adic place of F. We are assuming that $F_v/\mathbf{Q}_p$ is unramified. Let $\mathcal{MF}_{\mathcal{O}}$ be the category of finite $\mathcal{O}_{F_v} \otimes_{\mathbf{Z}_p} \mathcal{O}$-modules M equipped with the following data:

- A decreasing filtration $\mathrm{Fil}^i M$ of $\mathcal{O}_{F_v} \otimes_{\mathbf{Z}_p} \mathcal{O}$-submodules that are direct summands as $\mathcal{O}_{F_v}$-modules. For an embedding $\tau\colon F_v \hookrightarrow E$, define the filtered $\mathcal{O}$-module $M_\tau = M \otimes_{\mathcal{O}_{F_v}\otimes_{\mathbf{Z}_p}\mathcal{O}} \mathcal{O}$, where we view $\mathcal{O}$ as an $\mathcal{O}_{F_v} \otimes_{\mathbf{Z}_p} \mathcal{O}$-algebra via $\tau \otimes 1$. We assume that for each τ, there is an integer a_τ such that $\mathrm{Fil}^{a_\tau} M_\tau = M_\tau$ and $\mathrm{Fil}^{a_\tau+p-1} M_\tau = 0$.
- $\mathrm{Frob}_p^{-1} \otimes 1$-linear maps $\Phi^i\colon \mathrm{Fil}^i M \to M$ such that $\Phi^i|_{\mathrm{Fil}^{i+1} M} = p\Phi^{i+1}$ and $M = \sum_i \Phi^i \mathrm{Fil}^i M$.

Note that for $M \in \mathcal{MF}_{\mathcal{O}}$,

$$\mathrm{Fil}^i M = \prod_\tau \mathrm{Fil}^i M_\tau \quad \text{and} \quad \Phi^i = \prod_\tau \Phi^i_\tau \quad \text{with} \quad \Phi^i_\tau\colon \mathrm{Fil}^i M_\tau \to M_{\tau\circ\mathrm{Frob}_p^{-1}}.$$

Given a tuple of integers $a = (a_\tau) \in \mathbf{Z}^{\mathrm{Hom}(F_v,E)}$, we let $\mathcal{MF}^a_{\mathcal{O}}$ be the full subcategory of $\mathcal{MF}_{\mathcal{O}}$ consisting of objects M such that for each τ, $\mathrm{Fil}^{a_\tau} M_\tau = M_\tau$

and $\mathrm{Fil}^{a_\tau+p-1} M_\tau = 0$. We write $\mathcal{MF}^0_{\mathcal{O}}$ for $\mathcal{MF}^{(0,\dots,0)}_{\mathcal{O}}$. We let $\mathcal{MF}_k$ and $\mathcal{MF}^a_k$ be the full subcategories of $\mathcal{MF}_{\mathcal{O}}$ and $\mathcal{MF}^a_{\mathcal{O}}$, respectively, of objects annihilated by ϖ.

When $p > 2$, there is an exact, fully faithful, covariant functor $\mathbf{G}^0$ from $\mathcal{MF}^0_{\mathcal{O}}$ to the category of finite $\mathcal{O}$-modules with continuous $\mathcal{O}$-linear G_{F_v}-action (see [CHT08, §2.4.1], where $\mathbf{G}^0$ is denoted $\mathbf{G}$). The essential image of $\mathbf{G}^0$ is closed under subquotients, and the restriction of $\mathbf{G}^0$ to $\mathcal{MF}^0_k$ takes values in the category of continuous G_{F_v}-representations on finite-dimensional k-vector spaces. Moreover, if M_1 and M_2 are objects of $\mathcal{MF}^0_{\mathcal{O}}$ such that $M_1 \otimes_{\mathcal{O}_{F_v}\otimes_{\mathbf{Z}_p}\mathcal{O}} M_2$ also lies in $\mathcal{MF}^0_{\mathcal{O}}$, then

$$\mathbf{G}^0(M_1 \otimes_{\mathcal{O}_{F_v}\otimes_{\mathbf{Z}_p}\mathcal{O}} M_2) = \mathbf{G}^0(M_1) \otimes_{\mathcal{O}} \mathbf{G}^0(M_2). \tag{4.1.1}$$

We extend $\mathbf{G}^0$ to a functor $\mathbf{G}$ on $\mathcal{MF}_{\mathcal{O}}$ by twisting as follows. Fix $M \in \mathcal{MF}_{\mathcal{O}}$ and $a = (a_\tau) \in \mathbf{Z}^{\mathrm{Hom}(F_v,E)}$ such that $M \in \mathcal{MF}^a_{\mathcal{O}}$. Define the crystalline character $\psi_a \colon G_{F_v} \to \mathcal{O}^\times$ by

$$\psi_a \circ \mathrm{Art}_{F_v}(x) = \prod_\tau \tau(x)^{-a_\tau} \quad \text{for} \quad x \in \mathcal{O}^\times_{F_v} \quad \text{and} \quad \psi_a \circ \mathrm{Art}_{F_v}(p) = 1,$$

and the object $M(a) \in \mathcal{MF}^0_{\mathcal{O}}$ by $\mathrm{Fil}^i M(a)_\tau = \mathrm{Fil}^{i+a_\tau} M_\tau$ and $\Phi^i_{M(a),\tau} = \Phi^{i+a_\tau}_{M,\tau}$. We then set

$$\mathbf{G}(M) = \mathbf{G}^0(M(a)) \otimes_{\mathcal{O}} \psi_a.$$

Using (4.1.1), one checks that this is independent of a such that $M \in \mathcal{MF}^a_{\mathcal{O}}$. We will denote by $\mathbf{G}^a$ the restriction of $\mathbf{G}$ to $\mathcal{MF}^a_{\mathcal{O}}$. Any $\mathbf{G}^a$ is fully faithful and its essential image is stable under subquotients, but $\mathbf{G}$ is not full on all of $\mathcal{MF}_{\mathcal{O}}$. We note also that the essential image of $\mathbf{G}$ is stable under twists by crystalline characters.

Let $\overline{M}$ be an object of $\mathcal{MF}_k$. For each embedding $\tau \colon F_v \hookrightarrow E$, we let $\mathrm{FL}_\tau(\overline{M})$ be the multiset of integers i such that

$$\mathrm{gr}^i \overline{M} \otimes_{\mathcal{O}_{F_v}\otimes_{\mathbf{Z}_p} k} k \neq 0,$$

counted with multiplicity equal to the k-dimension of this space, where we view k as a $\mathcal{O}_{F_v} \otimes_{\mathbf{Z}_p} k$ algebra via $\tau \otimes 1$. If $p > 2$ and M is a p-torsion free object of $\mathcal{MF}_{\mathcal{O}}$, the representation $\mathbf{G}(M) \otimes_{\mathcal{O}} E$ is crystalline and for every embedding $\tau \colon F_v \hookrightarrow E$, we have

$$\mathrm{HT}_\tau(\mathbf{G}(M) \otimes_{\mathcal{O}} E) = \mathrm{FL}_\tau(M \otimes_{\mathcal{O}} k).$$

Moreover, if W is an $\mathcal{O}$-lattice in a crystalline representation of G_{F_v} such that every τ-Hodge–Tate weight lies in $[a_\tau, a_\tau + p - 2]$ for some integer a_τ, then W is in the essential image of $\mathbf{G}^a$.

We can now state the main theorem of this chapter (with the same numbering as it occurs again immediately before the proof).

THEOREM 4.5.1. *Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal. Suppose that $\mathbf{T}^S(K,\lambda)/\mathfrak{m} = k$ has residue characteristic p. Let $\overline{v}$ be a p-adic place of F^+, and suppose that the following additional conditions are satisfied*:

(1) *The prime p is unramified in F, and F contains an imaginary quadratic field in which p splits.*
(2) *Let w be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*
(3) *For each place $v|\overline{v}$ of F, $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.*
(4) *For every embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+,*

$$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} \leq p - 2n - 1.$$

(5) *$p > n^2$.*
(6) *There exists a p-adic place $\overline{v}' \neq \overline{v}$ of F^+ such that*

$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$

(7) *$\overline{\rho}_{\mathfrak{m}}$ is decomposed generic (Definition 4.3.1).*
(8) *Assume that one of the following holds*:
   (a) *$H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p] \neq 0$, or*
   (b) *for every embedding $\tau\colon F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+,*

$$-\lambda_{\tau c,n} - \lambda_{\tau,n} \leq p - 2n - 2 \quad \text{and} \quad -\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0.$$

Then there exist an integer $N \geq 1$ depending only on $[F^+ : \mathbf{Q}]$ and n, an ideal $J \subset \mathbf{T}^S(K,\lambda)$ satisfying $J^N = 0$, and a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/J)$$

satisfying the following conditions:

(a) *For each finite place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/J)[X]$.*
(b) *For each place $v|\overline{v}$ of F, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is in the essential image of $\mathbf{G}^a$ (with $a = (\lambda_{\tau,n}) \in \mathbf{Z}^{\mathrm{Hom}(F_v,E)}$).*
(c) *There is $\overline{M} \in \mathcal{MF}_k$ such that $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}} \cong \mathbf{G}(\overline{M})$ and for any embedding $\tau\colon F_v \hookrightarrow E$,*

$$\mathrm{FL}_\tau(\overline{M}) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\}.$$

The rest of this chapter is devoted to the proof of Theorem 4.5.1. The proof will be by reduction to known results for automorphic forms on $\widetilde{G}$ (in particular, Theorem 2.3.3).

4.2. *A direct summand of the boundary cohomology.* In this section, we show how to realize the cohomology of X_K as a direct summand of the cohomology of the boundary $\partial\widetilde{X}_{\widetilde{K}}$ of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$. This is the first step in relating the cohomology of X_K to automorphic forms on $\widetilde{G}$. We must first introduce some new notation, in addition to the notation introduced in Section 2.4.

We recall (cf. Section 2.2.1) that we write $\overline{S}_p$ for the set of p-adic places of F^+, S_p for the set of p-adic places of F, and that we have fixed a subset $\widetilde{S}_p = \{\widetilde{v} \mid \overline{v} \in \overline{S}_p\}$ with the property that $S_p = \widetilde{S}_p \sqcup \widetilde{S}_p^c$. Moreover, we write $\widetilde{I}_p$ for the set of embeddings $\tau : F \hookrightarrow E$ inducing a place of $\widetilde{S}_p$. For any $\overline{v} \in \overline{S}_p$, we write $\widetilde{I}_{\overline{v}}$ for the set of embeddings $\tau : F \hookrightarrow E$ inducing $\widetilde{v}$. Similarly, we write $I_{\overline{v}}$ for the set of embeddings $\tau : F^+ \hookrightarrow E$ inducing $\overline{v}$.

These choices determine an isomorphism $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E \cong \prod_{\tau\in\widetilde{I}_p}\mathrm{GL}_{2n}$. For any embedding $\tau : F^+ \hookrightarrow E$, we set

$$W_\tau = W(\widetilde{G}\otimes_{F^+,\tau} E, T\otimes_{F^+,\tau} E)$$

and

$$W_{P,\tau} = W(G\otimes_{F^+,\tau} E, T\otimes_{F^+,\tau} E);$$

these may be identified with the Weyl groups of GL_{2n} and $\mathrm{GL}_n\times\mathrm{GL}_n$, respectively. Since $\widetilde{G}$ is equipped with the Borel subgroup B, we may also define the subset $W_\tau^P \subset W_\tau$ of representatives for the quotient $W_{P,\tau}\backslash W_\tau$ (cf. Section 1.2). We write $\rho_\tau \in X^*(T\otimes_{F^+,\tau} E)$ for the half-sum of the $B\otimes_{F^+,\tau} E$-positive roots.

If $\overline{v}\in\overline{S}_p$, then we set $W_{\overline{v}} = \prod_{\tau\in I_{\overline{v}}} W_\tau$, $W_{P,\overline{v}} = \prod_{\tau\in I_{\overline{v}}} W_{P,\tau}$, and $W_{\overline{v}}^P = \prod_{\tau\in I_{\overline{v}}} W_\tau^P$. We define $\rho_{\overline{v}} \in X^*((\mathrm{Res}_{F^+_{\overline{v}}/\mathbf{Q}_p} T)_E)$ to be the half-sum of the $(\mathrm{Res}_{F^+_{\overline{v}}/\mathbf{Q}_p} B)_E$-positive roots; thus we can identify $\rho_{\overline{v}} = \sum_{\tau\in\mathrm{Hom}(F^+_{\overline{v}},E)}\rho_\tau$. Given a subset $\overline{T}\subset\overline{S}_p$, we set $W_{\overline{T}} = \prod_{\overline{v}\in\overline{T}} W_{\overline{v}}$, and we define $W_{P,\overline{T}}$ and $W_{\overline{T}}^P$ similarly. If $\overline{T} = \overline{S}_p$, then we drop $\overline{T}$ from the notation; thus W may be identified with the Weyl group $W((\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E, (\mathrm{Res}_{F^+/\mathbf{Q}} T)_E)$ of $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_E$. We write $l : W \to \mathbf{Z}_{\geq 0}$ for the length function with respect to the Borel subgroup B, and $\rho \in X^*((\mathrm{Res}_{F^+/\mathbf{Q}} T)_E)$ for the half-sum of the $(\mathrm{Res}_{F^+/\mathbf{Q}} B)_E$-positive roots; thus we can identify $\rho = \sum_{\overline{v}\in\overline{S}_p}\rho_{\overline{v}}$.

If $\widetilde{\lambda}\in(\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, and $\overline{v}\in\overline{S}_p$, then we set

$$\widetilde{\lambda}_{\overline{v}} = (\widetilde{\lambda}_\tau)_{\tau\in\mathrm{Hom}_{\mathbf{Q}_p}(F^+_{\overline{v}},E)} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+_{\overline{v}},E)}.$$

If $\lambda\in(\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, and $\overline{v}\in\overline{S}_p$, then we set

$$\lambda_{\overline{v}} = (\lambda_\tau)_{\tau\in\mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}},E)\sqcup\mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}^c},E)} \in (\mathbf{Z}_+^n)^{\mathrm{Hom}_{\mathbf{Q}_p}(F\otimes_{F^+}F^+_{\overline{v}},E)}.$$

THEOREM 4.2.1. *Let $\widetilde{K}\subset\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup that is decomposed with respect to P, and with the property that for each $\overline{v}\in\overline{S}_p$, $\widetilde{K}_{U,\overline{v}} = U(\mathcal{O}_{F^+_{\overline{v}}})$. Let $\mathfrak{m}\subset\mathbf{T}^S$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m})\subset\widetilde{\mathbf{T}}^S$.*

Choose a partition $\overline{S}_p = \overline{S}_1 \sqcup \overline{S}_2$. *Let* $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ *and* $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ *be dominant weights for* $\widetilde{G}$ *and* G, *respectively. We assume that the following conditions are satisfied*:

(1) *For each* $\overline{v} \in \overline{S}_1$, $\widetilde{\lambda}_{\overline{v}} = \lambda_{\overline{v}}$ (*identification as in* (2.2.2)).
(2) *For each* $\overline{v} \in \overline{S}_2$, $\widetilde{\lambda}_{\overline{v}} = 0$.
(3) *For each* $\overline{v} \in \overline{S}_2$, *there exists* $w_{\overline{v}} \in W_{\overline{v}}^P$ *such that* $\lambda_{\overline{v}} = w_{\overline{v}}(\rho_{\overline{v}}) - \rho_{\overline{v}}$.
(4) $p > n^2$. (*We recall our blanket assumption throughout Section 4 that* p *is unramified in* F.)

If $\overline{v} \in \overline{S}_1$, *we let* $w_{\overline{v}}$ *denote the identity element of* $W_{\overline{v}}$. *We let* $w = (w_{\overline{v}})_{\overline{v} \in \overline{S}_p}$. *Then for any* $m \geq 1$, $R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}[-l(w)]$ *is a* $\widetilde{\mathbf{T}}^S$*-equivariant direct summand of* $R\Gamma(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)_{\widetilde{\mathfrak{m}}}$.

(If S is a ring and $A, B \in \mathbf{D}(S)$ are complexes equipped with homomorphisms of S-algebras

$$f_A : R \to \mathrm{End}_{\mathbf{D}(S)}(A),\ f_B : R \to \mathrm{End}_{\mathbf{D}(S)}(B),$$

then we say that A is an R-equivariant direct summand of B if there are a complex $C \in \mathbf{D}(S)$ equipped with a homomorphism of S-algebras

$$f_C : R \to \mathrm{End}_{\mathbf{D}(S)}(C)$$

and an isomorphism $\phi : B \cong A \oplus C$ in $\mathbf{D}(S)$ such that for each $r \in R$, we have $f_B(r) = \phi^{-1} \circ (f_A(r) \oplus f_C(r)) \circ \phi$.)

Proof. By Theorem 2.4.2, it is enough to show that $R\Gamma(X_K, \mathcal{V}_\lambda/\varpi^m)[-l(w)]$ is a $\widetilde{\mathbf{T}}^S$-equivariant direct summand of $R\Gamma(\widetilde{X}^P_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)$. We will argue in a similar way to the proof of Theorem 2.4.4.

Looking at the proof of Theorem 2.4.4, we see that there is a $\widetilde{\mathbf{T}}^S$-equivariant isomorphism

$$R\Gamma(\widetilde{X}^P_{\widetilde{K}_P}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \cong R\Gamma(\widetilde{K}^S_P \times K_S, R\Gamma(\mathrm{Inf}^{P^S \times K_S}_{G^S \times K_S} \mathfrak{X}_G, R1_*^{\widetilde{K}_{U,S}} \mathcal{V}_{\widetilde{\lambda}}/\varpi^m))$$

in $\mathbf{D}(\mathcal{O}/\varpi^m)$, where $\widetilde{\mathbf{T}}^S$ acts on both sides via the map r_P, and that the current theorem will be proved if we can establish the following claim:

- $R1_*^{\widetilde{K}_{U,S}} \mathcal{V}_{\widetilde{\lambda}}/\varpi^m$ admits $\mathcal{V}_\lambda/\varpi^m[-l(w)]$ as a direct summand in

$$\mathbf{D}(\mathrm{Sh}_{P^S \times K_S}(\mathfrak{X}_G)),$$

  the derived category of $P^S \times K_S$-equivariant sheaves of $\mathcal{O}/\varpi^m$-modules on $\mathfrak{X}_G$.

In fact, $R1_*^{\widetilde{K}_{U,S}} \mathcal{V}_{\widetilde{\lambda}}/\varpi^m$ is pulled back from $R\Gamma(\widetilde{K}_{U,S}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \in \mathbf{D}(\mathrm{Sh}_{K_S}(pt))$, so it suffices to show that $\mathcal{V}_\lambda/\varpi^m[-l(w)]$ is a direct summand of the complex $R\Gamma(\widetilde{K}_{U,S}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)$ in this category.

We observe that $\widetilde{K}_{U,S} = \prod_{\overline{v}\in\overline{S}} \widetilde{K}_{U,\overline{v}}$, and that $\mathcal{V}_{\widetilde{\lambda}}$ admits a corresponding decomposition $\mathcal{V}_{\widetilde{\lambda}} = \otimes_{\overline{v}\in\overline{S}_p} \mathcal{V}_{\widetilde{\lambda}_v}$. By the Künneth formula, it is therefore enough to show the following two claims:

(1) If $\overline{v} \in \overline{S}_1$, then $\mathcal{V}_{\lambda_{\overline{v}}}/\varpi^m$ is a direct summand of $R\Gamma(\widetilde{K}_{U,\overline{v}}, \mathcal{V}_{\widetilde{\lambda}_{\overline{v}}}/\varpi^m)$ in $\mathbf{D}(\mathcal{O}/\varpi^m[K_{\widetilde{v}} \times K_{\widetilde{v}^c}])$.

(2) If $\overline{v} \in \overline{S}_2$, then $\mathcal{V}_{\lambda_{\overline{v}}}/\varpi^m[-l(w_{\overline{v}})]$ is a direct summand of $R\Gamma(\widetilde{K}_{U,\overline{v}}, \mathcal{O}/\varpi^m)$ in $\mathbf{D}(\mathcal{O}/\varpi^m[K_{\widetilde{v}} \times K_{\widetilde{v}^c}])$.

The first claim can be proved using the same argument as in the end of the proof of Theorem 2.4.4. The second claim follows from Lemmas 4.2.2 and 4.2.3 below. (This is where we use our hypothesis $p > n^2$.) This completes the proof of the theorem. □

LEMMA 4.2.2. *Let $\overline{v} \in \overline{S}_p$, let $K = F^+_{\overline{v}}$, and fix an integer $m \geq 1$.*

(1) *For each $i \in \mathbf{Z}_{\geq 0}$ there is a $G(\mathcal{O}_K)$-equivariant isomorphism*

$$\begin{aligned} H^i(U(\mathcal{O}_K), \mathcal{O}/\varpi^m) &\cong \mathrm{Hom}_{\mathbf{Z}_p}(\wedge^i_{\mathbf{Z}_p} U(\mathcal{O}_K), \mathcal{O}/\varpi^m) \\ &= \mathrm{Hom}_{\mathcal{O}}(\wedge^i_{\mathcal{O}}(U(\mathcal{O}_K) \otimes_{\mathbf{Z}_p} \mathcal{O}), \mathcal{O}/\varpi^m) \end{aligned}$$

with $G(\mathcal{O}_K)$-action on the right-hand side induced by its conjugation action on $U(\mathcal{O}_K)$.

(2) *Suppose $p \geq 2n-1$. Given $w \in W^P_{\overline{v}}$, let*

$$\lambda_w = w(\rho_{\overline{v}}) - \rho_{\overline{v}} \in (\mathbf{Z}^n_+)^{\mathrm{Hom}_{\mathbf{Q}_p}(F \otimes_{F^+} F^+_{\overline{v}}, E)}$$

(using the identification (2.2.2)). For each $i \in \mathbf{Z}_{\geq 0}$, there is a $G(\mathcal{O}_K)$-equivariant isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\wedge^i_{\mathcal{O}}(U(\mathcal{O}_K) \otimes_{\mathbf{Z}_p} \mathcal{O}), \mathcal{O}) \cong \bigoplus_{\substack{w \in W^P_{\overline{v}} \\ l(w)=i}} \mathcal{V}_{\lambda_w}.$$

Proof. Note that $U(\mathcal{O}_K)$ is isomorphic (as an abstract group) to $\mathbf{Z}_p^{n^2[K:\mathbf{Q}_p]}$. The usual isomorphism $H^1(U(\mathcal{O}_K), \mathcal{O}/\varpi^m) \cong \mathrm{Hom}_{\mathbf{Z}_p}(U(\mathcal{O}_K), \mathcal{O}/\varpi^m)$ extends, by cup product, to a morphism

$$\wedge^* \mathrm{Hom}_{\mathbf{Z}_p}(U(\mathcal{O}_K), \mathcal{O}/\varpi^m) \to H^*(U(\mathcal{O}_K), \mathcal{O}/\varpi^m).$$

This can be seen to be an isomorphism using the Künneth formula. This proves the first part of the lemma.

For the second part, given $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(K, E)$ and $w \in W^P_\tau$, let $\lambda_w = w(\rho_\tau) - \rho_\tau \in (\mathbf{Z}^n_+)^2$. It is enough for us to show that for each $i \in \mathbf{Z}_{\geq 0}$, there is a $G(\mathcal{O}_K)$-equivariant isomorphism

$$\mathrm{Hom}_{\mathcal{O}}(\wedge^i_{\mathcal{O}}(U(\mathcal{O}_K) \otimes_{\mathcal{O}_K,\tau} \mathcal{O}), \mathcal{O}) \cong \bigoplus_{\substack{w \in W^P_\tau \\ l(w)=i}} \mathcal{V}_{\lambda_w}.$$

After tensoring up to E we do have such an isomorphism, by [Kos61]:

$$\mathrm{Hom}_{\mathcal{O}}(\wedge^i_{\mathcal{O}}(U(\mathcal{O}_K)\otimes_{\mathcal{O}_K,\tau}\mathcal{O}),E)\cong\bigoplus_{\substack{w\in W^P_\tau\\ l(w)=i}}V_{\lambda_w}.$$

Since $p\geq 2n-1$, it follows from [Jan03, Cor. II.5.6] that $\mathcal{V}_{\lambda_w}\otimes_{\mathcal{O}}k$ is a simple $G\otimes_{\mathcal{O}_K,\tau}k$-module for all $w\in W^P_\tau$. It follows that intersecting the lattice $\mathrm{Hom}_{\mathcal{O}}(\wedge^i_{\mathcal{O}}(U(\mathcal{O}_K)\otimes_{\mathcal{O}_K,\tau}\mathcal{O}),\mathcal{O})$ with a copy of V_{λ_w} arising from the above decomposition gives a sublattice isomorphic to $\mathcal{V}_{\lambda_w}$. By the remark following [Jan03, Cor. II.5.6], we know that there are no non-trivial extensions between the simple modules $\mathcal{V}_{\lambda_w}\otimes_{\mathcal{O}}k$ with varying w. Combining this with the universal coefficient theorem [Jan03, Prop. I.4.18a] we deduce that there are also no non-trivial extensions between the $G\otimes_{\mathcal{O}_K,\tau}\mathcal{O}$-modules $\mathcal{V}_{\lambda_w}$. This implies the existence of the desired isomorphism. □

LEMMA 4.2.3. *Let $\overline{v}\in\overline{S}_p$, let $K=F^+_{\overline{v}}$, and fix an integer $m\geq 1$. Suppose that $p>n^2$. Then we have a natural isomorphism (inducing the identity on cohomology)*

$$R\Gamma(U(\mathcal{O}_K),\mathcal{O}/\varpi^m)\xrightarrow{\sim}\bigoplus_{i=0}^{n^2[K:\mathbf{Q}_p]}H^i(U(\mathcal{O}_K),\mathcal{O}/\varpi^m)[-i]$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[G(\mathcal{O}_K)])$.

Proof. We have already observed that there is an isomorphism

$$R\Gamma(U(\mathcal{O}_K),\mathcal{O}/\varpi^m)\xrightarrow{\sim}H^0(U(\mathcal{O}_K),\mathcal{O}/\varpi^m)\oplus\tau_{\geq 1}R\Gamma(U(\mathcal{O}_K),\mathcal{O}/\varpi^m);$$

see claim (1) in the proof of Theorem 4.2.1. Under the assumption that $p>n^2$, we can distinguish the remaining degrees of cohomology appearing in the above direct sum using the action of central elements of $G(\mathcal{O}_K)$. Let $f=[K:\mathbf{Q}_p]$. The center of $G(\mathcal{O}_K)$ is $(\mathcal{O}_F\otimes_{\mathcal{O}_{F^+}}\mathcal{O}_K)^\times$, and an element $z\in(\mathcal{O}_F\otimes_{\mathcal{O}_{F^+}}\mathcal{O}_K)^\times$ acts on $U(\mathcal{O}_K)$ as multiplication by $(N_{F/F^+}\otimes\mathrm{id})(z)\in\mathcal{O}_K^\times$. We denote by ζ a primitive p^f-1 root of unity in $\mathcal{O}_K^\times$. We can choose an element z of the center of $G(\mathcal{O}_K)$ of order p^f-1 that acts as multiplication by ζ on $U(\mathcal{O}_K)$. It follows from Lemma 4.2.2 and the decomposition

$$U(\mathcal{O}_K)\otimes_{\mathbf{Z}_p}\mathcal{O}=\bigoplus_{\sigma:\mathcal{O}_K\hookrightarrow\mathcal{O}}U(\mathcal{O}_K)\otimes_{\mathcal{O}_K,\sigma}\mathcal{O}$$

that for each degree i, we have a decomposition of $H^i(U(\mathcal{O}_K),\mathcal{O}/\varpi^m)$ into a direct sum of $G(\mathcal{O}_K)$-modules

$$M_{(i_\sigma)}=\mathrm{Hom}_{\mathcal{O}}\left(\bigotimes_\sigma\wedge^{i_\sigma}_{\mathcal{O}}(U(\mathcal{O}_K)\otimes_{\mathcal{O}_K,\sigma}\mathcal{O}),\mathcal{O}/\varpi^m\right)$$

indexed by f-tuples of integers

$$\{(i_\sigma)_{\sigma:\mathcal{O}_K\hookrightarrow\mathcal{O}}:0\leq i_\sigma\leq n^2,\sum_\sigma i_\sigma=i\}.$$

The action of z on $M_{(i_\sigma)}$ is multiplication by $\prod_\sigma \sigma(\zeta)^{-i_\sigma}$, so if we fix an embedding σ_0 and write i_j for the jth Frobenius twist of σ_0, then z acts as multiplication by $\sigma_0(\zeta)^{-\sum_{j=0}^{f-1} i_j p^j}$. Since we are assuming $p > n^2$, the value of $\sum_{j=0}^{f-1} i_j p^j \bmod p^f - 1$ determines the integers i_j uniquely, with the exception (only occurring if $p = n^2 + 1$) of when this value is $0 \bmod p^f - 1$, in which case there are two possibilities: $i_\sigma = 0$ for all σ and $i_\sigma = p - 1$ for all σ. As a consequence, for each degree $1 \leq i \leq n^2 f$, we can write down an idempotent $e_i \in \mathcal{O}[z]$ that induces the identity on $H^i(U(\mathcal{O}_K), \mathcal{O}/\varpi^m)$ and the zero map on other degrees $i' \neq i$. There is a homomorphism $\mathcal{O}[z] \to \mathrm{End}_{\mathbf{D}(\mathcal{O}/\varpi^m[G(\mathcal{O}_K)])}(\tau_{\geq 1} R\Gamma(U(\mathcal{O}_K), \mathcal{O}/\varpi^m))$, so the idempotent-completeness of the derived category implies the existence of a natural decomposition

$$\tau_{\geq 1} R\Gamma(U(\mathcal{O}_K), \mathcal{O}/\varpi^m) = \bigoplus_{i=1}^{n^2 f} e_i R\Gamma(U(\mathcal{O}_K), \mathcal{O}/\varpi^m).$$

This completes the proof. □

4.3. *Cohomology in the middle degree.* In this section we state the fundamental result that we need to study cohomology in the middle degree using automorphic representations of $\widetilde{G}$. We first need to recall a definition; see [CS17, Def. 1.9] — although note that since our representations are in characteristic p, the roles of p and l are reversed.

Definition 4.3.1. Let k be a finite field of characteristic p.

(1) Let $l \neq p$ be a prime, and let $L/\mathbf{Q}_l$ be a finite extension. We say that a continuous representation $\overline{r} : G_L \to \mathrm{GL}_n(k)$ is generic if it is unramified and the eigenvalues (with multiplicity) $\alpha_1, \ldots, \alpha_n \in \overline{k}$ of $\overline{r}(\mathrm{Frob}_L)$ satisfy $\alpha_i/\alpha_j \neq |\mathcal{O}_L/\mathfrak{m}_L|$ for all $i \neq j$.
(2) Let L be a number field, and let $\overline{r} : G_L \to \mathrm{GL}_n(k)$ be a continuous representation. We say that a prime $l \neq p$ is decomposed generic for $\overline{r}$ if l splits completely in L and for all places $v|l$ of L, $\overline{r}|_{G_{L_v}}$ is generic.
(3) Let L be a number field, and let $\overline{r} : G_L \to \mathrm{GL}_n(k)$ be a continuous representation. We say that $\overline{r}$ is decomposed generic if there exists a prime $l \neq p$ that is decomposed generic for $\overline{r}$.

Note that if $\overline{r}$ and $\overline{r}'$ give rise to the same projective representation, then one is (decomposed) generic if and only if the other is.

Lemma 4.3.2. *Let L be a number field, and let $\overline{r} : G_L \to \mathrm{GL}_n(k)$ be a continuous representation. Suppose that $\overline{r}$ is decomposed generic. Then there exist infinitely many primes $l \neq p$ that are decomposed generic for $\overline{r}$.*

Proof. Let $K'/\mathbf{Q}$ denote the Galois closure of the extension of $L(\zeta_p)$ cut out by $\overline{r}$. Let l_0 be a prime that is decomposed generic for $\overline{r}$; then any other

prime l that is unramified in K' and such that Frob_l, Frob_{l_0} lie in the same conjugacy class of $\mathrm{Gal}(K'/\mathbf{Q})$ is also decomposed generic for $\overline{r}$. There are infinitely many such primes, by the Chebotarev density theorem. $\square$

Let $d = n^2[F^+ : \mathbf{Q}] = \frac{1}{2}\dim_{\mathbf{R}} \widetilde{X} = \dim_{\mathbf{R}} X + 1$.

THEOREM 4.3.3. *Suppose that $[F^+ : \mathbf{Q}] > 1$. Let $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbf{T}}^S(\widetilde{K}, \widetilde{\lambda})$ be a maximal ideal, and suppose that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ has length at most 2. Suppose that S satisfies the following condition*:

- *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

Suppose that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic, in the sense of Definition 4.3.1. Then we have

$$H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}[1/p])_{\widetilde{\mathfrak{m}}} \hookleftarrow H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \twoheadrightarrow H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}.$$

Proof. This is an immediate consequence of the main result in [CS19]. This states that

$$H^i(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi)_{\widetilde{\mathfrak{m}}} = 0 \text{ if } i < d, \text{ and } H^i_c(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi)_{\widetilde{\mathfrak{m}}} = 0 \text{ if } i > d,$$

under the assumptions on $\widetilde{\mathfrak{m}}$ in the statement of the theorem. By considering the short exact sequence of sheaves of $\mathcal{O}$-modules on $\widetilde{X}_{\widetilde{K}}$

$$0 \to \mathcal{V}_{\widetilde{\lambda}} \to \mathcal{V}_{\widetilde{\lambda}} \to \mathcal{V}_{\widetilde{\lambda}}/\varpi \to 0$$

and taking cohomology, we see that $H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}[\varpi] = 0$, since $H^{d-1}(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi) = 0$. By considering the excision sequence for

$$\widetilde{X}_{\widetilde{K}} \hookrightarrow \overline{\widetilde{X}}_{\widetilde{K}},$$

we see that the cokernel of the map $H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \to H^d(\partial\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$ injects into $H^{d+1}_c(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}) = 0$. $\square$

PROPOSITION 4.3.4. *Suppose that $[F^+ : \mathbf{Q}] > 1$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ be a good subgroup that is decomposed with respect to P. Let $\widetilde{\lambda} \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$. Fix a decomposition $\overline{S}_p = \overline{S}_1 \sqcup \overline{S}_2$. Suppose that the following conditions are satisfied*:

(1) *For each $\overline{v} \in \overline{S}_2$, $\widetilde{\lambda}_{\overline{v}} = 0$.*
(2) *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*
(3) *$p > n^2$. (We remind the reader of our blanket assumption in Section 4 that p is unramified in F.)*

Let $w \in W^P_{\overline{S}_2}$, *and let* $\lambda_w = w(\widetilde{\lambda}+\rho)-\rho \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$. *Let* $\mathfrak{m} \subset \mathbf{T}^S$ *be a non-Eisenstein maximal ideal in the support of* $H^*(X_K, \mathcal{V}_{\lambda_w})$, *let* $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$, *and suppose that* $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ *is decomposed generic. Then the map* $\mathcal{S} : \widetilde{\mathbf{T}}^S \to \mathbf{T}^S$ *descends to a homomorphism*

$$\widetilde{\mathbf{T}}^S(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}} \to \mathbf{T}^S(H^{d-l(w)}(X_K, \mathcal{V}_{\lambda_w}))_{\mathfrak{m}}.$$

Moreover, the map

$$\widetilde{\mathbf{T}}^S(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}} \to \widetilde{\mathbf{T}}^S(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}}))_{\widetilde{\mathfrak{m}}}[1/p]$$

is injective.

Proof. This results on combining Theorems 4.3.3 and 4.2.1. □

We introduce some useful language.

Definition 4.3.5. A weight $\widetilde{\lambda} \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F,E)}$ will be said to be CTG ("cohomologically trivial for G") if it satisfies the following condition:

- Given $w \in W^P$, define $\lambda_w = w(\widetilde{\lambda}+\rho)-\rho \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$. Then for all $w \in W^P$ and for all $i_0 \in \mathbf{Z}$, there exists $\tau \in \mathrm{Hom}(F,E)$ such that $\lambda_{w,\tau} - \lambda^\vee_{w,\tau c} \neq (i_0, i_0, \ldots, i_0)$.

This definition will be useful to us because Proposition 4.3.4 shows how to relate a Hecke algebra for G acting on cohomology with integral coefficients to a Hecke algebra for $\widetilde{G}$ acting on cohomology with rational coefficients of weight $\widetilde{\lambda}$ (say). If the weight $\widetilde{\lambda}$ is moreover CTG, then Theorem 2.4.11 (together with the purity lemma [Clo90, Lem. 4.9]) shows that this rational cohomology can moreover be computed in terms of cuspidal automorphic forms for $\widetilde{G}$, which have associated Galois representations with well-understood local properties.

Exploiting this is not straightforward since the weight for G depends both on the chosen weight $\widetilde{\lambda}$ and the chosen Weyl group element w (which must be of a suitable length $l(w)$ in order to target a particular cohomological degree for X_K). This problem will be dealt with in the next section with a "degree shifting" argument.

We first state a lemma that shows that there are "many" dominant weights for $\widetilde{G}$ that are CTG:

Lemma 4.3.6. *Suppose that* $[F^+ : \mathbf{Q}] > 1$. *Let* $\widetilde{\lambda} \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$, *and fix a choice of embedding* $\tau_0 : F^+ \hookrightarrow E$. *Then there exists* $\widetilde{\lambda}' \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$ *satisfying the following conditions*:

(1) $\widetilde{\lambda}_\tau = \widetilde{\lambda}'_\tau$ *for all* $\tau \neq \tau_0$.
(2) $\widetilde{\lambda}'$ *is CTG.*

Proof. Let $\tau \neq \tau_0$ be another embedding $\tau : F^+ \hookrightarrow E$. Note that a dominant weight $\widetilde{\mu} \in (\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$ is CTG if it satisfies the following condition:

for all $w \in W^P$, we have

$$\sum_{i=1}^{n}(\mu_{w,\widetilde{\tau},i} - \mu_{w,\widetilde{\tau}c,i}) \neq \sum_{i=1}^{n}(\mu_{w,\widetilde{\tau}_0,i} - \mu_{w,\widetilde{\tau}_0 c,i}). \tag{4.3.7}$$

Let $a \in \mathbf{Z}_{\geq 0}$, and define $\widetilde{\lambda}' \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ by the formula $\widetilde{\lambda}'_\tau = \widetilde{\lambda}_\tau$ if $\tau \neq \tau_0$, $\widetilde{\lambda}'_{\tau_0,1} = \widetilde{\lambda}_{\tau_0,1} + a$, $\widetilde{\lambda}'_{\tau,i} = \widetilde{\lambda}_{\tau_0,i}$ if $i > 1$. Then $\widetilde{\lambda}'$ will satisfy condition (4.3.7) as soon as a is sufficiently large (in a way depending on $\widetilde{\lambda}$). □

4.4. *The degree shifting argument.* We are now going to show how to use Proposition 4.3.4 to control the Hecke algebra of G acting on the cohomology groups $H^q(X_K, \mathcal{V}_\lambda)$. We will do this "one place of F^+ above p at a time." The argument will involve induction on the cohomological degree q. Since the cohomology groups of locally symmetric spaces for G may contain torsion, one needs an inductive argument to pass from the cohomology groups with $\mathcal{O}$-coefficients (which appear in Proposition 4.3.4) to cohomology groups with $\mathcal{O}/\varpi^m$-coefficients (where one can use congruences to modify the weight).

The first step is the following proposition. Given a non-Eisenstein maximal ideal $\mathfrak{m} \subset \mathbf{T}^S$, we will set $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$. We will use the notation

$$A(K, \lambda, q) = \mathbf{T}^S(H^q(X_K, \mathcal{V}_\lambda)_\mathfrak{m}),$$

$$A(K, \lambda, q, m) = \mathbf{T}^S(H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_\mathfrak{m}),$$

and

$$\widetilde{A}(\widetilde{K}, \widetilde{\lambda}) = \widetilde{\mathbf{T}}^S(H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}).$$

Note that there is no natural morphism $A(K, \lambda, q) \to A(K, \lambda, q, m)$.

Proposition 4.4.1. *Let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, and let $\overline{v}, \overline{v}'$ be distinct places of $\overline{S}_p$. (The condition that $\overline{S}_p$ has at least two distinct places implies, in particular, that $F^+ \neq \mathbf{Q}$.) Fix an integer $m \geq 1$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup. Suppose that the following conditions are satisfied:*

(1) *For each embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$.*

(2) *We have*

$$\sum_{\substack{\overline{v}'' \in \overline{S}_p \\ \overline{v}'' \neq \overline{v}, \overline{v}'}} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$

(3) *For each p-adic place $\overline{v}''$ of F^+ not equal to $\overline{v}$, we have*

$$U(\mathcal{O}_{F^+_{\overline{v}''}}) \subset \widetilde{K}_{\overline{v}''} \subset \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \ mod\ \varpi_{\overline{v}''}^m \right\}.$$

We have $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$.

(4) $p > n^2$. (*We recall our blanket assumption in Section 4 that p is unramified in F.*)
(5) *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*
(6) $\mathfrak{m} \subset \mathbf{T}^S$ *is a non-Eisenstein maximal ideal such that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.*

Define a weight $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ as follows: if $\tau \in \mathrm{Hom}(F^+, E)$ does not induce either $\overline{v}$ or $\overline{v}'$, then $\widetilde{\lambda}_\tau = 0$. If τ induces $\overline{v}$, then we set

$$\widetilde{\lambda}_\tau = (-\lambda_{\widetilde{\tau}c,n}, \ldots, -\lambda_{\widetilde{\tau}c,1}, \lambda_{\widetilde{\tau},1}, \ldots, \lambda_{\widetilde{\tau},n}).$$

(*Note that this is dominant because of our assumption on λ.*) *If τ induces $\overline{v}'$, then we choose $\widetilde{\lambda}_\tau$ to be an arbitrary element of $\mathbf{Z}_+^{2n}$.*

Let $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d-1\right]$. Then there exist an integer $m' \geq m$, an integer $N \geq 1$, a nilpotent ideal $J \subset A(K, \lambda, q, m)$ satisfying $J^N = 0$, and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbf{T}}^S & \longrightarrow & \widetilde{A}(\widetilde{K}(m'), \widetilde{\lambda}) \\ {\scriptstyle \mathcal{S}}\big\downarrow & & \big\downarrow \\ \mathbf{T}^S & \longrightarrow & A(K, \lambda, q, m)/J, \end{array}$$

where $\widetilde{K}(m') \subset \widetilde{K}$ is the good subgroup defined by setting

$$\widetilde{K}(m')_{\overline{v}''} = \widetilde{K}_{\overline{v}''} \cap \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \ mod\ \varpi_{\overline{v}''}^{m'} \right\} \subset \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}''}})$$

if $\overline{v}''$ is a p-adic place of F^+ not equal to $\overline{v}$, and $\widetilde{K}(m')_{\overline{v}''} = \widetilde{K}_{\overline{v}''}$ otherwise. (*Thus $\widetilde{K} = \widetilde{K}(m)$, by hypothesis.*) *Moreover, the integer N can be chosen to depend only on n and $[F^+ : \mathbf{Q}]$.*

Proof. The idea of the proof is to choose a Weyl group element $w = w(q) \in W^P$ such that $l(w) = d-q$ and a weight $\widetilde{\lambda}$ such that $\lambda = w(\widetilde{\lambda}+\rho)-\rho$, and then apply Proposition 4.3.4. The actual argument is more subtle, because we need to work with $\mathcal{O}$-coefficients in order to access the Hecke algebras $\widetilde{A}(\widetilde{K}, \widetilde{\lambda})$, whilst the Hecke algebras $A(K, \lambda, q, m)$ act on cohomology with torsion coefficients. We argue by descending induction on q, the induction hypothesis being as follows:

Hypothesis 4.4.2. Let $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d-1\right]$. Then the proposition holds for every cohomological degree $i \in [q+1, d-1]$ and every $m \in \mathbf{Z}_{\geq 1}$. Moreover, the integer N can be chosen to depend only on n, $[F^+ : \mathbf{Q}]$, and q.

The induction hypothesis is always satisfied when $q = d-1$. Assume the induction hypothesis holds for some $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor + 1, d-1\right]$. We will prove that the induction hypothesis holds for $q-1$. Let us fix m, $\widetilde{K}$, and λ as in the statement of the proposition. Note that the $\mathbf{T}^S$-algebra $A(K,\lambda,q,m)$ is independent of $\lambda_{\overline{v}''}$ for $\overline{v}'' \in \overline{S}_p$, $\overline{v}'' \neq \overline{v}$, because K_S acts on $\mathcal{V}_\lambda/\varpi^m$ via the projection to $K_{\overline{v}}$. Modifying λ, we can therefore assume that in fact $\lambda_{\overline{v}'} = \widetilde{\lambda}_{\overline{v}'}$.

Let $\overline{S}_1 = \{\overline{v}, \overline{v}'\}$ and $\overline{S}_2 = \overline{S}_p - \overline{S}_1$. Let $w = w(q) \in W^P_{\overline{S}_2}$ be any element of length $l(w) = d-q$. Such an element exists because for any $\tau \in \mathrm{Hom}(F^+, E)$, $l(w_\tau)$ takes all integer values in $[0, n^2]$ as w_τ ranges over elements of W^P_τ. We have chosen our totally real field F^+ to satisfy

$$\sum_{\overline{v}'' \in \overline{S}_2} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$

This means that the desired sum can take any value in $[0, \frac{d}{2} + \frac{n^2}{2}]$. On the other hand, $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d\right]$, so $d - q \leq d - \left\lfloor \frac{d}{2} \right\rfloor$. Since $n \geq 1$, we can indeed make an appropriate choice of w.

Now we let $\lambda'(q) = w(q)(\widetilde{\lambda} + \rho) - \rho$. This can be different from λ precisely at those embeddings inducing a place of $\overline{S}_2$. In particular, the Hecke algebras $A(K, \lambda'(q), q, m)$ and $A(K, \lambda, q, m)$ are canonically isomorphic as $\mathbf{T}^S$-algebras, once again because K_S acts on both $\mathcal{V}_{\lambda'(q)}/\varpi^m$ and $\mathcal{V}_\lambda/\varpi^m$ via projection to $K_{\overline{v}}$.

There is a short exact sequence of $\mathbf{T}^S$-modules

$$\begin{aligned} 0 \to H^q(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}/\varpi^m \to H^q(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^m)_{\mathfrak{m}} \\ \to H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}[\varpi^m] \to 0. \end{aligned} \tag{4.4.3}$$

Note that the ϖ^m-torsion $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}[\varpi^m]$ does not, in general, inject into $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^m)_{\mathfrak{m}}$, so we cannot reduce to understanding the Hecke algebra $A(q+1, K, \lambda'(q), m)$. However, the cohomology group $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}$ is a finitely generated $\mathcal{O}$-module, so $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}[\varpi^m]$ does inject into $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}/\varpi^{m'}$ provided that $m' \geq m$ is chosen large enough for $\varpi^{m'}$ to annihilate the torsion submodule of $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}$. This, in turn, injects into $H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}}$. It follows that we have an inclusion

$$\begin{aligned} \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}} \cdot \mathrm{Ann}_{\mathbf{T}^S} H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}} \\ \subset \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}. \end{aligned} \tag{4.4.4}$$

Let $K(m') = \widetilde{K}(m') \cap G(\mathbf{A}^\infty_{F^+})$. Let $\mathfrak{m}^\vee = \iota(\mathfrak{m}) \subset \mathbf{T}^S$ (notation as in Section 2.2.20). Then $\mathfrak{m}^\vee$ is a non-Eisenstein maximal ideal. Poincaré duality implies that (cf. Corollary 2.2.22 and [NT16, Th. 4.2], noting that $\mathcal{O}/\varpi^m$ is an injective $\mathcal{O}/\varpi^m$-module) there is an equality

$$\mathrm{Ann}_{\mathbf{T}^S} H^i(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^m)_{\mathfrak{m}} = \iota(\mathrm{Ann}_{\mathbf{T}^S} H^{d-1-i}(X_K, \mathcal{V}^\vee_{\lambda'(q)}/\varpi^m)_{\mathfrak{m}^\vee})$$

of ideals of $\mathbf{T}^S$. The existence of the Hochschild–Serre spectral sequence

$$H^i\left(K/K(m'), H^j(X_{K(m')}, \mathcal{V}^\vee_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}^\vee}\right) \Rightarrow H^{i+j}(X_K, \mathcal{V}^\vee_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}^\vee}$$

implies that there is an inclusion

$$\prod_{i=0}^{d-q-2} \mathrm{Ann}_{\mathbf{T}^S} H^i(X_{K(m')}, \mathcal{V}^\vee_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}^\vee} \subset \mathrm{Ann}_{\mathbf{T}^S} H^{d-2-q}(X_K, \mathcal{V}^\vee_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}^\vee}.$$

Applying Corollary 2.2.22 once more, we see that there is an inclusion

$$\prod_{i=q+1}^{d-1} \mathrm{Ann}_{\mathbf{T}^S} H^i(X_{K(m')}, \mathcal{V}_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}} \subset \mathrm{Ann}_{\mathbf{T}^S} H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}},$$

or equivalently

$$\prod_{i=q+1}^{d-1} \mathrm{Ann}_{\mathbf{T}^S} H^i(X_{K(m')}, \mathcal{V}_{\lambda}/\varpi^{m'})_{\mathfrak{m}} \subset \mathrm{Ann}_{\mathbf{T}^S} H^{q+1}(X_K, \mathcal{V}_{\lambda'(q)}/\varpi^{m'})_{\mathfrak{m}}.$$

Combining this with (4.4.4), we deduce that there is an inclusion

$$\begin{aligned} \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}} \cdot \prod_{i=q+1}^{d-1} \mathrm{Ann}_{\mathbf{T}^S} H^i(X_{K(m')}, \mathcal{V}_{\lambda}/\varpi^{m'})_{\mathfrak{m}} \\ \subset \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_{\lambda}/\varpi^{m})_{\mathfrak{m}}. \end{aligned} \tag{4.4.5}$$

By induction, we can find an integer $N \geq 1$ and for each $i = q+1, \ldots, d-1$, an integer $m'_i \geq m'$ such that

$$\mathcal{S}\left(\mathrm{Ann}_{\widetilde{\mathbf{T}}^S} H^d(\widetilde{X}_{\widetilde{K}(m'_i)}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}\right)^N \subset \mathrm{Ann}_{\mathbf{T}^S} H^i(X_{K(m')}, \mathcal{V}_{\lambda}/\varpi^{m'})_{\mathfrak{m}}.$$

Moreover, Proposition 4.3.4 implies that there is an inclusion

$$\mathcal{S}\left(\mathrm{Ann}_{\widetilde{\mathbf{T}}^S} H^d(\widetilde{X}_{\widetilde{K}}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}\right) \subset \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_{\lambda'(q)})_{\mathfrak{m}}.$$

Let $m'' = \sup_i m'_i$, and note that for each i, we have

$$\mathrm{Ann}_{\widetilde{\mathbf{T}}^S} H^d(\widetilde{X}_{\widetilde{K}(m'')}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}} \subset \mathrm{Ann}_{\widetilde{\mathbf{T}}^S} H^d(\widetilde{X}_{\widetilde{K}(m'_i)}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}$$

(because this is true rationally, and the cohomology groups are torsion-free, by Theorem 4.3.3). Finally, let $N' = 1 + (d-q-1)N$, and let J denote the image of the ideal

$$\mathcal{S}\left(\mathrm{Ann}_{\widetilde{\mathbf{T}}^S} H^d(\widetilde{X}_{\widetilde{K}(m'')}, \mathcal{V}_{\widetilde{\lambda}})_{\widetilde{\mathfrak{m}}}\right)$$

in $A(K, q, \lambda, m)$. The existence of the inclusion (4.4.5) implies that $\mathcal{S}$ descends to a morphism

$$\widetilde{A}(\widetilde{K}(m''), \widetilde{\lambda}) \to A(K, \lambda, q, m)/J,$$

and that the ideal J satisfies $J^{N'} = 0$. This completes the proof. □

This proposition has the following consequence for Galois representations.

PROPOSITION 4.4.6. *Let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, and let $\overline{v}, \overline{v}'$ be distinct places of $\overline{S}_p$. Fix an integer $m \geq 1$. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup. Suppose that the following conditions are satisfied*:

(1) *For each embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$, we have $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$ and $-\lambda_{\tau c,n} - \lambda_{\tau,n} \leq p - 2n - 1$.*

(2) *We have*

$$\sum_{\substack{\overline{v}'' \in \overline{S}_p \\ \overline{v}'' \neq \overline{v}, \overline{v}'}} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$

(3) *For each p-adic place $\overline{v}''$ of F^+ not equal to $\overline{v}$, we have*

$$U(\mathcal{O}_{F^+_{\overline{v}''}}) \subset \widetilde{K}_{\overline{v}''} \subset \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \ mod\ \varpi_{\overline{v}''}^m \right\}.$$

We have $\widetilde{K}_{\overline{v}} = \widetilde{G}(\mathcal{O}_{F^+_{\overline{v}}})$.

(4) *$p > n^2$. (We recall our blanket assumption in Section 4 that p is unramified in F.)*

(5) *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*

(6) *$\mathfrak{m} \subset \mathbf{T}^S$ is a non-Eisenstein maximal ideal such that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.*

Let $q \in \left[\left\lfloor \frac{d}{2} \right\rfloor, d-1\right]$. Then there exist an integer $N \geq 1$ depending only on $[F : \mathbf{Q}]$ and n, an ideal $J \subset A(K, \lambda, q, m)$ satisfying $J^N = 0$, and a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(A(K, \lambda, q, m)/J)$$

satisfying the following conditions:

(a) *For each place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(A(K, \lambda, q, m)/J)[X]$.*

(b) *For each place $v|\overline{v}$ of F, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is in the essential image of the functor $\mathbf{G}^a$ for $a = (\lambda_{\tau,n}) \in \mathbf{Z}^{\mathrm{Hom}_{\mathbf{Q}_p}(F_v,E)}$.*

(c) *For each place $v|\overline{v}$ of F, there exists $\overline{N} \in \mathcal{MF}_k$ with $\overline{\rho}_{\widetilde{\mathfrak{m}}}|_{G_{F_v}} \cong \mathbf{G}(\overline{N})$ and*

$$\mathrm{FL}_\tau(\overline{N}) = \{-\lambda_{\tau c,n} + 2n - 1, \ldots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + (n-1), \ldots, \lambda_{\tau,n}\}$$

for each embedding $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$.

Proof. Our hypotheses include those of Proposition 4.4.1. We choose the weight $\widetilde{\lambda}$ of Proposition 4.4.1 to be CTG (as we may, using Lemma 4.3.6 and our freedom to specify $\widetilde{\lambda}_{\overline{v}'}$). Let N_0 be the integer denoted by N in the statement

of that proposition. Thus we can find an integer $m' \geq m$, a nilpotent ideal $J_0 \subset A(q, K, \lambda, m)$ satisfying $J_0^{N_0} = 0$, and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbf{T}}^S & \longrightarrow & \widetilde{A}(\widetilde{K}(m'), \widetilde{\lambda}) \\ {\scriptstyle \mathcal{S}}\downarrow & & \downarrow \\ \mathbf{T}^S & \longrightarrow & A(K, \lambda, q, m)/J_0. \end{array}$$

Let us abbreviate $\widetilde{A} = \widetilde{A}(\widetilde{K}(m'), \widetilde{\lambda})$ and $A = A(K, \lambda, q, m)$. By Theorem 4.3.3, $\widetilde{A}$ is $\mathcal{O}$-flat, and by Theorem 2.4.11, $\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$ is semisimple and can be computed in terms of cuspidal automorphic representations of $\widetilde{G}$. By Theorem 2.3.3, there exists a continuous homomorphism

$$\widetilde{\rho} : G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p)$$

such that for any homomorphism $f : \widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \to \overline{\mathbf{Q}}_p$, and for any finite place $v \notin S$ of F, $f \circ \widetilde{\rho}(\mathrm{Frob}_v)$ has characteristic polynomial equal to the image of $\widetilde{P}_v(X)$ in $\overline{\mathbf{Q}}_p[X]$; and for any place $v|\overline{v}$ of F, $(f \circ \widetilde{\rho})|_{G_{F_v}}$ is crystalline of Hodge–Tate weights

$$\mathrm{HT}_\tau(f \circ \widetilde{\rho}|_{G_{F_v}}) = \{-\lambda_{\tau c,n} + (2n-1), \ldots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + (n-1), \ldots, \lambda_{\tau,n}\}.$$

In particular, any G_{F_v}-invariant $\mathcal{O}$-lattice in $\widetilde{A}^{2n}$ is crystalline with all τ-Hodge–Tate weights in the interval $[\lambda_{\tau,n}, (2n-1) - \lambda_{\tau c,n}]$. Using our hypothesis that $-\lambda_{\tau c,n} + (2n-1) - \lambda_{\tau,n} \leq p-2$, we see that any G_{F_v}-invariant $\mathcal{O}$-lattice in $\widetilde{A}^{2n}$ is in the image of the functor $\mathbf{G}^a$ with $a = (\lambda_{\tau,n}) \in \mathbf{Z}^{\mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)}$ (cf. the discussion of the functor $\mathbf{G}^a$ at the beginning of Section 4).

This establishes part (c) of the proposition. Since for each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$ the integers

$$-\lambda_{\tau c,n} + (2n-1), \ldots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + (n-1), \ldots, \lambda_{\tau,n}$$

are all distinct, and $\overline{\rho}_{\widetilde{\mathfrak{m}}} \cong \overline{\rho}_{\mathfrak{m}} \oplus (\overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})$, it follows as well that $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}} \not\cong (\overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})|_{G_{F_v}}$.

Let $\widetilde{D} = \det \widetilde{\rho}$, a continuous determinant of $G_{F,S}$ of dimension $2n$ valued in $\widetilde{A}$ (by [Che14, Ex. 2.32]). Its kernel is a two-sided ideal of $\widetilde{A}[G_{F,S}]$. (See [Che14, §1.17] for the definition of the kernel of a determinant.) The formation of kernels commutes with flat base change over $\widetilde{A}$, so there is an algebra embedding

$$(\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p = (\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p)[G_{F,S}]/\ker(\widetilde{D} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p) \subset M_{2n}(\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p),$$

by [Che14, Th. 2.12]. This is, in particular, an embedding of left $\widetilde{A}[G_{F,S}]$-modules. It follows that $(\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$ is a subrepresentation of $\widetilde{\rho}^{2n}$, hence that for each $v|\overline{v}$, the G_{F_v}-representation $\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})$ is in the essential image of $\mathbf{G}^a$.

Theorem 2.3.7 implies that there there are an integer N_1 depending only on $[F:\mathbf{Q}]$ and n, a nilpotent ideal $J_1 \subset A(K,\lambda,q,m)$ satisfying $J_1^{N_1}=0$, and a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(A(K,\lambda,q,m)/J_1)$$

such that for each finite place $v \notin S$ of F, $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial equal to the image of $P_v(X)$ in $(A(K,\lambda,q,m)/J_1)[X]$. Let $J=(J_0,J_1)\subset A(K,\lambda,q,m)$; then $J^N=0$, where $N=N_0+N_1$. We will show that the proposition holds with this choice of J and this value of N. Let us now write $\rho_{\mathfrak{m}}$ for the projection of $\rho_{\mathfrak{m}}$ to a representation with coefficients in $A(K,\lambda,q,m)/J = A/J$.

Set $\widetilde{D}_{A/J} = \widetilde{D}\otimes_{\widetilde{A}} A/J$. Then $\widetilde{D}_{A/J} = \det(\rho_{\mathfrak{m}} \oplus \rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n})$, hence

$$(\ker\det\rho_{\mathfrak{m}}) \cap (\ker\det\rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n}) \subset \ker\widetilde{D}_{A/J}.$$

The representation $\rho_{\mathfrak{m}} \oplus (\rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n})$ induces an A-algebra homomorphism

$$(A/J)[G_{F,S}] \to M_n(A/J)\oplus M_n(A/J)$$

which, by [Che14, Th. 2.22(i)], is surjective with kernel equal to $(\ker\det\rho_{\mathfrak{m}})\cap(\ker\det\rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n})$. We deduce that $(A/J)[G_{F,S}]/\ker(\widetilde{D}_{A/J})$ is a quotient A/J-algebra of $M_n(A/J)\times M_n(A/J)$. By [Che14, Th. 2.22(ii)], this forces $(A/J)[G_{F,S}]/\ker(\widetilde{D}_{A/J}) = M_n(A/J)\times M_n(A/J)$.

The surjection $\widetilde{A}[G_{F,S}] \to (A/J)[G_{F,S}]/\ker(\widetilde{D}_{A/J})$ factors through the quotient $\widetilde{A}[G_{F,S}]/\ker(\widetilde{D})$; see [Che14, Lem. 1.18]. For each place $v|\overline{v}$ of F, it follows that $M_n(A/J)\times M_n(A/J)$, viewed as a left $(A/J)[G_{F_v}]$-module, is in the essential image of the functor $\mathbf{G}^a$. (The essential image is stable under passage to subquotients.) Since $M_n(A/J)\times M_n(A/J)$ contains $\rho_{\mathfrak{m}}$ as a subobject, it follows that $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is in the essential image of $\mathbf{G}^a$, as desired. □

Remark 4.4.7. Ideas similar to, and more general than, those used in the proof above were developed by Wake–Wang-Erickson [WWE19].

We now extend the range of cohomological degrees and allowable level subgroups to which Proposition 4.4.6 applies.

Corollary 4.4.8. *Let $\overline{v}\in S_p$, and let $K\subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup. Let $\lambda\in(\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, and let $\mathfrak{m}\subset \mathbf{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal. Suppose that the following conditions are satisfied*:

(1) *For each place $v|\overline{v}$ of F, we have $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.*
(2) *There exists a place $\overline{v}'\in\overline{S}_p$ such that $\overline{v}'\neq\overline{v}$ and*

$$\sum_{\substack{\overline{v}''\in\overline{S}_p\\ \overline{v}''\neq\overline{v},\overline{v}'}} [F^+_{\overline{v}''}:\mathbf{Q}_p] > \frac{1}{2}[F^+:\mathbf{Q}].$$

(3) *For each embedding* $\tau : F \hookrightarrow E$ *inducing the place* $\overline{v}$ *of* F^+, *we have* $-\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0$ *and* $-\lambda_{\tau c,n} - \lambda_{\tau,n} \leq p - 1 - 2n$.

(4) $p > n^2$. (*We recall our blanket assumption in Section 4 that* p *is unramified in* F.)

(5) *Let* v *be a finite place of* F *not contained in* S, *and let* l *be its residue characteristic. Then either* S *contains no* l*-adic places of* F *and* l *is unramified in* F, *or there exists an imaginary quadratic field* $F_0 \subset F$ *in which* l *splits.*

(6) $\overline{\rho}_{\mathfrak{m}}$ *is decomposed generic.*

Let $q \in [0, d-1]$ *and* $m \geq 1$ *be integers. Then there exist an integer* $N \geq 1$ *depending only on* $[F : \mathbf{Q}]$ *and* n, *an ideal* $J \subset A(K, \lambda, q, m)$ *satisfying* $J^N = 0$, *and a continuous representation*

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(A(K, \lambda, q, m)/J)$$

satisfying the following conditions:

(a) *For each place* $v \notin S$ *of* F, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ *is equal to the image of* $P_v(X)$ *in* $(A(K, \lambda, q, m)/J)[X]$.

(b) *For each place* $v|\overline{v}$ *of* F, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ *is in the essential image of the functor* $\mathbf{G}^a$ *for* $a = (\lambda_{\tau,n}) \in \mathbf{Z}^{\mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)}$.

(c) *For each place* $v|\overline{v}$ *of* F, *there exists* $\overline{N} \in \mathcal{MF}_k$ *with* $\overline{\rho}_{\widetilde{\mathfrak{m}}}|_{G_{F_v}} \cong \mathbf{G}(\overline{N})$ *and*

$$\mathrm{FL}_\tau(\overline{N}) = \{-\lambda_{\tau c,n} + (2n-1), \ldots, -\lambda_{\tau c,1} + n, \lambda_{\tau,1} + (n-1), \ldots, \lambda_{\tau,n}\}$$

for each embedding $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$.

Proof. Note that the existence of a $\rho_{\mathfrak{m}}$ satisfying only condition (a) (local-global compatibility at unramified places) is already known (Theorem 2.3.5). We are therefore free to enlarge S if necessary. We first prove the corollary with hypothesis (6) replaced by the stronger assumption that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic. Let $K' \subset K$ be the good normal subgroup defined by the formula $K'_v = K_v$ if $v \nmid p$ or $v|\overline{v}$, and $K'_v = K_v \cap \ker(\mathrm{GL}_n(\mathcal{O}_{F_v}) \to \mathrm{GL}_n(\mathcal{O}_{F_v}/\varpi_v^m))$ otherwise. Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup satisfying the following conditions:

- $\widetilde{K} \cap G(\mathbf{A}_{F^+}^\infty) = K$.
- $\widetilde{K}^S = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^S)$.
- For each place $\overline{v}''|p$ of F^+, $U(\mathcal{O}_K) \subset \widetilde{K}_{\overline{v}''}$.
- $\widetilde{K}_{\overline{v}} = G(\mathcal{O}_{F_{\overline{v}}^+})$.

Let $\widetilde{K}' = \widetilde{K}(m)$ be the good subgroup defined as follows: if $\overline{v}''$ is a finite place of F^+ that is prime to p or equal to $\overline{v}$, then $\widetilde{K}'_{\overline{v}''} = \widetilde{K}_{\overline{v}''}$. Otherwise, we set

$$\widetilde{K}'_{\overline{v}''} = \widetilde{K}_{\overline{v}''} \cap \left\{ \begin{pmatrix} 1_n & * \\ 0 & 1_n \end{pmatrix} \bmod \varpi_{\overline{v}''}^m \right\}.$$

Note that the triple $(\widetilde{K}', \lambda, m)$ satisfies the hypotheses of Proposition 4.4.6. We let $K' = \widetilde{K}' \cap G(\mathbf{A}_{F^+}^{\infty})$. There is a Hochschild–Serre spectral sequence

$$H^i(K/K', H^j(X_{K'}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}) \Rightarrow H^{i+j}(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}.$$

It follows that we have an inclusion

$$\prod_{i=0}^{q} \mathrm{Ann}_{\mathbf{T}^S} H^{q-i}(X_{K'}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}} \subset \mathrm{Ann}_{\mathbf{T}^S} H^q(X_K, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}}.$$

Suppose we could show that there is an integer N_0 depending only on $[F^+ : \mathbf{Q}]$ and n and for each $i = 0, \dots, q$ an ideal $J_i \subset A(K', \lambda, q-i, m)$ satisfying $J_i^{N_0} = 0$ and a continuous representation $\rho_{\mathfrak{m},i} : G_{F,S} \to \mathrm{GL}_n(A(K', \lambda, q-i, m)/J_i)$ satisfying the conditions the same conditions as $\rho_{\mathfrak{m}}$. Then the corollary would follow, with J equal to the image in $A(K, \lambda, q, m)$ of the intersection of the pre-images of $J_0, \dots, J_q$ in $\mathbf{T}^S$, and $N = qN_0$. A theorem of Carayol, [Car94, Th. 2], implies that the product representation

$$\prod_{i=0}^{q} \rho_{\mathfrak{m},i} : G_{F,S} \to \mathrm{GL}_n\left(\prod_{i=0}^{q} A(K', \lambda, q-i, m)/J_i\right)$$

can be conjugated to take values in $\mathrm{GL}_n(\mathrm{im}(\mathbf{T}^S \to \prod_{i=0}^{q} A(K', \lambda, q-i, m)/J_i))$, and the ring $\mathrm{im}(\mathbf{T}^S \to \prod_{i=0}^{q} A(K', \lambda, q-i, m)/J_i)$ has $A(K, \lambda, q, m)/J$ as a quotient.

We are therefore free to assume that $K = K'$ and $\widetilde{K} = \widetilde{K}'$, which we now do. In this case, we can moreover assume that $\lambda_{\overline{v}''} = 0$ if $\overline{v}'' \in \overline{S}_p$ and $\overline{v}'' \neq \overline{v}$. Note that $\widetilde{K}$ satisfies the conditions of Proposition 4.4.6, so if $q - i \geq \lfloor d/2 \rfloor$, there is nothing to do. Suppose instead that $q - i < \lfloor d/2 \rfloor$. Then $d - 1 - q + i \geq \lfloor d/2 \rfloor$.

Our condition on $\lambda_{\overline{v}}$ then implies, together with [Jan03, Cor. II.5.6], that there is an isomorphism $\mathcal{V}_{\lambda^\vee} \cong \mathcal{V}_\lambda^\vee$. Let $n_0 = (2n + 1 - p)/2$, and let $\mu_0 \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ be defined by $\mu_{0,\tau} = (n_0, \dots, n_0)$ for each τ. Then the maximal ideal $\mathfrak{m}^\vee(\epsilon^{-n_0})$ of $\mathbf{T}^S$ (cf. Section 2.2.20) is in the support of $H^*(X_K, \mathcal{V}_{\lambda^\vee + \mu_0})$, and the weight $\lambda^\vee + \mu_0$ also satisfies the hypothesis (3) of the corollary.

Proposition 4.4.6 implies the existence of an ideal

$$J_i' \subset \mathbf{T}^S(H^{d-1-q+i}(X_K, \mathcal{V}_{\lambda^\vee + \mu_0}/\varpi^m))_{\mathfrak{m}^\vee(\epsilon^{-n_0})}$$

and a continuous representation

$$\rho'_{\mathfrak{m},i} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(H^{d-1-q+i}(X_K, \mathcal{V}_{\lambda^\vee + \mu_0}/\varpi^m))_{\mathfrak{m}^\vee(\epsilon^{-n_0})}/J_i')$$

satisfying the same conditions as $\rho_{\mathfrak{m}}$. Propositions 2.2.21 and 2.2.23 together imply that the isomorphism

$$\mathbf{T}^S \to \mathbf{T}^S,\ [K^S g K^S] \mapsto \epsilon(\mathrm{Art}_K(\det(g)))^{-n_0}[K^S g^{-1} K^S]$$

descends to an isomorphism

$$f : \mathbf{T}^S(H^{d-1-q+i}(X_K, \mathcal{V}_{\lambda^\vee+\mu_0}/\varpi^m))_{\mathfrak{m}^\vee(\epsilon^{-n_0})} \to A(K, \lambda, q-i, m).$$

The proof in this case is completed by taking $J_i = f(J_i')$ and $\rho_{\mathfrak{m},i} = (f \circ \rho'_{\mathfrak{m},i})^\vee \otimes \epsilon^{1-2n+(p-1)/2}$.

We now remove the assumption that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic, assuming instead only that $\overline{\rho}_{\mathfrak{m}}$ is decomposed generic. After possibly enlarging k, we can find a character $\overline{\psi} : G_F \to k^\times$ such that

$$(\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi}) \oplus ((\overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi})^{c,\vee} \otimes \epsilon^{1-2n})$$

is decomposed generic, and $\overline{\psi}|_{G_{F_v}}$ is trivial for each place $v \in S$ of F. Let $\psi : G_F \to \mathcal{O}^\times$ denote the Teichmüller lift of $\overline{\psi}$.

Choose a finite set S' containing S and the set of places where ψ is ramified and a good normal subgroup $K' \subset K$, all satisfying the following conditions:

- $(K')^{S'-S} = K^{S'-S}$.
- The quotient K'/K is abelian of order prime to p.
- For each place v of F, the restriction of $\psi|_{G_{F_v}} \circ \mathrm{Art}_{F_v}$ to $\det(K_v')$ is trivial.
- S' satisfies the analogue of hypothesis (5) of the corollary.

Then there is a surjection $A(K', \lambda, q, m) \to A(K, \lambda, q, m)$ of $\mathbf{T}^{S'}$-algebras, so it suffices to establish the corollary for $A(K', \lambda, q, m)$. We write $\mathfrak{m}(\psi) \subset \mathbf{T}^{S'}$ for the non-Eisenstein maximal ideal with $\overline{\rho}_{\mathfrak{m}(\psi)} \cong \overline{\rho}_{\mathfrak{m}} \otimes \overline{\psi}$.

Let $\widetilde{\mathfrak{m}}(\psi) = \mathcal{S}^*(\mathfrak{m}(\psi))$. Then $\overline{\rho}_{\widetilde{\mathfrak{m}}(\psi)}$ is decomposed generic, so the already established case of the corollary implies that we can find an integer $N \geq 1$ depending only on $[F^+ : \mathbf{Q}]$ and n, an ideal $J' \subset \mathbf{T}^{S'}(H^q(X_{K'}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}(\psi)})$ satisfying $J'^n = 0$, and a continuous representation

$$\rho_{\mathfrak{m}(\psi)} : G_{F,S'} \to \mathrm{GL}_n(\mathbf{T}^{S'}(H^q(X_{K'}, \mathcal{V}_\lambda/\varpi^m)_{\mathfrak{m}(\psi)})/J')$$

satisfying the conditions (a)–(c) of the corollary. Proposition 2.2.23 implies that the isomorphism

$$\mathbf{T}^{S'} \to \mathbf{T}^{S'},\ [K'^S g K'^S] \mapsto \psi(\mathrm{Art}_F(\det(g)))[K'^S g K'^S]$$

descends to an isomorphism

$$f : \mathbf{T}^{S'}(H^q(X_{K'}, \mathcal{V}_\lambda/\varpi^m))_{\mathfrak{m}(\psi)} \to A(K', \lambda, q, m).$$

The proof is completed on taking $J = f(J')$ and $\rho_{\mathfrak{m}} = (f \circ \rho_{\mathfrak{m}(\psi)}) \otimes \psi^{-1}$. □

4.5. *The end of the proof.* We can now prove the main theorem of this chapter. (For the reader's convenience, we repeat the statement here.) To avoid confusion, we also restate the standing hypotheses for this chapter in the statement of the theorem.

THEOREM 4.5.1. *Let $\mathfrak{m} \subset \mathbf{T}^S(K,\lambda)$ be a non-Eisenstein maximal ideal. Suppose that $\mathbf{T}^S(K,\lambda)/\mathfrak{m} = k$ has residue characteristic p. Let $\overline{v}$ be a p-adic place of F^+, and suppose that the following additional conditions are satisfied*:

(1) *The prime p is unramified in F, and F contains an imaginary quadratic field in which p splits.*
(2) *Let w be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0 \subset F$ in which l splits.*
(3) *For each place $v|\overline{v}$ of F, $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.*
(4) *For every embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+,*
$$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} \leq p - 2n - 1.$$
(5) *$p > n^2$.*
(6) *There exists a p-adic place $\overline{v}' \neq \overline{v}$ of F^+ such that*
$$\sum_{\substack{\overline{v}'' \in \overline{S}_p \\ \overline{v}'' \neq \overline{v}, \overline{v}'}} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}].$$
(7) *$\overline{\rho}_{\mathfrak{m}}$ is decomposed generic (Definition 4.3.1).*
(8) *Assume that one of the following holds*:
   (a) *$H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p] \neq 0$, or*
   (b) *for every embedding $\tau\colon F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+,*
$$-\lambda_{\tau c,n} - \lambda_{\tau,n} \leq p - 2n - 2 \quad \text{and} \quad -\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0.$$

Then there exist an integer $N \geq 1$ depending only on $[F^+ : \mathbf{Q}]$ and n, an ideal $J \subset \mathbf{T}^S(K,\lambda)$ satisfying $J^N = 0$, and a continuous representation
$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/J)$$
satisfying the following conditions:

(a) *For each finite place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $(\mathbf{T}^S(K,\lambda)_{\mathfrak{m}}/J)[X]$.*
(b) *For each place $v|\overline{v}$ of F, $\rho_{\mathfrak{m}}|_{G_{F_v}}$ is in the essential image of $\mathbf{G}^a$ (with $a = (\lambda_{\tau,n}) \in \mathbf{Z}^{\mathrm{Hom}(F_v,E)}$).*
(c) *There is $\overline{M} \in \mathcal{MF}_k$ such that $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}} \cong \mathbf{G}(\overline{M})$ and for any embedding $\tau\colon F_v \hookrightarrow E$,*
$$\mathrm{FL}_\tau(\overline{M}) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\}.$$

Proof. Note that the existence of a $\rho_{\mathfrak{m}}$ satisfying only condition (a) is already known (Theorem 2.3.5). We are therefore free to enlarge S if necessary. We first prove the theorem under the assumption that $H^*(X_K, \mathcal{V}_\lambda)_{\mathfrak{m}}[1/p] \neq 0$. By Theorem 2.4.10, there exist an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ and a cuspidal

automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$ such that $(\pi^\infty)^K \neq 0$ and such that $\overline{r_\iota(\pi)} \cong \overline{\rho}_{\mathfrak{m}}$. By [Clo90, Lem. 4.9], there is an integer $w \in \mathbf{Z}$ such that for each embedding $\tau\colon F \hookrightarrow E$ and for each $i = 1, \ldots, n$, we have $\lambda_{\tau,i} + \lambda_{\tau c, n+1-i} = w$. Fix an embedding $\tau_0 \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$ such that $\lambda_{\tau_0,1} + \lambda_{\tau_0 c,1}$ is maximal. (Recall that $\widetilde{v}$ is a fixed choice of place of F lying above $\overline{v}$.)

After possibly enlarging E, we can (cf. [HSBT10, Lem. 2.2]) find a continuous character $\psi\colon G_F \to \mathcal{O}^\times$ satisfying

- ψ is crystalline at each $v \mid p$,
- ψ is unramified at $\widetilde{v}$ and at each $v \in S - S_p$,
- $\psi \circ \mathrm{Art}_{F_{\widetilde{v}^c}}|_{\mathcal{O}^\times_{F_{\widetilde{v}^c}}} = \prod_{\tau\colon F_{\widetilde{v}} \hookrightarrow E} (\tau c)^{\lambda_{\tau_0,1} + \lambda_{\tau_0 c,1}}$.

Define a weight $\mu = (\mu_{\tau,1}, \ldots, \mu_{\tau,n}) \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ by letting $\mu_{\tau,i}$ be the unique τ-Hodge–Tate weight for ψ for each $1 \leq i \leq n$. Note that for τ inducing $\widetilde{v}$, $\mu_{\tau,i} = 0$ and $\mu_{\tau c,i} = -\lambda_{\tau_0,1} - \lambda_{\tau_0 c,1}$ for all $1 \leq i \leq n$.

Choose a finite set S' containing S and the set of places where ψ is ramified and a good normal subgroup $K' \subset K$, all satisfying the following conditions:

- $(K')^{S'-S} = K^{S'-S}$.
- The quotient K'/K is abelian.
- For each finite place $v \nmid p$ of F, the restriction of $\psi|_{G_{F_v}} \circ \mathrm{Art}_{F_v}$ to $\det(K'_v)$ is trivial.
- S' satisfies the analogue of hypothesis (2) of the theorem.

By an argument with the Hochschild–Serre spectral sequence, just as in the proof of Corollary 4.4.8, we are free to assume that $K = K'$ and $S = S'$, and we now do this. Let $\lambda' = \lambda + \mu$. By Proposition 2.2.23, the map

$$\mathbf{T}^S \to \mathbf{T}^S,\ [K^S g K^S] \mapsto \psi(\mathrm{Art}_F(\det(g)))[K^S g K^S]$$

descends to an isomorphism $f : \mathbf{T}^S(K, \lambda')_{\mathfrak{m}(\psi)} \to \mathbf{T}^S(K, \lambda)_{\mathfrak{m}}$. We observe that for any $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, we have

$$-\lambda'_{\tau c,1} - \lambda'_{\tau,1} = -\lambda_{\tau c,1} - \lambda_{\tau,1} + \lambda_{\tau_0 c,1} + \lambda_{\tau_0,1} \geq 0$$

and (using that $\lambda_{\tau,i} + \lambda_{\tau c, n+1-i} = w$ is independent of τ and i)

$$\begin{aligned} -\lambda'_{\tau c,n} - \lambda'_{\tau,n} &= -\lambda_{\tau c,n} - \lambda_{\tau,n} + \lambda_{\tau_0 c,1} + \lambda_{\tau_0,1} = \lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau_0,n} - \lambda_{\tau_0 c,n} \\ &\leq \lambda_{\tau_0,1} + \lambda_{\tau_0 c,1} - \lambda_{\tau_0 c,n} - \lambda_{\tau_0,n} \leq p - 1 - 2n. \end{aligned}$$

In particular, λ' satisfies the assumptions of Corollary 4.4.8.

We recall (Lemma 2.1.7) that $R\Gamma(X_K, \mathcal{V}_{\lambda'})$ is a perfect complex, with cohomology concentrated in the range $[0, d-1]$. It follows (cf. [NT16, Lem. 3.11]) that the map

$$\mathbf{T}^S(K, \lambda') \to \varprojlim_{m \geq 1} \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_{\lambda'}/\varpi^m))$$

is an isomorphism. On the other hand, [KT17, Lem. 2.5] shows that for any $m \geq 1$, the kernel of the map

$$\mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_{\lambda'}/\varpi^m)) \to \prod_q \mathbf{T}^S(H^q(X_K, \mathcal{V}_{\lambda'}/\varpi^m))$$

is a nilpotent ideal I satisfying $I^d = 0$. Applying Corollary 4.4.8, we see that we can find an integer $N \geq 1$ depending only on $[F^+ : \mathbf{Q}]$ and n, an ideal $J' \subset \mathbf{T}^S(K, \lambda')_{\mathfrak{m}(\psi)}$ satisfying $(J')^N = 0$, and a continuous representation

$$\rho_{\mathfrak{m}(\psi)} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K, \lambda')_{\mathfrak{m}(\psi)}/J')$$

satisfying the following conditions:

(a′) For each place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}(\psi)}(\mathrm{Frob}_v)$ is equal to the image of $P_v(X)$ in $(\mathbf{T}^S(K, \lambda')_{\mathfrak{m}(\psi)}/J')[X]$.

(b′) For each place $v|\overline{v}$ of F, $\rho_{\mathfrak{m}(\psi)}|_{G_{F_v}}$ is in the essential image of the functor $\mathbf{G}^{a'}$ for $a' = (\lambda'_{\tau,n}) \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$.

(c′) For each place $v|\overline{v}$ of F, there exists $\overline{N} \in \mathcal{MF}_k$ with

$$\left(\overline{\rho}_{\mathfrak{m}(\psi)} \oplus (\overline{\rho}^{c,\vee}_{\mathfrak{m}(\psi)} \otimes \epsilon^{1-2n})\right)|_{G_{F_v}} \cong \mathbf{G}(\overline{N})$$

and

$$\mathrm{FL}_\tau(\overline{N}) = \{-\lambda'_{\tau c,n} + (2n-1), \ldots, -\lambda'_{\tau c,1} + n, \lambda'_{\tau,1} + (n-1), \ldots, \lambda'_{\tau,n}\}$$

for each embedding $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$.

Let us define $J = f(J')$ and $\rho_{\mathfrak{m}} = (f \circ \rho_{\mathfrak{m}(\psi)}) \otimes \psi^{-1}$. We see immediately that $\rho_{\mathfrak{m}}$ satisfies the requirements (a) and (b) of the theorem; it remains to establish requirement (c), in other words to recover the Fontaine–Laffaille weights of $\overline{\rho}_{\mathfrak{m}}$.

By the above, there is $\overline{M} \in \mathcal{MF}^a_k$ such that $\overline{\rho}_{\mathfrak{m}} \cong \mathbf{G}^a(\overline{M})$. Let $x\colon \mathbf{T}(K, \lambda)_{\mathfrak{m}} \to \overline{\mathbf{Q}}_p$ denote the homomorphism that gives the action of Hecke operators on $\iota^{-1}(\pi^\infty)^K$. The pushforward $\rho_x = x \circ \rho_{\mathfrak{m}}$ via x is a continuous representation of $G_{F,S}$ that is crystalline at $\widetilde{v}$ and $\widetilde{v}^c$, satisfying $\mathrm{HT}_\tau(\rho_x) = \mathrm{FL}_\tau(\overline{M})$ for each $\tau \in \mathrm{Hom}(F, E)$ inducing the place $\overline{v}$ of F^+. It therefore suffices to show that

$$\mathrm{HT}_\tau(\rho_x) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\}$$

for each $\tau \in \mathrm{Hom}(F, E)$ inducing the place $\overline{v}$ of F^+, or equivalently that

$$\mathrm{HT}_\tau(\rho_x \otimes \psi) = \{\lambda'_{\tau,1} + n - 1, \lambda'_{\tau,2} + n - 2, \ldots, \lambda'_{\tau,n}\}.$$

Let $\omega_\pi : \mathbf{A}_F^\times \to \mathbf{C}^\times$ denote the central character of π. Then ω_π is a character of type A_0 and for each embedding $\tau : F \hookrightarrow E$ inducing the place $\overline{v}$ of F^+, we have

$$\mathrm{HT}_\tau(r_\iota(\omega_\pi)) = \left\{\sum_{i=1}^n \lambda_{\tau,i}\right\}.$$

Moreover, $\det \rho_x = r_\iota(\omega_\pi)\epsilon^{n(1-n)/2}$, hence $\det(\rho_x \otimes \psi) = r_\iota(\omega_\pi)\epsilon^{n(1-n)/2}\psi^n$, as this can be checked on Frobenius elements at unramified places. We are now done: $\mathrm{HT}_\tau(\rho_x \otimes \psi)$ is an n-element subset of

$$\mathrm{FL}_\tau(\overline{N}) = \{-\lambda'_{\tau c,n} + (2n-1), \dots, -\lambda'_{\tau c,1} + n, \lambda'_{\tau,1} + (n-1), \dots, \lambda'_{\tau,n}\}$$

with sum equal to $\sum_{i=1}^n (\lambda'_{\tau,i} + n - i)$. By construction, we have

$$-\lambda'_{\tau c,n} + (2n-1) > \cdots > -\lambda'_{\tau c,1} + n > \lambda'_{\tau,1} + (n-1) > \cdots > \lambda'_{\tau,n}.$$

The only possibility is that $\mathrm{HT}_\tau(\rho_x \otimes \psi)$ has the required form. This completes the proof of the theorem in the case $H^*(X_K, \mathcal{V}_\lambda)_\mathfrak{m}[1/p] \neq 0$.

We now treat the second case, assuming that for every embedding $\tau \in \mathrm{Hom}(F, E)$ inducing the place $\overline{v}$ of F^+, we have

$$-\lambda_{\tau c,n} - \lambda_{\tau,n} \leq p - 2n - 2 \quad \text{and} \quad -\lambda_{\tau c,1} - \lambda_{\tau,1} \geq 0.$$

In this case Corollary 4.4.8 applies directly, and it only remains to identify the Fontaine–Laffaille weights of $\overline{\rho}_\mathfrak{m}$ for each place $v|\overline{v}$ of F. There are $\overline{M}, \overline{M}' \in \mathcal{MF}_k^a$ such that $\overline{\rho}_\mathfrak{m}|_{G_{F_v}} \cong \mathbf{G}(\overline{M})$ and $(\overline{\rho}_\mathfrak{m}^{c,\vee} \otimes \epsilon^{1-2n})|_{G_{F_v}} \cong \mathbf{G}(\overline{M}')$. We choose a continuous character $\psi\colon G_F \to \mathcal{O}^\times$ satisfying

- ψ is crystalline at each $v' \mid p$,
- ψ is unramified at $\widetilde{v}$,
- $\psi \circ \mathrm{Art}_{F_{\widetilde{v}^c}} = \prod_{\tau\colon F_{\widetilde{v}} \hookrightarrow E} (\tau c)$ on $\mathcal{O}_{F_{\widetilde{v}^c}}^\times$.

After enlarging S, as in the first part of the proof, we can assume that ψ is unramified outside S, in which case the maximal ideal $\mathfrak{m}(\psi)$ of $\mathbf{T}^S$ is defined and occurs in the support of $H^*(X_K, \mathcal{V}_{\lambda'})$, where the weight $\lambda' \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ is defined by the formula $\lambda'_\tau = \lambda_\tau$ if τ does not induce the place $\widetilde{v}^c$ of F, and $\lambda'_\tau = \lambda_\tau - (1, \dots, 1)$ if τ does induce the place $\widetilde{v}^c$ of F. We observe that the weight λ' also satisfies the assumptions of Corollary 4.4.8.

We can now conclude. Let $\tau \in \mathrm{Hom}(F, E)$ be an embedding inducing the place $\widetilde{v}$ of F. The sets $\mathrm{FL}_\tau(\overline{M})$ and $\mathrm{FL}_\tau(\overline{M}')$ partition the $2n$ distinct integers

$$-\lambda_{\tau c,n} + (2n-1) > \cdots > -\lambda_{\tau c,1} + n > \lambda_{\tau,1} + (n-1) > \cdots > \lambda_{\tau,n}$$

and $\mathrm{FL}_\tau(\overline{M})$ and $\mathrm{FL}_\tau(\overline{M}') + 1$ partition the $2n$ distinct integers

$$-\lambda_{\tau c,n} + 2n > \cdots > -\lambda_{\tau c,1} + (n+1) > \lambda_{\tau,1} + (n-1) > \cdots > \lambda_{\tau,n}.$$

Using Lemma 4.5.2, this forces

$$\mathrm{FL}_\tau(\overline{M}) = \{\lambda_{\tau,1} + (n-1), \lambda_{\tau,2} + (n-2), \dots, \lambda_{\tau,n}\}$$

and

$$\mathrm{FL}_\tau(\overline{M}') = \{-\lambda_{\tau c,n} + (2n-1), \dots, -\lambda_{\tau c,1} + n\}.$$

Since $\mathbf{G}(\overline{M}') = (\overline{\rho}_\mathfrak{m}^{c,\vee} \otimes \epsilon^{1-2n})|_{G_{F_v}}$, this implies that for each place $v|\overline{v}$ of F, $\overline{\rho}_\mathfrak{m}|_{G_{F_v}}$ has the correct Fontaine–Laffaille weights. $\square$

LEMMA 4.5.2. *Let $m \geq 1$ be an integer, and let A, B, C, D be sets of integers each of size m. Assume that for any $c \in C$ and $d \in D$, we have $c > d$. If $A \cup B = C \cup D$ and $(A+1) \cup B = (C+1) \cup D$ and both of these sets have $2m$ elements, then $A = C$ and $B = D$.*

Proof. We induct on m. Let c be the largest element of C, and let d be the smallest element of D. Since $A \cup B = C \cup D$ and $(A+1) \cup B = (C+1) \cup D$, we must have $c \in A$ and $d \in B$. We can then apply the inductive hypothesis to $A' = A \setminus \{c\}$, $B' = B \setminus \{d\}$, $C' = C \setminus \{c\}$, and $D' = D \setminus \{d\}$. □

5. Local-global compatibility, $l = p$ (ordinary case)

5.1. *Statements.* Let F be a CM field, and fix an integer $n \geq 1$. Let p be a prime, and let E be a finite extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ large enough to contain the images of all embeddings of F in $\overline{\mathbf{Q}}_p$. We assume throughout this chapter that F satisfies the following standing hypothesis:

- F contains an imaginary quadratic field in which p splits.

In contrast to Section 4, we do not assume that p is unramified in F. As in Section 4, our goal in this chapter is to establish local-global compatibility for some Hecke algebra-valued Galois representations at the p-adic places of F. More precisely, we will show that after projection to the ordinary Hecke algebra, these Galois representations satisfy an ordinariness condition. (See (b) and (c) in the statement of Theorem 5.5.1 below — the consequences of this condition will be explored in Section 6.2.6.) Before formulating the main theorem of this chapter, we must define these ordinary Hecke algebras.

Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Let S be a finite set of finite places of F, containing the p-adic places, stable under complex conjugation. We assume that the following conditions are satisfied:

- Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic subfield of F in which l splits.
- For each place $v|p$ of F, $K_v = \mathrm{Iw}_v$. For each finite place $v \notin S$ of F, $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.

If $c \geq b \geq 0$ are integers with $c \geq 1$, then we define a good subgroup $K(b,c) \subset K$ by the formula $K(b,c)_v = K_v$ if $v \nmid p$ and $K_v = \mathrm{Iw}_v(b,c)$ if $v|p$. Thus $K(0,1) = K$. Then there is an isomorphism $K(0,c)/K(b,c) \cong \prod_{v|p} T_n(\mathcal{O}_{F_v}/\varpi_v^b)$. (Here we are using notation for open compact subgroups and Hecke operators that has been defined in Section 2.2.5.)

We define a Hecke algebra

$$\mathbf{T}^{S,\mathrm{ord}} = \mathbf{T}^S \otimes_{\mathcal{O}} \mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!][\{U_{v,1}, \ldots, U_{v,n}, U_{v,n}^{-1}\}_{v|p}]$$

(where the $U_{v,i}$ are viewed as formal variables). We write $U_v = U_{v,1}U_{v,2}\cdots U_{v,n-1} \in \mathbf{T}^{S,\mathrm{ord}}$ and $U_p = \prod_{v|p} U_v$. We observe that there is a canonical surjective $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!] \to \mathcal{O}[K(0,c)/K(b,c)]$. This extends to a homomorphism

$$\mathbf{T}^{S,\mathrm{ord}} \to \mathrm{End}_{\mathbf{D}(\mathcal{O}[K(0,c)/K(b,c)])}(R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)),$$

where each element $U_{v,i}$ of $\mathbf{T}^{S,\mathrm{ord}}$ acts on the complex $R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)$ by the Hecke operator of the same name. By the theory of ordinary parts (cf. [KT17, §2.4]), there is a well-defined direct summand

$$R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}$$

of $R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)$ in $\mathbf{D}(\mathcal{O}[K(0,c)/K(b,c)])$ on which U_p acts invertibly, and we define $\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$ to be the image of the associated homomorphism

$$\mathbf{T}^{S,\mathrm{ord}} \to \mathrm{End}_{\mathbf{D}(\mathcal{O}[K(0,c)/K(b,c)])}(R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}})$$

or equivalently, extending our usage for the Hecke algebra $\mathbf{T}^S$,

$$\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}} = \mathbf{T}^{S,\mathrm{ord}}(R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}).$$

We observe that there is a canonical homomorphism $\mathbf{T}^S(K(0,c)/K(b,c),\mathcal{V}_\lambda) \to \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$. The Hecke algebra in the source is defined in Section 2.2.1. In general this homomorphism is neither injective nor surjective. However, we do see from the existence of this homomorphism that for any maximal ideal $\mathfrak{m}$ of $\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$, there exists an associated Galois representation $\overline{\rho}_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}/\mathfrak{m})$. We call a maximal ideal $\mathfrak{m}$ of $\mathbf{T}^{S,\mathrm{ord}}$ with residue field a finite extension of k *of Galois type* (resp. *non-Eisenstein*) if its pullback to $\mathbf{T}^S$ is of Galois type (resp. non-Eisenstein) in the sense of Definition 2.3.6.

The Hecke operators $U_{v,i} \in \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$ are invertible (because U_p is). For each place $v|p$ and for each $i = 1,\ldots,n$, we define a character $\chi_{\lambda,v,i} : G_{F_v} \to \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord},\times}$ as the unique continuous character satisfying the identities

$$\begin{aligned}&\chi_{\lambda,v,i} \circ \mathrm{Art}_{F_v}(u)\\ &= \epsilon^{1-i}(\mathrm{Art}_{F_v}(u)) \left(\prod_\tau \tau(u)^{-(w_0^G\lambda)_{\tau,i}}\right) \langle \mathrm{diag}(1,\ldots,u,\ldots,1)\rangle \ (u \in \mathcal{O}_{F_v}^\times)\end{aligned}$$

(the product being over $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, E)$) and

$$\chi_{\lambda,v,i} \circ \mathrm{Art}_{F_v}(\varpi_v) = \epsilon^{1-i}(\mathrm{Art}_{F_v}(\varpi_v)) \frac{U_{v,i}}{U_{v,i-1}}.$$

We can now state the main theorem of this chapter. (As with Theorem 4.5.1 in Section 4.1, we will repeat the statement immediately before its proof with the same numbering.)

THEOREM 5.5.1. *Suppose that* $[F^+ : \mathbf{Q}] > 1$. *Let* $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ *be a good subgroup such that for each place* $v \in S_p$ *of* F, $K_v = \mathrm{Iw}_v$. *Let* $c \geq b \geq 0$ *be integers with* $c \geq 1$, *let* $\lambda \in (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$, *and let* $\mathfrak{m} \subset \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$ *be a non-Eisenstein maximal ideal. Suppose that the following conditions are satisfied*:

(1) *Let* v *be a finite place of* F *not contained in* S, *and let* l *be its residue characteristic. Then either* S *contains no* l*-adic places of* F *and* l *is unramified in* F, *or there exists an imaginary quadratic field* $F_0 \subset F$ *in which* l *splits.*

(2) $\overline{\rho}_{\mathfrak{m}}$ *is decomposed generic.*

Then we can find an integer $N \geq 1$, *which depends only on* $[F^+ : \mathbf{Q}]$ *and* n, *an ideal* $J \subset \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}$ *such that* $J^N = 0$, *and a continuous representation*

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}/J)$$

satisfying the following conditions:

(a) *For each finite place* $v \notin S$ *of* F, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ *equals the image of* $P_v(X)$ *in* $(\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}/J)[X]$.

(b) *For each* $v \in S_p$, *and for each* $g \in G_{F_v}$, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(g)$ *equals* $\prod_{i=1}^n (X - \chi_{\lambda,v,i}(g))$.

(c) *For each* $v \in S_p$, *and for each* $g_1, \ldots, g_n \in G_{F_v}$, *we have*

$$(\rho_{\mathfrak{m}}(g_1) - \chi_{\lambda,v,1}(g_1))(\rho_{\mathfrak{m}}(g_2) - \chi_{\lambda,v,2}(g_2)) \ldots (\rho_{\mathfrak{m}}(g_n) - \chi_{\lambda,v,n}(g_n)) = 0.$$

We refer the reader to Lemma 6.2.11 for the comparison between the condition (c) and the usual notion of an ordinary Galois representation. In short, they coincide for representations with coefficients in a field and distinct diagonal characters.

The rest of Section 5 is devoted to the proof of Theorem 5.5.1. (After proving the theorem, we record a local-global compatibility result for a single ordinary automorphic representation as a corollary.) In the rest of the chapter, we make the following additional standing hypothesis:

- For each place $v|p$ of F, our fixed choices of uniformizer satisfy $\varpi_{v^c} = \varpi_v^c$.

This simplifies notation once we introduce the group $\widetilde{G}$. It is important to note that while the definition of the operators $U_{v,i}$ above depends on the choice of uniformizer ϖ_v, neither the complex $R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}$, nor the Hecke algebra $\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$, nor the truth of Theorem 5.5.1 depend on this choice.

5.2. *Hida theory.* In the previous section we introduced the ordinary Hecke algebras $\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$. In Section 5.2, we recall the basic results about these Hecke algebras and the complexes on which they act; this material goes under the name "Hida theory." We also describe how this theory is related to the corresponding theory for the group $\widetilde{G}$.

5.2.1. *The ordinary part of a smooth representation.* Our first goal is to show, following Emerton [Eme10a], [Eme10b], how to define ordinary parts in a more representation-theoretic way. We will work throughout with $\mathcal{O}/\varpi^m$ coefficients (for some fixed $m \geq 1$) in order to avoid topological issues. We first need to introduce some more notation. If G is a locally profinite group, then we write $\mathrm{Mod}(\mathcal{O}/\varpi^m[\mathrm{G}])$ for the category of $\mathcal{O}/\varpi^m[\mathrm{G}]$-modules, and

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\mathrm{G}]) \subset \mathrm{Mod}(\mathcal{O}/\varpi^m[\mathrm{G}]) \tag{5.2.2}$$

for the full subcategory of smooth modules. More generally, if $\Delta \subset \mathrm{G}$ is an open submonoid that contains an open compact subgroup of G, then we write

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta]) \subset \mathrm{Mod}(\mathcal{O}/\varpi^m[\Delta]) \tag{5.2.3}$$

for the full subcategory of smooth modules (by definition, those for which every vector is fixed by an open subgroup of Δ). We write

$$M \mapsto M^{\mathrm{sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[\Delta]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta])$$

for the functor of smooth vectors; it is right adjoint to the inclusion (5.2.3).

Lemma 5.2.4.

(1) *The category* $\mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[\Delta])$ *is abelian and has enough injectives.*
(2) *Let* $\Delta' \subset \Delta$ *be a subgroup that is either compact or open* (Δ' *is therefore a locally profinite group*). *Then the forgetful functor*

$$\mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[\Delta]) \to \mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[\Delta'])$$

preserves injectives.

Proof. The functor $M \mapsto M^{\mathrm{sm}}$ has an exact left adjoint, and so preserves injectives. Since the category $\mathrm{Mod}(\mathcal{O}/\varpi^m[\Delta])$ has enough injectives, so does $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta])$.

For the second part of the lemma, we split into cases. Suppose first that $\Delta' \subset \Delta$ is an open subgroup. Then compact induction $\text{c-Ind}_{\Delta'}^{\Delta}$ is an exact left adjoint to the forgetful functor. Suppose instead that $\Delta' \subset \Delta$ is a compact subgroup. In this case, we can find a compact open subgroup of Δ that contains Δ'. Using what we have already proved, we can assume that $\Delta = \mathrm{G}$, in which case the result follows from [Eme10b, Prop. 2.1.11]. □

We write $\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta])$ for the derived category of $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta])$.

We introduce some monoids, with the aim of studying the theory for $\mathrm{G} = \mathrm{GL}_n(F_p)$. We write $T_n(F_p)^+ \subset T_n(F_p)$ for the open submonoid consisting of those elements $t \in T_n(F_p)$ with $tN_n(\mathcal{O}_{F,p})t^{-1} \subset N_n(\mathcal{O}_{F,p})$, and $T_n(F_v)^+ = T_n(F_v) \cap T_n(F_p)^+$. We recall (Section 2.2.5) that $\Delta_p \subset \mathrm{GL}_n(F_p)$ denotes the

monoid $\prod_{v|p} \mathrm{Iw}_v T_n(F_v)^+ \mathrm{Iw}_v$. If $b \geq 0$ is an integer, we define

$$T_n(\mathcal{O}_{F,p})(b) = \prod_{v \in S_p} \ker(T_n(\mathcal{O}_{F,v}) \to T_n(\mathcal{O}_{F,v}/\varpi_v^b)),$$
$$T_n(\mathcal{O}_{F,p})_b = T_n(\mathcal{O}_{F,p})/T_n(\mathcal{O}_{F,p})(b),$$
$$T_n(F_p)_b^+ = T_n(F_p)^+/T_n(\mathcal{O}_{F,p})(b),$$

and

$$T_n(F_p)_b = T_n(F_p)/T_n(\mathcal{O}_{F,p})(b).$$

We write $u_p \in T_n(\mathbf{Q}_p) \subset T_n(F_p)$ for the element $(p^{n-1}, p^{n-2}, \ldots, 1)$. It lies in $T_n(F_p)^+$. We define $B_n(F_p)^+ = N_n(\mathcal{O}_{F,p}) \cdot T_n(F_p)^+ \subset B_n(F_p)$. Note that $B_n(F_p)^+ \subset \Delta_p$. We write $B_n(\mathcal{O}_{F,p})(b)$ for the pre-image in $B_n(\mathcal{O}_{F,p})$ of $T_n(\mathcal{O}_{F,p})(b)$. It will be important for us to note that a complex $C \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])$ comes equipped with a functorial homomorphism

$$\mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!][\{U_{v,1}, \ldots, U_{v,n}, U_{v,n}^{-1}\}_{v \in S_p}] \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])}(C)$$

via the map that is the canonical homomorphism

$$\mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!] \to \mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})/T_n(\mathcal{O}_{F,p})(b)]$$

on this subalgebra and that sends $U_{v,i}$ to the matrix

$$\mathrm{diag}(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1) \in T_n(F_v) \subset T_n(F_p)$$

(with i occurrences of ϖ_v). Consequently, if $\mathbf{T}^S$ acts on a complex C, then we can extend this to an action of the algebra $\mathbf{T}^{S,\mathrm{ord}}$.

If $\lambda \in X^*((\mathrm{Res}_{F/\mathbf{Q}} T_n)_E) = (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$, we write $\mathcal{O}(\lambda)$ for the $\mathcal{O}[T_n(F_p)]$-module defined as follows: it is a free rank 1 $\mathcal{O}$-module on which an element $u \in T_n(\mathcal{O}_{F,p})$ acts as multiplication by the scalar $\prod_{\tau \in \mathrm{Hom}(F,E)} \prod_{i=1}^n \tau(u_i)^{\lambda_{\tau,i}}$ and on which any element $\mathrm{diag}(\varpi_v^{a_1}, \ldots, \varpi_v^{a_n})$ ($a_i \in \mathbf{Z}$) acts trivially.

We recall that in Section 2.2.5 we have defined, for any $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, a twisted action $(\delta, v) \mapsto \delta \cdot_p v$ of Δ_p on $\mathcal{V}_\lambda$. Projection to the lowest weight space determines an $\mathcal{O}$-module homomorphism $\mathcal{V}_\lambda \to \mathcal{O}(w_0^G \lambda)$ that is equivariant for the action of $B_n(F_p)^+$ (where $B_n(F_p)^+$ acts through the $\cdot_p$-action on the source and through its projection to $T_n(F_p)$ on the target). We write $\mathcal{K}_\lambda$ for the kernel of the projection $\mathcal{V}_\lambda \to \mathcal{O}(w_0^G \lambda)$; it is again an $\mathcal{O}[B_n(F_p)^+]$-module, finite free as $\mathcal{O}$-module.

We now define various functors that together will allow us to study ordinary parts using completed cohomology. We write

$$\Gamma(N_n(\mathcal{O}_{F,p}), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+])$$

for the functor of $N_n(\mathcal{O}_{F,p})$-invariants. If $V \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$, then the action of an element $t \in T_n(F_p)^+$ on $v \in \Gamma(N_n(\mathcal{O}_{F,p}), V)$ is given by the formula

$$t \cdot v = \sum_{n \in N_n(\mathcal{O}_{F,p})/tN_n(\mathcal{O}_{F,p})t^{-1}} ntv; \tag{5.2.5}$$

cf. [Eme10a, §3], and note that the action of t is by the "double coset operator" $[N_n(\mathcal{O}_{F,p})tN_n(\mathcal{O}_{F,p})]$. We write

$$\Gamma(B_n(\mathcal{O}_{F,p})(b), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])$$

for the functor of $B_n(\mathcal{O}_{F,p})(b)$-invariants. The action of an element $t \in T_n(F_p)_b^+$ is given by the same formula (5.2.5).

If $c \geq b \geq 0$ are integers with $c \geq 1$, we define $\mathrm{Iw}_p(b,c) = \prod_{v \in S_p} \mathrm{Iw}_v(b,c) \subset \mathrm{GL}_n(F_p)$. We write

$$\Gamma(\mathrm{Iw}_p(b,c), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])$$

for the functor of $\mathrm{Iw}_p(b,c)$-invariants. If $V \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$, then the action of an element $t \in T_n(F_p)^+$ on $v \in \Gamma(\mathrm{Iw}_p(b,c), V)$ is given by the action of the Hecke operator $[\mathrm{Iw}_p(b,c)t\mathrm{Iw}_p(b,c)]$ (cf. Section 2.1.9).

For any $b \geq 0$, we consider the functors

$$\Gamma(T_n(\mathcal{O}_{F,p})(b), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])$$

and

$$\Gamma(T_n(\mathcal{O}_{F,p})(b), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b])$$

of $T_n(\mathcal{O}_{F,p})(b)$-invariants. Finally, we write

$$\mathrm{ord} : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)])$$

and

$$\mathrm{ord}_b : \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b])$$

respectively for the localization functors $- \otimes_{\mathcal{O}/\varpi^m[T_n(F_p)^+]} \mathcal{O}/\varpi^m[T_n(F_p)]$ and $- \otimes_{\mathcal{O}/\varpi^m[T_n(F_p)_b^+]} \mathcal{O}/\varpi^m[T_n(F_p)_b]$. (As the notation suggests, we will use localization to define "ordinary parts." The reader may object that the ordinary part usually denotes a direct summand, rather than a localization. At least in the context of $\mathcal{O}/\varpi^m[T_n(F_p)_b^+]$-modules that are finitely generated as $\mathcal{O}/\varpi^m$-modules, the two notions agree; cf. [Eme10b, Lem. 3.2.1] and also Proposition 5.2.15 below. We use localization here since it is easier to define without finiteness conditions.)

LEMMA 5.2.6. *The following diagram is commutative up to natural isomorphism*:

$$\begin{tikzcd}
\mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[T_n(F_p)^+]) \arrow[r, "\Gamma(T_n(\mathcal{O}_{F,p})(b),-)"] \arrow[d, "\mathrm{ord}"'] & \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+]) \arrow[d, "\mathrm{ord}_b"] \\
\mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[T_n(F_p)]) \arrow[r, "\Gamma(T_n(\mathcal{O}_{F,p})(b),-)"'] & \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b]).
\end{tikzcd}$$

Proof. Let $M \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+])$. There is a natural morphism

$$\mathrm{ord}_b\, \Gamma(T_n(\mathcal{O}_{F,p})(b), M) \to \Gamma(T_n(\mathcal{O}_{F,p})(b), \mathrm{ord}\, M),$$

or equivalently

$$M^{T_n(\mathcal{O}_{F,p})(b)} \otimes_{\mathcal{O}/\varpi^m[T_n(F_p)_b^+]} \mathcal{O}/\varpi^m[T_n(F_p)_b] \\ \to (M \otimes_{\mathcal{O}/\varpi^m[T_n(F_p)^+]} \mathcal{O}/\varpi^m[T_n(F_p)])^{T_n(\mathcal{O}_{F,p})(b)}.$$

We must show that it is an isomorphism. It is injective since $M^{T_n(\mathcal{O}_{F,p})(b)} \to M$ is injective and localization is exact. To show it is surjective, let $x \in M$, and suppose that $x \otimes 1 \in (M \otimes_{\mathcal{O}/\varpi^m[T_n(F_p)^+]} \mathcal{O}/\varpi^m[T_n(F_p)])^{T_n(\mathcal{O}_{F,p})(b)}$. We must show that there exists $n \geq 0$ such that $u_p^n x \in M^{T_n(\mathcal{O}_{F,p})(b)}$. Since M is smooth, there exists $c \geq b$ such that $x \in M^{T_n(\mathcal{O}_{F,p})(c)}$. On the other hand, our assumption on $x \otimes 1$ means that for any $t \in T_n(\mathcal{O}_{F,p})(b)$, there exists $n(t)$ such that $u_p^{n(t)}(t-1)x = 0$ in M. Choosing $n(t)$ to be as small as possible, we see that $n(t)$ depends only on the image of t in the (finite) quotient $T_n(\mathcal{O}_{F,p})(b)/T_n(\mathcal{O}_{F,p})(c)$. We can therefore take $n = \sup_t n(t)$. □

LEMMA 5.2.7.

(1) *Each functor* $\Gamma(N_n(\mathcal{O}_{F,p}),-)$, $\Gamma(B_n(\mathcal{O}_{F,p})(b),-)$, *and* $\Gamma(\mathrm{Iw}_p(b,c),-)$ *is left exact. For any* $b \geq 0$, *the functor* $\Gamma(N_n(\mathcal{O}_{F,p}),-)$ *sends injectives to* $\Gamma(T_n(\mathcal{O}_{F,p})(b),-)$*-acyclics.*

(2) *The functors* ord *and* ord_b *are exact and preserve injectives.*

Proof. It is immediate from the definitions that the three functors in the first part are left exact. We now show that the functor $\Gamma(N_n(\mathcal{O}_{F,p}),-)$ sends injectives to $\Gamma(T_n(\mathcal{O}_{F,p})(b),-)$-acyclics.

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p]) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+]) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+]) \arrow[d, "\gamma"'] \\
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B_n(\mathcal{O}_{F,p})(b)]) \arrow[r] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})(b)]) \arrow[r] & \mathrm{Mod}(\mathcal{O}/\varpi^m)
\end{tikzcd}$$

where the horizontal arrows are taking invariants and the vertical arrows are restriction to compact or open subgroups. By Lemma 5.2.4, the vertical arrows are exact and preserve injectives. We must show that if $\mathcal{I} \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$ is injective, then for each $i > 0$, $R^i\Gamma(T_n(\mathcal{O}_{F,p})(b), \Gamma(N_n(\mathcal{O}_{F,p}), \mathcal{I})) = 0$. Equivalently (using the formula for a composition of derived functors, [Wei94, Cor. 10.8.3]), we must show that

$$\gamma R^i\Gamma(T_n(\mathcal{O}_{F,p})(b), \Gamma(N_n(\mathcal{O}_{F,p}), \mathcal{I})) = R^i\Gamma(T_n(\mathcal{O}_{F,p})(b), \Gamma(N_n(\mathcal{O}_{F,p}), \alpha\mathcal{I})) = 0.$$

However, $\alpha\mathcal{I}$ is injective, so this follows from the fact that the functor

$$\Gamma(N_n(\mathcal{O}_{F,p}), -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B_n(\mathcal{O}_{F,p})(b)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})(b)])$$

preserves injectives (because it has an exact left adjoint, given by inflation). This proves the first part of the lemma.

We now prove the second part of the lemma. Both ord and ord_b are exact because localization is an exact functor. Since localization preserves injectives in the case of a Noetherian base ring, ord_b preserves injectives. To show that ord preserves injectives, we go back to the definitions. Let $\mathcal{I}$ be an injective object of $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+])$, let $M \hookrightarrow N$ be an inclusion in $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)])$, and let $\alpha : M \to \mathrm{ord}(\mathcal{I})$ be a morphism. We must show that α extends to N.

For any $b \geq 0$, passing to $T_n(\mathcal{O}_{F,p})(b)$-fixed vectors gives a morphism (cf. Lemma 5.2.6)

$$\alpha(b) : M^{T_n(\mathcal{O}_{F,p})(b)} \to \mathrm{ord}(\mathcal{I})^{T_n(\mathcal{O}_{F,p})(b)} \cong \mathrm{ord}_b(\mathcal{I}^{T_n(\mathcal{O}_{F,p})(b)}).$$

The object $\mathcal{I}^{T_n(\mathcal{O}_{F,p})(b)} \in \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b^+])$ is injective, showing that we can extend $\alpha(b)$ to a morphism $\alpha(b)' : N^{T_n(\mathcal{O}_{F,p})(b)} \to \mathrm{ord}(\mathcal{I})^{T_n(\mathcal{O}_{F,p})(b)}$. Zorn's lemma implies that there exists a maximal extension $\alpha' : L_{\max} \to \mathrm{ord}(\mathcal{I})$ of α. The preceding argument shows that we can extend the map induced by α' on $T_n(\mathcal{O}_{F,p})(b)$-invariants from $L_{\max}^{T_n(\mathcal{O}_{F,p})(b)}$ to $N^{T_n(\mathcal{O}_{F,p})(b)}$. It follows that we can extend α' to $L_{\max} + N^{T_n(\mathcal{O}_{F,p})(b)}$. By maximality, and since $N = \cup_{b\geq 0} N^{T_n(\mathcal{O}_{F,p})(b)}$, we have $L_{\max} = N$, as desired. □

Lemma 5.2.8. *For any $c \geq b \geq 0$ with $c \geq 1$, there is a natural isomorphism*

$$\mathrm{ord}_b \circ \Gamma(\mathrm{Iw}_p(b,c), -) \cong \mathrm{ord}_b \circ \Gamma(B_n(\mathcal{O}_{F,p})(b), -)$$

of functors

$$\mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[\Delta_p]) \to \mathrm{Mod}(\mathcal{O}/\varpi^m[T_n(F_p)_b]).$$

Proof. We first show that for any $V \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$, the natural inclusion $\Gamma(\mathrm{Iw}_p(b,c), V) \subset \Gamma(B_n(\mathcal{O}_{F,p})(b), V)$ is a morphism of $\mathcal{O}/\varpi^m[T_n(F_p)_b^+]$-modules. A given element $t \in T_n(F_p)_b^+$ acts on the source via the Hecke operator

$[\mathrm{Iw}_p(b,c)t\mathrm{Iw}_p(b,c)]$ and on the target by the formula (5.2.5). We see that we must show that the map

$$N(\mathcal{O}_{F,p})/tN(\mathcal{O}_{F,p})t^{-1} \to \mathrm{Iw}_p(b,c)/(\mathrm{Iw}_p(b,c) \cap t\mathrm{Iw}_p(b,c)t^{-1})$$

is bijective. This is true, because $\mathrm{Iw}_p(b,c)$ admits an Iwahori decomposition with respect to B_n (cf. Section 2.1.9).

The exactness of ord_b implies that for any $V \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$, there is an inclusion $\mathrm{ord}_b\,\Gamma(\mathrm{Iw}_p(b,c),V) \subset \mathrm{ord}_b\,\Gamma(B_n(\mathcal{O}_{F,p})(b),V)$. We must show that this is an equality.

We have $\mathcal{O}/\varpi^m[T_n(F_p)_b^+][u_p]^{-1} = \mathcal{O}/\varpi^m[T_n(F_p)_b]$. Consequently, the lemma will follow if we can show that for any $v \in \Gamma(B_n(\mathcal{O}_{F,p})(b),V)$, there exists $n \geq 0$ such that $u_p^n \cdot v \in \Gamma(\mathrm{Iw}_p(b,c),V) = V^{\mathrm{Iw}_p(b,c)}$.

Since V is smooth, there exists $c' > c$ such that $v \in V^{\mathrm{Iw}_p(b,c')}$. By induction, it is enough to show that $U_p \cdot v \in V^{\mathrm{Iw}_p(b,c'-1)}$. The definition of the Hecke operator U_p shows that this will follow if the double coset $\mathrm{Iw}_p(b,c')u_p\mathrm{Iw}_p(b,c')$ is invariant under left multiplication by the group $\mathrm{Iw}_p(b,c'-1)$. This is true, as proved in, e.g., [Ger19, Lem. 2.19]. □

Lemma 5.2.9. *Let $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[\Delta_p])$ be a bounded below complex. Then for any $c \geq b \geq 0$, $c \geq 1$, there is a natural isomorphism*

$$R\Gamma(T_n(\mathcal{O}_{F,p})(b), \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}),\pi)) \cong \mathrm{ord}_b\, R\Gamma(\mathrm{Iw}_p(b,c),\pi)$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(F_p)_b])$.

Proof. We will use [Wei94, Cor. 10.8.3] (composition formula for derived functors) repeatedly. Since ord preserves injectives, this implies the existence of a natural isomorphism

$$\begin{aligned} R\Gamma(T_n(\mathcal{O}_{F,p})(b),-) \circ \mathrm{ord} &\cong R(\Gamma(T_n(\mathcal{O}_{F,p})(b), \mathrm{ord}(-)) \\ &\cong R(\mathrm{ord}_b \circ \Gamma(T_n(\mathcal{O}_{F,p})(b),-)) \\ &\cong \mathrm{ord}_b\, R\Gamma(T_n(\mathcal{O}_{F,p})(b),-). \end{aligned}$$

It follows that for π as in the statement of the lemma, there is a natural isomorphism

$$\begin{aligned} &R\Gamma(T_n(\mathcal{O}_{F,p})(b), \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}),\pi)) \\ \cong\ &\mathrm{ord}_b\, R\Gamma(T_n(\mathcal{O}_{F,p})(b), R\Gamma(N_n(\mathcal{O}_{F,p}),\pi)). \end{aligned}$$

Using the first part of Lemma 5.2.7, we see that there is a natural isomorphism

$$R\Gamma(T_n(\mathcal{O}_{F,p})(b), R\Gamma(N_n(\mathcal{O}_{F,p}),\pi)) \cong R\Gamma(B_n(\mathcal{O}_{F,p}(b),\pi)).$$

Lemma 5.2.8 implies the existence of a natural isomorphism

$$\begin{aligned}\operatorname{ord}_b R\Gamma(B_n(\mathcal{O}_{F,p})(b),\pi)) &\cong R(\operatorname{ord}_b \Gamma(B_n(\mathcal{O}_{F,p})(b),-))(\pi)\\ &\cong R(\operatorname{ord}_b \Gamma(\mathrm{Iw}_p(b,c),-))(\pi)\\ &\cong \operatorname{ord}_b R\Gamma(\mathrm{Iw}_p(b,c),\pi).\end{aligned}$$

This concludes the proof. □

5.2.10. *The ordinary part of completed cohomology.* We now apply the formalism developed in the previous section to the cohomology groups of the spaces X_K. If $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ is a good subgroup, then there are functors

$$\Gamma_{K^p,\mathrm{sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[G^\infty]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(F_p^+)])$$

and

$$\Gamma_{K^p,\mathrm{sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[G^{p,\infty}\times\Delta_p]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$$

that send a module M to $\Gamma(K^p,M)^{\mathrm{sm}}$. If $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, then we define the weight λ completed cohomology

$$\pi(K^p,\lambda,m) = R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G,\mathcal{V}_\lambda/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p]).$$

If $K^S = \prod_{v\notin S}\mathrm{GL}_n(\mathcal{O}_{F,v})$, then $\pi(K^p,\lambda,m)$ comes equipped with a homomorphism

$$\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])}(\pi(K^p,\lambda,m)) \tag{5.2.11}$$

and, if $K_p \subset \Delta_p$, a canonical $\mathbf{T}^S$-equivariant isomorphism

$$R\Gamma(K_p,\pi(K^p,\lambda,m)) \cong R\Gamma(X_K,\mathcal{V}_\lambda/\varpi^m) \tag{5.2.12}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m)$. We define

$$\pi(K^p,m) = R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G,\mathcal{O}/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(F_p^+)]);$$

this complex comes equipped with a homomorphism

$$\mathbf{T}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(F_p^+)])}(\pi(K^p,m)), \tag{5.2.13}$$

which recovers (5.2.11) in the case $\lambda = 0$ after applying the forgetful functor to $\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\Delta_p])$. We write $\mathbf{T}^S(K^p,m)$ for the image of (5.2.13).

Lemma 5.2.14. *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup. Then $\mathbf{T}^S(K^p,m)$ is a semi-local ring, complete with respect to the J-adic topology defined by its Jacobson radical J. For each maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K^p,m)$, there is a unique idempotent $e_\mathfrak{m} \in \mathbf{T}^S(K^p,m)$ with the property $e_\mathfrak{m} H^*(\pi(K^p,m)) = H^*(\pi(K^p,m))_\mathfrak{m}$.*

Proof. See [GN22, Lem. 2.1.14]. □

One important consequence of Lemma 5.2.14 is that the localization

$$\pi(K^p, m)_{\mathfrak{m}} \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(F_p^+)])$$

is defined.

We define the ordinary part of completed cohomology

$$\pi^{\mathrm{ord}}(K^p, \lambda, m) = \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), \pi(K^p, \lambda, m)) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)]).$$

(If $\lambda = 0$, then we write simply $\pi^{\mathrm{ord}}(K^p, m)$.) Its relation to the complex $R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}$ defined in Section 5.1 is the expected one:

Proposition 5.2.15. *Let $K \subset G^\infty$ be a good subgroup with $K_v = \mathrm{Iw}_v$ for each $v|p$ and $K^S = \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_{F_v})$. Let $c \geq b \geq 0$ be integers with $c \geq 1$. Then for any $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, there is a $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism*

$$R\Gamma(T_n(\mathcal{O}_{F,p})(b), \pi^{\mathrm{ord}}(K^p, \lambda, m)) \cong R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[K(0,c)/K(b,c)])$. (Recall that we may identify $K(0,c)/K(b,c)$ with $T_n(\mathcal{O}_{F,p})_b$.)

Proof. We compute. We have a $\mathbf{T}^S$-equivariant isomorphism

$$\begin{aligned} &R\Gamma(T_n(\mathcal{O}_{F,p})(b), \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), \pi(K^p, \lambda, m))) \\ \cong\ &\mathrm{ord}_b\, R\Gamma(\mathrm{Iw}_p(b,c), \pi(K^p, \lambda, m)) \end{aligned}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(F_p)_b])$. We have a morphism

$$\begin{aligned} R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}} \to &R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m) \\ \to &\mathrm{ord}_b\, R\Gamma(\mathrm{Iw}_p(b,c), \pi(K^p, \lambda, m)) \end{aligned}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})_b])$. Note that we identify $R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m)$ and $R\Gamma(\mathrm{Iw}_p(b,c), \pi(K^p, \lambda, m))$ in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})_b])$. To complete the proof, we must show that our morphism induces an isomorphism on cohomology groups. This, in turn, reduces us to the problem of showing that if M is an $\mathcal{O}/\varpi^m[U]$-module, finite as $\mathcal{O}/\varpi^m$-module, and M^{ord} is the maximal direct summand of M on which U acts invertibly, then the natural map $M^{\mathrm{ord}} \to M \to M \otimes_{\mathcal{O}/\varpi^m[U]} \mathcal{O}/\varpi^m[U, U^{-1}]$ is an isomorphism of $\mathcal{O}/\varpi^m$-modules. This is true (cf. [Eme10b, Lem. 3.2.1]). □

Corollary 5.2.16 (Independence of level). *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup with $K_v = \mathrm{Iw}_v$ for each $v|p$ and $K^S = \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_{F_v})$. Let $c \geq b \geq 0$ be integers with $c \geq 1$. Then for any $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, the natural morphism*

$$\begin{aligned} &R\Gamma_{K(0,\max(1,b))/K(b,\max(1,b))}(X_{K(b,\max(1,b))}, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}} \\ &\qquad \to R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}} \end{aligned}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F,p})_b])$ is an isomorphism.

PROPOSITION 5.2.17. *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup with $K^S = \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_{F,v})$. Then there are the following $\mathbf{T}^S$-equivariant isomorphisms in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(F_p)])$:*

$$\begin{aligned}\pi^{\mathrm{ord}}(K^p, \lambda, m) &\cong \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), R\Gamma_{K^p,sm} R\Gamma(\mathfrak{X}_G, \mathcal{O}(w_0^G\lambda)/\varpi^m)) \\ &\cong \pi^{\mathrm{ord}}(K^p, m) \otimes_{\mathcal{O}} \mathcal{O}(w_0^G\lambda).\end{aligned}$$

Proof. By definition, we have

$$\pi^{\mathrm{ord}}(K^p, \lambda, m) = \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G, \mathcal{V}_\lambda/\varpi^m)).$$

This depends only on the image of $R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G, \mathcal{V}_\lambda/\varpi^m)$ in the category $\mathbf{D}(\mathcal{O}/\varpi^m[B_n(F_p)^+]$. In this category, the $B_n(F_p)^+$-equivariant morphism $\mathcal{V}_\lambda \to \mathcal{O}(w_0^G\lambda)$ induces a morphism

$$\pi^{\mathrm{ord}}(K^p, \lambda, m) \to \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G, \mathcal{O}(w_0^G\lambda)/\varpi^m)).$$

To show that this is an isomorphism, we just need to check that

$$\mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G, \mathcal{K}_\lambda/\varpi^m)) = 0,$$

where we recall that $\mathcal{K}_\lambda = \ker(\mathcal{V}_\lambda \to \mathcal{O}(w_0^G\lambda))$. This follows from the observation that for sufficiently large $N \geq 1$, we have $u_p^N \mathcal{K}_\lambda/\varpi^m = 0$ (cf. the proof of [Ger19, Prop. 2.22]). The existence of the second isomorphism follows from the fact that $N_n(\mathcal{O}_{F,p})$ acts trivially on $\mathcal{O}(w_0^G\lambda)$. □

COROLLARY 5.2.18 (Independence of weight). *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup with $K_v = \mathrm{Iw}_v$ for each $v|p$ and $K^S = \prod_{v \notin S} \mathrm{GL}_n(\mathcal{O}_{F_v})$. Let $c \geq b \geq 0$ be integers with $c \geq 1$. Then for any $\lambda, \lambda' \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ such that $\mathcal{O}(w_0^G\lambda)/\varpi^m \cong \mathcal{O}(w_0^G\lambda')/\varpi^m$ as $\mathcal{O}/\varpi^m[T_n(\mathcal{O}_{F_p})(b)]$-modules, there is an $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism*

$$\begin{aligned}&R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}} \\ &\qquad \cong R\Gamma_{K(0,c)/K(b,c)}(X_{K(b,c)}, \mathcal{V}_{\lambda'}/\varpi^m)^{\mathrm{ord}} \otimes_{\mathcal{O}} \mathcal{O}(w_0^G\lambda) \otimes_{\mathcal{O}} \mathcal{O}((w_0^G\lambda')^{-1})\end{aligned}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m[T_n(F_p)_b])$.

Proof. Combine Propositions 5.2.15 and 5.2.17. □

5.2.19. *Results for the group $\widetilde{G}$.* We recall that by assumption each p-adic place of F^+ splits in F, and that we have fixed for each place $\overline{v} \in \overline{S}_p$ of F^+ a lift $\widetilde{v} \in \widetilde{S}_p$ to a place of F. These choices determine an isomorphism

$$\prod_{\overline{v} \in \overline{S}_p} \iota_{\widetilde{v}} : \widetilde{G}(F_p^+) \cong \prod_{\overline{v} \in \overline{S}_p} \mathrm{GL}_{2n}(F_{\widetilde{v}}).$$

We have also fixed a maximal torus and Borel subgroup $T \subset B \subset \widetilde{G}$ that correspond under this isomorphism to $T_{2n} \subset B_{2n} \subset \mathrm{GL}_{2n}$. The theory of Section 5.2.1 can thus be easily generalized to study the completed cohomology

of $\widetilde{G}$. Since we will need to do this only in passing on our way to analyzing the complexes $\pi^{\mathrm{ord}}(K^p,\lambda,m)$, we just give some brief indications. We will use some of the Hecke operators and open compact subgroups defined in Section 2.2.5. We define

$$\widetilde{\mathbf{T}}^{S,\mathrm{ord}} = \frac{\widetilde{\mathbf{T}}^S \otimes_{\mathcal{O}} \mathcal{O}[\![T(\mathcal{O}_{F^+,p})]\!][\{\widetilde{U}_{v,1},\dots,\widetilde{U}_{v,2n},\widetilde{U}_{v,2n}^{-1}\}_{v\in S_p}]}{(\{\widetilde{U}_{v^c,i}-\widetilde{U}_{v,2n-i}\widetilde{U}_{v,2n}^{-1}\}_{\substack{v\in S_p\\ i=1,\dots,2n}})}.$$

We define $\widetilde{U}_v=\widetilde{U}_{v,1}\widetilde{U}_{v,2}\cdots\widetilde{U}_{v,n-1}$ and $\widetilde{U}_p=\prod_{v\in S_p}\widetilde{U}_v\in\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$. If $\widetilde{K}\subset\widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup with $\widetilde{K}_{\overline{v}}=\widetilde{\mathrm{Iw}}_{\overline{v}}$ for each $\overline{v}\in\overline{S}_p$, and $c\geq b\geq 0$ are integers with $c\geq 1$, then we define $\widetilde{K}(b,c)$ to be the good subgroup with $\widetilde{K}(b,c)_{\overline{v}}=\widetilde{K}_{\overline{v}}$ if $\overline{v}\notin\overline{S}_p$ and $\widetilde{K}(b,c)_{\overline{v}}=\widetilde{\mathrm{Iw}}_{\overline{v}}(b,c)$ otherwise. If $\widetilde{\lambda}\in(\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, there is a well-defined direct summand $R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}})^{\mathrm{ord}}$ of $R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}})$ on which $\widetilde{U}_p$ acts invertibly, and we define

$$\widetilde{\mathbf{T}}(\widetilde{K}(b,c),\widetilde{\lambda})^{\mathrm{ord}}=\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}})^{\mathrm{ord}})$$

(i.e., the image of the $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$ in the endomorphism algebra in

$$\mathbf{D}(\mathcal{O}[\widetilde{K}(0,c)/\widetilde{K}(b,c)])$$

of this direct summand).

To compare Hida theory for $\widetilde{G}$ and for GL_n, we recall that the Levi subgroup G of $\widetilde{G}$ is identified with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}}\mathrm{GL}_n$ which, in particular, identifies T with $\mathrm{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}}T_n$. We extend the homomorphism $\mathcal{S}:\widetilde{\mathbf{T}}^S\to\mathbf{T}^S$ (defined by equation (2.1.8)) to a homomorphism $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}\to\mathbf{T}^{S,\mathrm{ord}}$, also denoted $\mathcal{S}$, using the identification

$$\mathcal{O}[\![T(\mathcal{O}_{F^+,p})]\!]\cong\mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!],$$

and by sending each operator $\widetilde{U}_{v,i}$ to the operator $U_{v^c,n-i}U_{v^c,n}^{-1}$ (if $1\leq i\leq n$) and $U_{v^c,n}^{-1}U_{v,i-n}$ (if $n+1\leq i\leq 2n$). Note that these respective Hecke operators are double coset operators for elements of $T(F_p^+)$ and $T_n(F_p)$ that match under our identification $T(F_p^+)=T_n(F_p)$.

We write $T(F_p^+)^+\subset T(F_p^+)$ for the submonoid of elements that are contracting on $N(\mathcal{O}_{F^+,p})$. Under our identification $T(F_p^+)=T_n(F_p)$, we have $T(F_p^+)^+\subset T_n(F_p)^+$ (and the inclusion is strict provided $n\geq 2$). Let $\widetilde{\mathrm{Iw}}_p(b,c)=\prod_{\overline{v}\in\overline{S}_p}\widetilde{\mathrm{Iw}}_{\overline{v}}(b,c)$. We recall (Section 2.2.5) that we have defined $\widetilde{\Delta}_p=\widetilde{\mathrm{Iw}}_p(b,c)T(F_p^+)^+\widetilde{\mathrm{Iw}}_p(b,c)$, an open submonoid of $\widetilde{G}(F_p^+)$, and that we have defined an action $\cdot_p$ of this monoid on $\mathcal{V}_{\widetilde{\lambda}}$. If $b\geq 0$ is an integer, then we define $T(\mathcal{O}_{F^+,p})(b)=T_n(\mathcal{O}_{F,p})(b)$ and write $B(\mathcal{O}_{F^+,p})(b)$ for the pre-image in $B(\mathcal{O}_{F^+,p})$ of $T(\mathcal{O}_{F^+,p})(b)$ under the natural projection to T. We define $B(F_p^+)^+=N(\mathcal{O}_{F^+,p})\cdot T(F_p^+)^+$.

Fix $m \geq 1$. If $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ is a good subgroup with $\widetilde{K}_{\overline{v}} = \widetilde{\mathrm{Iw}}_{\overline{v}}$ for each $\overline{v} \in \overline{S}_p$ and $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, then we define

$$\widetilde{\pi}(\widetilde{K}^p, \widetilde{\lambda}, m) = R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_p]). \tag{5.2.20}$$

If $\widetilde{K}^S = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^S)$, then this complex comes equipped with a homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_p])}(\widetilde{\pi}(\widetilde{K}^p, \lambda, m)). \tag{5.2.21}$$

We define $\widetilde{\pi}(\widetilde{K}^p, m) = R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_{\widetilde{G}}, \mathcal{O}/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])$; this complex comes equipped with a homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])}(\widetilde{\pi}(\widetilde{K}^p, m)) \tag{5.2.22}$$

that recovers (5.2.21) after applying the forgetful functor induced by the inclusion $\widetilde{\Delta}_p \subset \widetilde{G}(F_p^+)$. We also need the completed boundary cohomology. We thus define

$$\widetilde{\pi}_\partial(\widetilde{K}^p, \widetilde{\lambda}, m) = R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_p]). \tag{5.2.23}$$

This complex comes equipped with a homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{\Delta}_p])}(\widetilde{\pi}_\partial(\widetilde{K}^p, \widetilde{\lambda}, m)). \tag{5.2.24}$$

We define $\widetilde{\pi}_\partial(\widetilde{K}^p, m) = R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{O}/\varpi^m) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])$; this complex comes equipped with a homomorphism

$$\widetilde{\mathbf{T}}^S \to \mathrm{End}_{\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])}(\widetilde{\pi}_\partial(\widetilde{K}^p, m)). \tag{5.2.25}$$

If $c \geq b \geq 0$ are integers with $c \geq 1$, then there are canonical $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-equivariant isomorphisms

$$R\Gamma(\widetilde{\mathrm{Iw}}_p(b,c), \widetilde{\pi}(\widetilde{K}^p, \widetilde{\lambda}, m)) \cong R\Gamma(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \tag{5.2.26}$$

and

$$R\Gamma(\widetilde{\mathrm{Iw}}_p(b,c), \widetilde{\pi}_\partial(\widetilde{K}^p, \widetilde{\lambda}, m)) \cong R\Gamma(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m) \tag{5.2.27}$$

in $\mathbf{D}(\mathcal{O}/\varpi^m)$. We define the ordinary part of completed and completed boundary cohomology:

$$\widetilde{\pi}^{\mathrm{ord}}(\widetilde{K}^p, \widetilde{\lambda}, m) = \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), \widetilde{\pi}(\widetilde{K}^p, \widetilde{\lambda}, m)) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T(F_p^+)])$$

and

$$\widetilde{\pi}_\partial^{\mathrm{ord}}(\widetilde{K}^p, \widetilde{\lambda}, m) = \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), \widetilde{\pi}_\partial(\widetilde{K}^p, \widetilde{\lambda}, m)) \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T(F_p^+)]).$$

If $\widetilde{\lambda} = 0$, then we omit it from the notation. We have the following result, which contains the analogues of some of the results in Section 5.2.10 for the group $\widetilde{G}$. The proofs are entirely similar, and so are omitted.

PROPOSITION 5.2.28. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup with $\widetilde{K}_{\overline{v}} = \widetilde{\mathrm{Iw}}_{\overline{v}}$ for each $\overline{S}_p$ and $\widetilde{K}^S = \widetilde{G}(\widehat{\mathcal{O}}_{F^+}^S)$. Let $c \geq b \geq 0$ be integers with $c \geq 1$. Then for any $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, there are $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-equivariant isomorphisms*

$$\begin{aligned} R\Gamma(T(\mathcal{O}_{F^+,p})(b), \widetilde{\pi}^{\mathrm{ord}}(\widetilde{K}^p, \widetilde{\lambda}, m)) &\cong R\Gamma(T(\mathcal{O}_{F^+,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} \widetilde{\pi}^{\mathrm{ord}}(\widetilde{K}^p, m)) \\ &\cong R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)^{\mathrm{ord}} \end{aligned}$$

and

$$\begin{aligned} R\Gamma(T(\mathcal{O}_{F^+,p})(b), \widetilde{\pi}_\partial^{\mathrm{ord}}(\widetilde{K}^p, \widetilde{\lambda}, m)) &\cong R\Gamma(T(\mathcal{O}_{F^+,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} \widetilde{\pi}_\partial^{\mathrm{ord}}(\widetilde{K}^p, m)) \\ &\cong R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)^{\mathrm{ord}} \end{aligned}$$

in $\mathbf{D}_{sm}(\mathcal{O}/\varpi^m[\widetilde{K}(0,c)/\widetilde{K}(b,c)])$.

5.3. *The ordinary part of a parabolic induction.* In this subsection, we compute the ordinary part (in the sense defined above) of a parabolic induction from G to $\widetilde{G}$, with the aim of understanding the ordinary part of the cohomology of the boundary of the Borel–Serre compactification of $\widetilde{X}_{\widetilde{K}}$ in terms of the ordinary part of the cohomology of X_K. Our calculations here are purely local; the global application will be carried out in Section 5.4 below.

Let $\overline{v}$ be a p-adic place of F^+. In this section, we write ${}^rW_{\overline{v}} = W(\widetilde{G}_{F_{\overline{v}}^+}, T_{F_{\overline{v}}^+})$, ${}^rW_{P,\overline{v}} = W(G_{F_{\overline{v}}^+}, T_{F_{\overline{v}}^+})$, and ${}^rW_{\overline{v}}^P \subset {}^rW_{\overline{v}}$ for the set of representatives for the quotient ${}^rW_{P,\overline{v}} \backslash {}^rW_{\overline{v}}$ that is associated to the choice of Borel subgroup $B_{F_{\overline{v}}^+}$. We define ${}^rW = \prod_{\overline{v} \in \overline{S}_p} {}^rW_{\overline{v}}$, ${}^rW_P = \prod_{\overline{v} \in \overline{S}_p} {}^rW_{P,\overline{v}}$, and ${}^rW^P = \prod_{\overline{v} \in \overline{S}_p} {}^rW_{\overline{v}}^P$. Thus ${}^rW$ is the relative Weyl group of the group $(\mathrm{Res}_{F^+/\mathbf{Q}} G)_{\mathbf{Q}_p}$. Note that in Section 4 we made use of the absolute Weyl group W; there is a natural inclusion ${}^rW \subset W$, by which ${}^rW$ acts on, e.g., the group $X^*((\mathrm{Res}_{F^+/\mathbf{Q}} T)_E)$. We write $l_r(w)$ for the length of an element $w \in {}^rW$ as an element of the relative Weyl group, and $l(w)$ for its length as an element of the absolute Weyl group. Thus w_0^P, the longest element of W^P (equivalently, of ${}^rW^P$), has $l_r(w_0^P) = |S_p|n^2$ and $l(w_0^P) = [F^+ : \mathbf{Q}]n^2$. As in Section 4, we write $\rho \in X^*((\mathrm{Res}_{F^+/\mathbf{Q}} T)_E)$ for the half-sum of the $(\mathrm{Res}_{F^+/\mathbf{Q}} B)_E$-positive roots.

We recall (cf. Section 2.2.1) that P denotes the Siegel parabolic of $\widetilde{G}$, which has unipotent radical U, while the Borel subgroup B has unipotent radical N. We identify G with $\mathrm{Res}_{F/F^+} \mathrm{GL}_n$; this group has standard Borel $\mathrm{Res}_{F/F^+} B_n$ with unipotent radical $\mathrm{Res}_{F/F^+} N_n$. The parabolic induction functor

$$\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])$$

is exact and preserves injectives. (It is right adjoint to the exact restriction functor $\mathrm{Res}_{P(F_p^+)}^{\widetilde{G}(F_p^+)}$.) We now define several more functors that are related to parabolic induction.

We identify ${}^rW$ with the subgroup of permutation matrices of $\widetilde{G}(F_p^+) = \prod_{\widetilde{v}\in\widetilde{S}_p} \mathrm{GL}_{2n}(F_{\widetilde{v}})$. We recall (cf. [BT65, Cor. 5.20]) that there is a (set-theoretic) decomposition

$$\widetilde{G}(F_p^+) = \bigsqcup_{w\in {}^rW^P} P(F_p^+)wB(F_p^+).$$

If $w\in {}^rW^P$, then we define $S_w = P(F_p^+)wN(F_p^+)$ and $S_w^\circ = P(F_p^+)wN(\mathcal{O}_{F^+,p}) \subset S_w$. The closure $\overline{S}_w$ of S_w in $\widetilde{G}(F_p^+)$ can be described in terms of the Bruhat ordering of ${}^rW^P$:

$$\overline{S}_w = \bigsqcup_{w'\leq w} S_{w'}.$$

Note that if $w' < w$, then $l_r(w') < l_r(w)$. For an integer $i \geq 0$, we define

$$\widetilde{G}_{\geq i} = \bigsqcup_{\substack{w\in {}^rW^P \\ l_r(w)\geq i}} S_w.$$

It is an open subset of $\widetilde{G}(F_p^+)$ that is invariant under left multiplication by $P(F_p^+)$ and right multiplication by $B(F_p^+)$.

If $i \geq 0$, then we define a functor

$$I_{\geq i} : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(F_p^+)])$$

by sending π to

$$I_{\geq i}(\pi) = \{f : \widetilde{G}_{\geq i} \to \pi \mid f \text{ locally constant, of compact support modulo } P(F_p^+), \forall p \in P(F_p^+), g \in \widetilde{G}_{\geq i}, f(pg) = pf(g)\},$$

where $B(F_p^+)$ acts by right translation. If $w \in {}^rW^P$, then we define a functor

$$I_w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(F_p^+)])$$

by sending π to

$$I_w(\pi) = \{f : S_w \to \pi \mid f \text{ locally constant, of compact support modulo } P(F_p^+) \quad \forall p \in P(F_p^+), g \in S_w, f(pg) = pf(g)\},$$

where again $B(F_p^+)$ acts by right translation. We define a functor

$$I_w^\circ : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(F_p^+)^+])$$

by defining $I_w^\circ(\pi) \subset I_w(\pi)$ to be the set of functions with support in S_w°.

Proposition 5.3.1.

(1) $I_{\geq 0} = \mathrm{Res}_{B(F_p^+)}^{\widetilde{G}(F_p^+)} \circ \mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)}$.

(2) *Each functor $I_{\geq i}$, I_w and I_w° is exact.*

(3) *For each integer* $i \geq 0$ *and each* $\pi \in \mathrm{Mod}_{sm}(\mathcal{O}/\varpi^m[P(F_p^+)])$, *there is a functorial exact sequence*

$$0 \to I_{\geq i+1}(\pi) \to I_{\geq i}(\pi) \to \oplus_{\substack{w \in {}^rW^P \\ l_r(w)=i}} I_w(\pi) \to 0.$$

Proof. The first part is the definition of induction. For the second part, denote by I any of the functors appearing in the statement. To see the exactness of I, choose a continuous section to the map $\widetilde{G}(F_p^+) \to P(F_p^+)\backslash\widetilde{G}(F_p^+)$. (The existence of such a section is explained in [Hau16, §2.1].) This allows us to functorially identify $I(\pi)$ with the space of locally constant and compactly supported functions from a subset $C \subset P(F_p^+)\backslash\widetilde{G}(F_p^+)$ to π. The formation of locally constant and compactly supported functions is exact. The third part is proved in the same way as [Hau16, Prop. 2.1.3]. □

It follows that for any $\pi \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$, there is a functorial distinguished triangle

$$I_{\geq i+1}(\pi) \to I_{\geq i}(\pi) \to \oplus_{\substack{w \in {}^rW^P \\ l_r(w)=i}} I_w(\pi) \to I_{\geq i+1}(\pi)[1] \tag{5.3.2}$$

in $\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(F_p^+)])$.

Lemma 5.3.3. *Let* $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[P(F_p^+)])$ *be a bounded below complex, and fix an integer* $b \geq 0$. *Let* $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$. *Then for any* $i \geq 0$ *and any* $j \in \mathbf{Z}$, *the sequence*

$$\begin{aligned} 0 &\to R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{\geq i+1}(\pi)) \\ &\to R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{\geq i}(\pi)) \\ &\to R^j\Gamma(B(\mathcal{O}_{F,p})(b), \oplus_{\substack{w \in {}^rW^P \\ l_r(w)=i}} \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_w(\pi)) \to 0 \end{aligned}$$

in $\mathrm{Mod}(\mathcal{O}/\varpi^m[T(F_p^+)_b^+])$ *associated to* (5.3.2) *is exact.*

Proof. It suffices to show exactness after applying the exact forgetful functor to $\mathrm{Mod}(\mathcal{O}/\varpi^m)$. We consider decompositions $\widetilde{G}_{\geq i} = U_1 \sqcup U_2$ where U_1, U_2 are open sets that are invariant under left multiplication by $P(F_p^+)$ and right multiplication by $B(\mathcal{O}_{F^+,p})$, and such that $U_1 \subset \widetilde{G}_{\geq i+1}$. Any such decomposition determines a functorial decomposition $I_{\geq i}(\pi) = I_{U_1}(\pi) \oplus I_{U_2}(\pi)$, where I_{U_1} denotes functions with support in U_1, and similarly for U_2. This decomposition exists in the category $\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(\mathcal{O}_{F^+,p})])$. We see, in particular, that for any bounded below complex $\pi \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$, the associated morphism

$$R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{U_1}(\pi)) \to R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{\geq i}(\pi))$$

in $\mathrm{Mod}(\mathcal{O}/\varpi^m)$ is injective. Since $I_{\geq i+1}$ is the filtered direct limit of the I_{U_1} (which can be proven by following the same technique as in the proof of [Hau16, Prop. 2.2.3]), it follows that the morphism

$$R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{\geq i+1}(\pi)) \to R^j\Gamma(B(\mathcal{O}_{F,p})(b), \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} I_{\geq i}(\pi))$$

is injective. Since this applies for any $j \in \mathbf{Z}$, the exactness of the long exact sequence in cohomology attached to the distinguished triangle (5.3.2) implies that the sequence in the statement of the lemma is indeed a short exact sequence. □

Lemma 5.3.4. *Let $w \in {}^rW^P$. Then*

(1) *I_w° takes injectives to $\Gamma(N(\mathcal{O}_{F^+,p}), -)$-acyclics.*
(2) *Let $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[P(F_p^+)])$ be a bounded below complex. Then there is a natural isomorphism*

$$\mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(\pi)) \cong \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), I_w(\pi)).$$

Proof. For the first part, let $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$, and fix an $\mathcal{O}/\varpi^m$-embedding $\pi \hookrightarrow I$, where I is an injective $\mathcal{O}/\varpi^m$-module. Then there is an embedding $\pi \hookrightarrow \mathrm{Ind}_1^{P(F_p^+)} I$ of $\mathcal{O}/\varpi^m[P(F_p^+)]$-modules. We will show that $I_w^\circ(\mathrm{Ind}_1^{P(F_p^+)} I)$ is an injective smooth $\mathcal{O}/\varpi^m[N(\mathcal{O}_{F^+,p})]$-module. By [Eme10b, Lem. 2.1.10], this will show the first part of the lemma.

Let $\mathcal{C}^\infty(P(F_p^+)wN(\mathcal{O}_{F^+,p}), I)$ denote the set of locally constant functions $F : P(F_p^+)wN(\mathcal{O}_{F^+,p}) \to I$. It is an injective smooth $\mathcal{O}/\varpi^m[N(\mathcal{O}_{F^+,p})]$-module when $N(\mathcal{O}_{F^+,p})$ acts by right translation. There is a natural isomorphism

$$I_w^\circ(\mathrm{Ind}_1^{P(F_p^+)} I) \cong \mathcal{C}^\infty(P(F_p^+)wN(\mathcal{O}_{F^+,p}), I),$$

which sends a function $f : P(F_p^+)wN(\mathcal{O}_{F_p^+}) \to \mathrm{Ind}_1^{P(F_p^+)} I$ to the function $F : P(F_p^+)wN(\mathcal{O}_{F_p^+}) \to I$ given by the formula $F(x) = f(x)(1)$. This proves the first part of the lemma.

For the second part, we note that we may define an exact functor

$$J_w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[B(F_p^+)^+])$$

by the formula $J_w(\pi) = I_w(\pi)/I_w^\circ(\pi)$. Then for a bounded below complex $\pi \in \mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$, there is a natural distinguished triangle

$$\begin{aligned}\mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(\pi)) &\to \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), I_w(\pi)) \\ &\to \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), J_w(\pi)) \\ &\to \mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(\pi))[1].\end{aligned}$$

To prove the desired result, it is therefore enough to show that

$$\mathrm{ord}\, R\Gamma(N(\mathcal{O}_{F^+,p}), J_w(\pi)) = 0.$$

It is even enough to show that for any $\pi \in \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$ and for any $j \in \mathbf{Z}$, we have $\operatorname{ord} H^j(N(\mathcal{O}_{F^+,p}), J_w(\pi)) = 0$, and this can be proved in the same way as [Hau16, Lem. 3.3.1]. Indeed, it suffices to choose an element $t \in T(F_p^+)^+$, as in [Hau16, Lem. 3.1.3], such that $S_w = \cup_{k\geq 0} t^{-k} S_w^\circ t^k$. It follows that t acts locally nilpotently on $J_w(\pi)$, and consequently that each element of $H^i(N(\mathcal{O}_{F^+,p}), J_w(\pi))$ is annihilated by the Hecke action of a sufficiently high power of t. $\square$

If $w \in {}^rW^P$, we define $N_w = P(F_p^+) \cap wN(\mathcal{O}_{F^+,p})w^{-1}$. It is a compact subgroup of $P(F_p^+)$ that contains $N_n(\mathcal{O}_{F,p})$. We define a functor

$$\Gamma(N_w, -) : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T(F_p^+)^+]),$$

where an element $t \in T(F_p^+)^+$ acts by the formula $t \cdot v = \mathrm{tr}_{t^w N_w (t^w)^{-1}/N_w}(t^w v)$ ($t \in T(F_p^+)^+$). Note that this makes sense because $t^w N_w (t^w)^{-1} = P(F_p^+) \cap wtN(\mathcal{O}_{F^+,p})t^{-1}w^{-1} \subset N_w$. Note as well that $wT(F_p^+)^+w^{-1} \subset T_n(F_p)^+$ (by definition of ${}^rW^P$).

Lemma 5.3.5. *Let $w \in {}^rW^P$, and let $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[P(F_p^+)])$ be a bounded below complex. Then there is a natural isomorphism*

$$R\Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(\pi)) \cong R\Gamma(N_w, \pi).$$

Proof. By the first part of Lemma 5.3.4, it is enough to show that there is a natural isomorphism of underived functors

$$\Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(-)) \cong \Gamma(N_w, -).$$

The map sends an $N(\mathcal{O}_{F^+,p})$-invariant function $f : P(F_p^+)wN(\mathcal{O}_{F^+,p}) \to \pi$ to the value $f(w) \in \pi^{N_w}$. It is easy to see that this is an isomorphism of $\mathcal{O}/\varpi^m$-modules; what we need to check is that it is equivariant for the action of $T(F_p^+)^+$. In other words, we need to check that for any $f \in \Gamma(N(\mathcal{O}_{F^+,p}), I_w^\circ(\pi))$, we have

$$\sum_{n \in N(\mathcal{O}_{F^+,p})/tN(\mathcal{O}_{F^+,p})t^{-1}} f(wnt) = \sum_{m \in N_w/t^w N_w (t^w)^{-1}} mwtw^{-1} f(w). \tag{5.3.6}$$

Conjugation by w^{-1} determines a map

$$N_w/t^w N_w (t^w)^{-1} \to N(\mathcal{O}_{F^+,p})/tN(\mathcal{O}_{F^+,p})t^{-1},$$

which is easily seen to be injective. On the other hand, if $n \in N(\mathcal{O}_{F^+,p})$ and $f(wnt) \neq 0$, then the class of n is in the image of this map; indeed, $f(wnt)$ can be non-zero only if $wnt \in P(F_p^+)wN(\mathcal{O}_{F^+,p})$, in which case we write $wnt = qwm$, with $q \in P(F_p^+)$ and $m \in N(\mathcal{O}_{F^+,p})$, hence $n = w^{-1}qwt^{-1}tmt^{-1}$. As $w^{-1}qwt^{-1} \in w^{-1}P(F_p^+)w \cap N(\mathcal{O}_{F^+,p})$, this shows that n is in the image of

this map. It follows that we can rewrite the left-hand side of (5.3.6) as

$$\sum_{m\in N_w/t^wN_w(t^w)^{-1}} f(mwt) = \sum_{m\in N_w/t^wN_w(t^w)^{-1}} mwtw^{-1}f(w),$$

which equals the right-hand side of (5.3.6). □

For the statement of the next lemma, for any $w \in {}^rW^P$, we define a character $\chi_w : T(F_p^+) \to \mathcal{O}^\times$ by the formula

$$\chi_w(t) = \frac{\mathbf{N}_{F_p^+/\mathbf{Q}_p} \det_{F_p^+}(\mathrm{Ad}(t^w)|_{\mathrm{Lie}\, U(F_p^+)\cap wN(F_p^+)w^{-1}})^{-1}}{|\mathbf{N}_{F_p^+/\mathbf{Q}_p} \det_{F_p^+}(\mathrm{Ad}(t^w)|_{\mathrm{Lie}\, U(F_p^+)\cap wN(F_p^+)w^{-1}})|_p}.$$

Note that there is an isomorphism $\mathcal{O}(\chi_w) \cong \mathcal{O}(-\rho + w^{-1}w_0^P(\rho)) \otimes_{\mathcal{O}} \mathcal{O}(\alpha_w)$ of $\mathcal{O}[T(F_p^+)]$-modules, where $w_0^P = w_0^G w_0^{\widetilde{G}}$ is the longest element of ${}^rW^P$, and where $\alpha_w : T(F_p^+) \to \mathcal{O}^\times$ is the character that is trivial on $T(\mathcal{O}_{F^+,p})$ and that satisfies the identity $\alpha_w(t) = \chi_w(t)$ for any element of the form $t = \iota_v^{-1}(\mathrm{diag}(\varpi_v^{a_1}, \dots, \varpi_v^{a_{2n}}))$ $(a_i \in \mathbf{Z})$. We also write

$$\tau_w : \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)])$$

for the functor that sends a module π to $\tau_w(\pi) = \pi$, with action $\tau_w(\pi)(t)(v) = \pi(t^{w^{-1}})(v)$.

Lemma 5.3.7. *Let $w \in {}^rW^P$, and let $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[G(F_p^+)])$ be a bounded below complex. Then there is a natural isomorphism between the following two complexes in $\mathbf{D}_{sm}(\mathcal{O}/\varpi^m[T_n(F_p)])$:*

$$\mathrm{ord}\, R\Gamma(N_w, \mathrm{Inf}_{G(F_p^+)}^{P(F_p^+)}\pi)$$

and

$$\mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1}\, \mathrm{ord}\, R\Gamma(N_n(\mathcal{O}_{F,p}), \pi)[-[F^+ : \mathbf{Q}]n^2 + l(w)].$$

Proof. Let $N_w \rtimes_w T(F_p^+)^+$ denote the monoid $N_w \times T(F_p^+)^+$, equipped with multiplication $(t^w n(t^w)^{-1}, 1)(1, t) = (1, t)(n, 1)$ (where the product $t^w n(t^w)^{-1}$ is formed using the usual multiplication of the group $\widetilde{G}(F_p^+)$). Let $N_{w,U} = N_w \cap U(F_p^+)$. Then there is a short exact sequence

$$0 \to N_{w,U} \to N_w \to N_n(\mathcal{O}_{F,p}) \to 0$$

that is equivariant for the conjugation action of $T(F_p^+)^+$ via the map $T(F_p^+)^+ \to T_n(F_p)^+$, $t \mapsto t^w$. We consider the diagram, commutative up to natural

isomorphism:

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]) \arrow[d, "\mathrm{Res}^w"'] & & \\
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_w \rtimes_w T(F_p^+)^+]) \arrow[d, "\Gamma_{N_{w,U}}"'] & & \\
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes_w T(F_p^+)^+]) \arrow[r, "\alpha"] \arrow[d, "\Gamma_{N_n(\mathcal{O}_{F,p})}"'] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes T_n(F_p)^+]) \arrow[d, "\Gamma_{N_n(\mathcal{O}_{F,p})}"] \\
\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T(F_p^+)^+]) \arrow[r, "\beta"] \arrow[dr, "\tau_w \circ \mathrm{ord}"] & \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+]) \arrow[d, "\mathrm{ord}"] \\
& \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)]).
\end{tikzcd}$$

In this diagram we have abbreviated, e.g., $\Gamma(N_{w,U}, -) = \Gamma_{N_{w,U}}$. We also abbreviate $\mathrm{Inf}_G^P = \mathrm{Inf}_{G(F_p^+)}^{P(F_p^+)}$. The torus action on, e.g., $\Gamma_{N_{w,U}}$ is defined in the usual way; cf. [Hau16, §3.2]. The exact functor Res^w is defined by taking $\mathrm{Res}^w(\pi) = \pi$ as an $\mathcal{O}/\varpi^m$-module, with $\mathrm{Res}^w(\pi)(nt)(v) = \pi(nt^w)(v)$. We also use Res^w to denote the functor $\mathrm{Res}^w \circ \mathrm{Inf}_G^P$. The α is the composite of the equivalence

$$\begin{aligned}
&\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes_w T(F_p^+)^+]) \\
&\qquad \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes wT(F_p^+)^+w^{-1}])
\end{aligned}$$

induced by the map $nt \in N_n(\mathcal{O}_{F,p}) \rtimes wT(F_p^+)^+w^{-1} \mapsto (n, t^{w^{-1}}) \in N_n(\mathcal{O}_{F,p}) \rtimes_w T(F_p^+)^+$ with the localization

$$\begin{aligned}
&\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes wT(F_p^+)^+w^{-1}]) \\
&\qquad \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes T_n(F_p)^+])
\end{aligned}$$

induced by the inclusion $wT(F_p^+)^+w^{-1} \subset T_n(F_p)^+$. Similarly, the functor β is the composite of the equivalence

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T(F_p^+)^+]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[wT(F_p^+)^+w^{-1}])$$

with the localization

$$\mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[wT(F_p^+)^+w^{-1}]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[T_n(F_p)^+]).$$

Note that α takes injectives to $\Gamma_{N_n(\mathcal{O}_{F,p})}$-acyclics; this can be deduced from [Eme10b, Prop. 2.1.3], using the compactness of $N_n(\mathcal{O}_{F,p})$ and the observation that this localization can be thought of as a direct limit. Note that the composite of all left vertical arrows is the functor Γ_{N_w}.

Let π now be as in the statement of the lemma. We compute

$$\begin{aligned}\operatorname{ord} R\Gamma(N_w, \operatorname{Inf}_G^P \pi) &= \operatorname{ord} \beta R\Gamma_{N_n(\mathcal{O}_{F,p})} R\Gamma_{N_{w,U}} \operatorname{Res}^w \operatorname{Inf}_G^P \pi \\ &= \operatorname{ord} R\Gamma_{N_n(\mathcal{O}_{F,p})} \alpha R\Gamma_{N_{w,U}} \operatorname{Res}^w \operatorname{Inf}_G^P \pi.\end{aligned}$$

Since $U(F_p^+)$ acts trivially on π, there is an isomorphism

$$R\Gamma_{N_{w,U}} \operatorname{Res}^w \operatorname{Inf}_G^P \pi \cong \operatorname{Res}^w(\pi) \otimes_{\mathcal{O}/\varpi^m} R\Gamma(N_{w,U}, \mathcal{O}/\varpi^m)$$

in $\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[N_n(\mathcal{O}_{F,p}) \rtimes_w T(F_p^+)^+])$. To go further, we need to compute the complex $\alpha R\Gamma(N_{w,U}, \mathcal{O}/\varpi^m)$. To this end, we consider the action of the element

$$z_p = \operatorname{diag}(p, \dots, p, 1, \dots, 1) \in T(F_p^+)^+$$

(where there are n entries equal to p and n entries equal to 1; note that this element depends on our choice of set $\widetilde{S}_p$, which determines the identification of $\widetilde{G}(F_p^+)$ with $\prod_{\overline{v} \in \overline{S}_p} \mathrm{GL}_{2n}(F_{\overline{v}}^+)$). It is in the center of $G(F_p^+)$, and is therefore invertible in $T_n(F_p)^+$. Its action on the cohomology groups $H^i(N_{w,U}, \mathcal{O}/\varpi^m)$ is the one induced by its natural conjugation action on $N_{w,U}$; in other words, multiplication by p on this abelian group. The group $N_{w,U}$ has rank $n^2[F^+ : \mathbf{Q}] - l(w)$ as $\mathbf{Z}_p$-module, from which it follows that the Hecke action of z_p on $H^i(N_{w,U}, \mathcal{O}/\varpi^m)$ factors through multiplication by $p^{n^2[F^+:\mathbf{Q}]-l(w)-i}$ $(0 \le i \le n^2[F^+ : \mathbf{Q}] - l(w))$. The cohomology groups below the top degree $i = n^2[F^+ : \mathbf{Q}] - l(w)$ therefore vanish after applying the functor α, and it follows from [Hau16, Prop. 3.1.8] that

$$\begin{aligned}\alpha R\Gamma(N_{w,U}, \mathcal{O}/\varpi^m) &\cong \mathcal{O}/\varpi^m((\chi_w)^w)[-[F^+ : \mathbf{Q}]n^2 + l(w)] \\ &\cong \alpha\mathcal{O}/\varpi^m(\chi_w)[-[F^+ : \mathbf{Q}]n^2 + l(w)],\end{aligned}$$

hence that

$$\begin{aligned}&\beta R\Gamma_{N_w} \operatorname{Res}^w \operatorname{Inf}_G^P \pi \\ &\cong \beta R\Gamma_{N_n(\mathcal{O}_{F,p})} \operatorname{Res}^w \pi \otimes_{\mathcal{O}/\varpi^m} \mathcal{O}/\varpi^m(\chi_w)[-[F^+ : \mathbf{Q}]n^2 + l(w)].\end{aligned}$$

We finally see that $\operatorname{ord} R\Gamma(N_w, \operatorname{Inf}_G^P \pi)$ is isomorphic to

$$\begin{aligned}&\tau_w^{-1} \operatorname{ord} \beta R\Gamma_{N_n(\mathcal{O}_{F,p})} \operatorname{Res}^w \pi \otimes_{\mathcal{O}/\varpi^m} \mathcal{O}/\varpi^m(\chi_w)[-[F^+ : \mathbf{Q}]n^2 + l(w)] \\ &\cong \mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_w^{-1} \operatorname{ord} R\Gamma_{N_n(\mathcal{O}_{F,p})} \pi[-[F^+ : \mathbf{Q}]n^2 + l(w)].\end{aligned}$$ □

PROPOSITION 5.3.8. *Let $w \in {}^rW^P$, and let $\pi \in \mathbf{D}_{sm}(\mathcal{O}/\varpi^m[G(F_p^+)])$ be a bounded below complex. Then there is a natural isomorphism between the following two complexes in $\mathbf{D}_{sm}(\mathcal{O}/\varpi^m[T_n(F_p)])$:*

$$\operatorname{ord} R\Gamma(N(\mathcal{O}_{F^+,p}), I_w(\operatorname{Inf}_{G(F_p^+)}^{P(F_p^+)} \pi))$$

and

$$\mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}/\varpi^m} \tau_{w^{-1}} \operatorname{ord} R\Gamma(N_n(\mathcal{O}_{F,p}), \pi)[-[F^+ : \mathbf{Q}]n^2 + l(w)].$$

Proof. This follows on combining Lemmas 5.3.4, 5.3.5, and 5.3.7. □

5.4. *The degree shifting argument.* In this section, we give the analogue for completed cohomology of the results of Section 4.2, by relating the completed cohomology of X to the completed cohomology of the boundary $\partial\widetilde{X}$. The statement is simpler for completed cohomology than for cohomology at finite level because the contribution of the unipotent radical of the Siegel parabolic vanishes in the limit.

THEOREM 5.4.1. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^{\infty})$ be a good subgroup that is decomposed with respect to P. Let $\mathfrak{m} \subset \mathbf{T}^S$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$. Then the complex $\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} \pi(K^p, m)_{\mathfrak{m}}$ is a $\widetilde{\mathbf{T}}^S$-equivariant direct summand of the complex $\widetilde{\pi}_{\partial}(\widetilde{K}^p, m)_{\widetilde{\mathfrak{m}}}$ in $\mathbf{D}_{sm}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])$.*

Here and below we have written $\pi(K^p, m)_{\mathfrak{m}}$ for the complex previously denoted $\mathrm{Inf}_{G(F_p^+)}^{P(F_p^+)} \pi(K^p, m)_{\mathfrak{m}}$ in order to lighten the notation.

Proof. We first show that there is a $\widetilde{\mathbf{T}}^S$-equivariant isomorphism

$$\begin{aligned}(R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathrm{Ind}_{P^\infty}^{\widetilde{G}^\infty} \mathfrak{X}_P, \mathcal{O}/\varpi^m))_{\widetilde{\mathfrak{m}}} &\cong (R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\partial\mathfrak{X}_{\widetilde{G}}, \mathcal{O}/\varpi^m))_{\widetilde{\mathfrak{m}}} \\ &= \widetilde{\pi}_{\partial}(\widetilde{K}^p, m)_{\widetilde{\mathfrak{m}}}.\end{aligned}$$

As in the proof of Theorem 2.4.2, it suffices to show that for each standard proper parabolic subgroup $Q \subset \widetilde{G}$ with $Q \neq P$, we have

$$H^*(R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathrm{Ind}_{Q^\infty}^{\widetilde{G}^\infty} \mathfrak{X}_Q, k))_{\widetilde{\mathfrak{m}}}) = \varinjlim_{\widetilde{K}'_p} H^*(\widetilde{X}^Q_{\widetilde{K}^p\widetilde{K}'_p}, k)_{\widetilde{\mathfrak{m}}} = 0.$$

This follows from the corresponding finite level statement, which has already been proved in the course of the proof of Theorem 2.4.2.

We therefore need to compute $R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathrm{Ind}_{P^\infty}^{\widetilde{G}^\infty} \mathfrak{X}_P, \mathcal{O}/\varpi^m)$. We will in fact show that this complex admits $\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} R\Gamma_{K^p,\mathrm{sm}} R\Gamma(\mathfrak{X}_G, \mathcal{O}/\varpi^m)$ as a $\widetilde{\mathbf{T}}^S$-equivariant direct summand in $\mathbf{D}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[\widetilde{G}(F_p^+)])$, where $\widetilde{\mathbf{T}}^S$ acts on the latter complex via the map $\mathcal{S}$.

To see this, we compute

$$\begin{aligned}&R\Gamma_{\widetilde{K}^p,\mathrm{sm}} R\Gamma(\mathrm{Ind}_{P^\infty}^{\widetilde{G}^\infty} \mathfrak{X}_P, \mathcal{O}/\varpi^m)) \cong R\Gamma_{\widetilde{K}^p,\mathrm{sm}} \mathrm{Ind}_{P^\infty}^{\widetilde{G}^\infty} R\Gamma(\mathfrak{X}_P, \mathcal{O}/\varpi^m) \\ &\cong \oplus_{g \in P(F^+)\backslash \widetilde{G}_{S-S_p}/\widetilde{K}_{S-S_p}} \mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} R\Gamma_{\widetilde{K}_P^p,\mathrm{sm}} \mathrm{Res}_{P^{S-S_p} \times g\widetilde{K}_{P,S-S_p}g^{-1}}^{P^\infty} R\Gamma(\mathfrak{X}_P, \mathcal{O}/\varpi^m).\end{aligned}$$

Taking the summand corresponding to $g = 1$, we see that it will be enough to exhibit an isomorphism

$$R\Gamma_{\widetilde{K}_P^p,\mathrm{sm}} \operatorname{Res}^{P^\infty}_{P^{S-S_p}\times \widetilde{K}_{P,S-S_p}} R\Gamma(\mathfrak{X}_P, \mathcal{O}/\varpi^m)$$
$$\cong \operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)} R\Gamma_{K^p,\mathrm{sm}} \operatorname{Res}^{G^\infty}_{G^{S-S_p}\times K_{S-S_p}} R\Gamma(\mathfrak{X}_G, \mathcal{O}/\varpi^m).$$

Let us write

$$\Gamma_{P\text{-sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]),$$

$$\Gamma_{U\text{-sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[P(F_p^+)]) \to \mathrm{Mod}_{U\text{-sm}}(\mathcal{O}/\varpi^m[P(F_p^+)]),$$

and

$$\Gamma_{G\text{-sm}} : \mathrm{Mod}(\mathcal{O}/\varpi^m[G(F_p^+)]) \to \mathrm{Mod}_{\mathrm{sm}}(\mathcal{O}/\varpi^m[G(F_p^+)])$$

for the functors of P, U and G-smooth vectors, respectively. The target category for the second functor is $\mathcal{O}/\varpi^m[P(F_p^+)]$-modules with a smooth action of $U(F_p^+)$. These functors are all right adjoint to forgetful functors, and therefore preserve injectives. The restriction of $\Gamma_{P\text{-sm}}$ to $\mathrm{Mod}_{U\text{-sm}}(\mathcal{O}/\varpi^m[P(F_p^+)])$ is the same as taking G-smooth vectors.

Unpacking the above, we see that it is enough to construct a Hecke-equivariant isomorphism

(5.4.2)
$$\operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)} R\Gamma_{G\text{-sm}} H^0(G(F^+)\backslash G(\mathbf{A}^\infty_{F^+})/K^p, \mathcal{O}/\varpi^m)$$
$$\to R\Gamma_{P\text{-sm}} H^0(P(F^+)\backslash P(\mathbf{A}^\infty_{F^+})/\widetilde{K}_P^p, \mathcal{O}/\varpi^m).$$

The morphism (5.4.2) is constructed using the canonical natural transformation $\operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)} \circ R\Gamma_{G\text{-sm}} \to R\Gamma_{P\text{-sm}} \circ \operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)}$ ([NT16, Lem. 2.1]), and the morphism $\operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)} H^0(G(F^+)\backslash G(\mathbf{A}^\infty_{F^+})/K^p, \mathcal{O}/\varpi^m) \to H^0(P(F^+)\backslash P(\mathbf{A}^\infty_{F^+})/\widetilde{K}_P^p, \mathcal{O}/\varpi^m)$ given by inflation of functions. The Hecke-equivariance follows from [NT16, Cor. 2.8].

To show that (5.4.2) is an isomorphism, it will be enough to show that

$$R\Gamma_{U\text{-sm}} H^0(P(F^+)\backslash P(\mathbf{A}^\infty_{F^+})/\widetilde{K}_P^p, \mathcal{O}/\varpi^m)$$
$$\cong \operatorname{Inf}^{P(F_p^+)}_{G(F_p^+)} H^0(G(F^+)\backslash G(\mathbf{A}^\infty_{F^+})/K^p, \mathcal{O}/\varpi^m).$$

Indeed, we can then take the derived functor of G-smooth vectors on both sides to obtain (5.4.2) — this operation commutes with inflation from G to P, since (the inflation of) a G-injective is acyclic for the functor of G-smooth vectors.

However, the cohomology groups of the left-hand side here can be computed as

$$\varinjlim_{V_p \subset U(F_p^+)} \prod_{g \in G(F^+)\backslash G(\mathbf{A}_{F^+}^\infty)/K^p} H^i(V_p, H^0(U(F^+)\backslash U(\mathbf{A}_{F^+}^\infty)/\widetilde{K}_U^p, \mathcal{O}/\varpi^m),$$

the limit running over all open compact subgroups $V_p \subset U(F_p^+)$.

Using strong approximation, we compute

$$H^i(V_p, H^0(U(F^+)\backslash U(\mathbf{A}_{F^+}^\infty)/\widetilde{K}_U^p, \mathcal{O}/\varpi^m)) = H^i(U(F^+) \cap (\widetilde{K}_U^p V_p), \mathcal{O}/\varpi^m).$$

Taking the limit, we get a product of copies of $\mathcal{O}/\varpi^m$ in degree 0, and 0 in all higher degrees. This completes the proof. □

Combining this theorem with the results of the previous section, we obtain the following.

THEOREM 5.4.3. *Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup that is decomposed with respect to P, and such that $\widetilde{K}_{\overline{v}} = \mathrm{Iw}_{\overline{v}}$ for each place $\overline{v} \in \overline{S}_p$. Let $\widetilde{\lambda} \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$, let $w \in {}^rW^P$, and let $\lambda_w = w(\widetilde{\lambda} + \rho) - \rho \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Let $\mathfrak{m} \subset \mathbf{T}^S$ be a non-Eisenstein maximal ideal, and let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m})$. Fix integers $c \geq b \geq 0$ with $c \geq 1$. Then for any $j \in \mathbf{Z}$, $\mathcal{S}$ descends to a surjective homomorphism*

$$\begin{aligned}&\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^j(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})\\ &\qquad \to \mathbf{T}^{S,\mathrm{ord}}(\mathcal{O}(\alpha_{w_0^G w w_0^{\widetilde{G}}}) \otimes_{\mathcal{O}} \tau^{-1}_{w_0^G w w_0^{\widetilde{G}}} H^{j-l(w)}(X_{K(b,c)}, \mathcal{V}_{\lambda_w})^{\mathrm{ord}}_{\mathfrak{m}}).\end{aligned}$$

Proof. Let $m \geq 1$. To save space, we abbreviate functors $\Gamma(H, -)$ of H-invariants as Γ_H. By Theorem 5.4.1, Lemma 5.2.6, and Proposition 5.2.28, the complex

$$R\Gamma_{\widetilde{K}(0,c)/\widetilde{K}(b,c)}(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}}/\varpi^m)^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}$$

admits the complex

$$\mathrm{ord}_b\, R\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} \mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} \pi(K^p, m)_{\widetilde{\mathfrak{m}}}$$

as a $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-equivariant direct summand. These direct sum decompositions are compatible as m varies, so after passing to the inverse limit we get a surjection of $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-algebras:

(5.4.4)

$$\begin{aligned}&\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^j(\partial\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})\\ \to &\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(\varprojlim_m \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} \mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)} \pi(K^p, m)_{\widetilde{\mathfrak{m}}}).\end{aligned}$$

On the other hand, it follows from Lemma 5.3.3 that for any $i \geq 0$, we have a short exact sequence of $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-modules:

$$\begin{aligned} 0 &\to \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_{\geq i+1}\pi(K^p,m)_{\widetilde{\mathfrak{m}}} \\ &\to \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_{\geq i}\pi(K^p,m)_{\widetilde{\mathfrak{m}}} \\ &\to \oplus_{\substack{w\in {}^rW^P \\ l_r(w)=i}} \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_w\pi(K^p,m)_{\widetilde{\mathfrak{m}}} \to 0. \end{aligned}$$

These are compatible as m varies, and the cohomology groups are finitely generated $\mathcal{O}$-modules, so we can pass to the limit to obtain short exact sequences of $\mathcal{O}$-modules. It follows that for any $i \geq 0$ and any element $w \in {}^rW^P$ of length $l_r(w) = i$, there are surjective homomorphisms of $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-algebras

(5.4.5)

$$\begin{aligned} &\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(\varprojlim_m \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_{\geq i}\pi(K^p,m)_{\widetilde{\mathfrak{m}}}) \\ &\to \widetilde{\mathbf{T}}^{S,\mathrm{ord}}(\varprojlim_m \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_{\geq i+1}\pi(K^p,m)_{\widetilde{\mathfrak{m}}}) \end{aligned}$$

and

(5.4.6)

$$\begin{aligned} &\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(\varprojlim_m \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_{\geq i}\pi(K^p,m)_{\widetilde{\mathfrak{m}}}) \\ &\qquad \to \widetilde{\mathbf{T}}^{S,\mathrm{ord}}(\varprojlim_m \mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_w\pi(K^p,m)_{\widetilde{\mathfrak{m}}}). \end{aligned}$$

By definition, $I_{\geq 0}\pi(K^p,m)$ is (the restriction to $B(F_p^+)$ of) $\mathrm{Ind}_{P(F_p^+)}^{\widetilde{G}(F_p^+)}\pi(K^p,m)$. On the other hand, Proposition 5.3.8 shows that there is a $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-equivariant isomorphism

(5.4.7)

$$\begin{aligned} &\mathrm{ord}_b\, R^j\Gamma_{T(\mathcal{O}_{F^+,p})(b)}\mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} R\Gamma_{N(\mathcal{O}_{F^+,p})} I_w\pi(K^p,m)_{\widetilde{\mathfrak{m}}} \\ &\cong R^{j-[F^+:\mathbf{Q}]n^2+l(w)}\Gamma_{T_n(\mathcal{O}_{F,p})(b)}\mathcal{O}/\varpi^m(\chi_w) \otimes_{\mathcal{O}} \mathcal{O}(w_0^{\widetilde{G}}\widetilde{\lambda}) \otimes_{\mathcal{O}} \tau_w^{-1}\pi^{\mathrm{ord}}(K^p,m). \end{aligned}$$

We recall that there is an isomorphism $\mathcal{O}(\chi_w) \cong \mathcal{O}(-\rho + w^{-1}w_0^P\rho) \otimes_{\mathcal{O}} \mathcal{O}(\alpha_w)$. We have $(-\rho + w^{-1}w_0^P\rho + w_0^{\widetilde{G}}\widetilde{\lambda})^w = w_0^G\lambda_x$, where $x = w_0^G w w_0^{\widetilde{G}}$. Here we write w_0^G for the longest element of W_P, $w_0^{\widetilde{G}}$ for the longest element of W, and note that the map $w \mapsto w_0^G w w_0^{\widetilde{G}}$ is an involution of ${}^rW^P$ that satisfies $l(w_0^G w w_0^{\widetilde{G}}) = [F^+ : \mathbf{Q}]n^2 - l(w)$. Applying Propositions 5.2.15 and 5.2.17, it follows that the cohomology group in (5.4.7) may be identified with

$$\mathcal{O}(\alpha_{w_0^G x w_0^{\widetilde{G}}}) \otimes_{\mathcal{O}} \tau_{w_0^G x w_0^{\widetilde{G}}}^{-1} H^{j-l(x)}(X_{K(b,c)}, \mathcal{V}_{\lambda_x}/\varpi^m)_{\widetilde{\mathfrak{m}}}^{\mathrm{ord}}.$$

Putting all of this together, we see that we can chain together the surjections (5.4.4), (5.4.5) and (5.4.6) to obtain a surjection homomorphism of $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}$-algebras

$$\begin{aligned}&\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^j(\partial\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})\\ &\qquad\to \mathbf{T}^{S,\mathrm{ord}}(\mathcal{O}(\alpha_{w_0^G x w_0^{\widetilde{G}}})\otimes_{\mathcal{O}}\tau^{-1}_{w_0^G x w_0^{\widetilde{G}}}H^{j-l(x)}(X_{K(b,c)},\mathcal{V}_{\lambda_x})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}).\end{aligned}$$

The proof is complete on noting that $H^*(X_{K(b,c)},\mathcal{V}_{\lambda_x})^{\mathrm{ord}}_{\mathfrak{m}}$ is a $\mathbf{T}^{S,\mathrm{ord}}$-invariant direct summand of $H^*(X_{K(b,c)},\mathcal{V}_{\lambda_x})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}$. □

In order to apply Theorem 5.4.3, we will make use of the following combinatorial lemma. We use the following notation: if $\lambda\in(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ and $a\in\mathbf{Z}$, then $\lambda(a)\in(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ is the highest weight defined by the formula $\lambda(a)_{\tau,i}=\lambda_{\tau,i}+a$ for all $\tau\in\mathrm{Hom}(F,E)$, $i=1,\dots,n$. We recall as well that we have previously fixed the notation $\widetilde{S}_p$ for a set of p-adic places of F lifting $\overline{S}_p$, and $\widetilde{I}_p$ for the set of embeddings $\tau:F\hookrightarrow E$ inducing a place of $\widetilde{S}_p$; cf. Section 2.2.1.

Lemma 5.4.8. *Fix $m\geq 1$. Then we can find $\lambda\in(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$ with the following properties*:

(1) *There is an isomorphism $\mathcal{O}(\lambda)/\varpi^m\cong\mathcal{O}/\varpi^m$ of $T_n(F_p)$-modules.*
(2) *The sum $\sum_{i=1}^n(\lambda_{\tau,i}+\lambda_{\tau c,i})$ is independent of the choice of $\tau\in\mathrm{Hom}(F,E)$.*
(3) *For each $i=0,\dots,n^2$, there exist an element $w_i=(w_{i,\overline{v}})_{\overline{v}\in\overline{S}_p}\in{}^rW^P$, an integer $a_i\in(p-1)\mathbf{Z}$, and a dominant weight $\widetilde{\lambda}_i\in(\mathbf{Z}^{2n}_+)^{\mathrm{Hom}(F^+,E)}$, all satisfying the following conditions*:
   (a) *$\widetilde{\lambda}_i$ is CTG; cf. Definition 4.3.5.*
   (b) *For each $\overline{v}\in\overline{S}_p$, $l_r(w_{i,\overline{v}})=n^2-i$. Consequently, $l(w_i)=[F^+:\mathbf{Q}](n^2-i)$.*
   (c) *We have $w_i(\widetilde{\lambda}_i+\rho)-\rho=\lambda(a_i)$.*

Proof. Let $M>16n$ be a non-negative integer which is divisible by the quantity $8(p-1)\#(\mathcal{O}/\varpi^m)^\times$. We will show that we can take λ to be the dominant weight defined by the formulae

$$\lambda_\tau=\begin{cases}(-nM,-2nM,\dots,-n^2M) & \text{if } \tau\in\widetilde{I}_p,\\ (0,-M,\dots,(1-n)M) & \text{if } \tau c\in\widetilde{I}_p.\end{cases}$$

If $\widetilde{\lambda}(a)$ denotes the element of $(\mathbf{Z}^{2n})^{\mathrm{Hom}(F^+,E)}$ that corresponds to $\lambda(a)$ under our identifications, then we have

$$\widetilde{\lambda}(a)=((n-1)M-a,\dots,-a,-nM+a,\dots,-n^2M+a).$$

In order to construct the elements w_i and a_i, we make everything explicit. Our choice of the set $\widetilde{S}_p$ determines an isomorphism of the group $(\mathrm{Res}_{F^+/\mathbf{Q}}\widetilde{G})_{\mathbf{Q}_p}$ with the group $\prod_{\overline{v}\in\overline{S}_p}\mathrm{Res}_{F^+_{\overline{v}}/\mathbf{Q}_p}\mathrm{GL}_{2n}$, hence an identification of ${}^rW_{\overline{v}}$ with S_{2n} and of ${}^rW_{P,\overline{v}}$ with the subgroup $S_n\times S_n$. We can identify the set ${}^rW^P_{\overline{v}}$ of

representatives for the quotient ${}^rW_{P,\overline{v}}\backslash{}^rW_{\overline{v}}$ with the set of n-element subsets of $\{1,\dots,2n\}$. Given such a subset X, there is a unique permutation τ of $\{1,\dots,2n\}$ with $\tau(\{1,\dots,n\}) = X$ and with the property that τ is increasing on both $\{1,\dots,n\}$ and $\{n+1,\dots,2n\}$. The corresponding element of ${}^rW^P_{\overline{v}}$ is $\sigma_X = \tau^{-1}$. The length of a permutation $w \in S_{2n}$ is given by the formula $l(w) = \#\{1 \le i < j \le 2n \mid w(i) > w(j)\}$.

Given i, we choose integers $r, x \ge 0$ with $nx+n-r = n^2-i$ and $1 \le r \le n$ (the choice is unique). We define w_i by setting $w_{i,\overline{v}} = \sigma_{X_i}$ for each $\overline{v} \in \overline{S}_p$, where $X_i = \{x+1, x+2, \dots, x+r, x+r+2, x+r+3, \dots, x+n+1\}$. We have

$$\begin{aligned}(w_{i,\overline{v}}(1),\dots,w_{i,\overline{v}}(2n)) = (&n+1, n+2, \dots, n+x, 1, 2, \dots, r, n+x+1, r+1,\\ &r+2, \dots, n, n+x+2, n+x+3, \dots, 2n).\end{aligned}$$

We observe that indeed $l_r(w_{i,\overline{v}}) = n^2 - i$. We need to choose a_i so that the weight $\widetilde{\lambda}_i = w_i^{-1}(\widetilde{\lambda}(a_i)+\rho) - \rho$ is dominant. We first calculate $w_i^{-1}(\widetilde{\lambda}(a_i))$. For any $\tau \in \mathrm{Hom}(F^+, E)$, we have $w_i^{-1}(\widetilde{\lambda}(a_i))_{\tau,j} = \widetilde{\lambda}(a_i)_{\tau,w_i(j)}$, hence the τ component of $w_i^{-1}(\widetilde{\lambda}(a_i))$ is equal to

$$\begin{aligned}&\big(\widetilde{\lambda}(a_i)_{\tau,n+1},\dots,\widetilde{\lambda}(a_i)_{\tau,n+x},\widetilde{\lambda}(a_i)_{\tau,1},\dots,\widetilde{\lambda}(a_i)_{\tau,r},\widetilde{\lambda}(a_i)_{\tau,n+x+1},\\ &\widetilde{\lambda}_{\tau,r+1}(a_i),\dots,\widetilde{\lambda}_{\tau,n}(a_i),\widetilde{\lambda}_{\tau,n+x+2}(a_i),\dots,\widetilde{\lambda}_{\tau,2n}(a_i)\big)\\ =\ &\big(-nM+a_i,\dots,-nxM+a_i,(n-1)M-a_i,\dots,(n-r)M-a_i,\\ &-n(x+1)M+a_i,(n-r-1)M-a_i,\dots,-a_i,\\ &-n(x+2)M+a_i,\dots,-n^2M+a_i\big).\end{aligned}$$

We see that $w_i^{-1}(\widetilde{\lambda}(a_i))$ is dominant if and only if the following four inequalities are satisfied:

$$-nxM + a_i \ge (n-1)M - a_i, \tag{5.4.9}$$

$$(n-r)M - a_i \ge -n(x+1)M + a_i, \tag{5.4.10}$$

$$-n(x+1)M + a_i \ge (n-r-1)M - a_i, \tag{5.4.11}$$

$$-a_i \ge -n(x+2)M + a_i. \tag{5.4.12}$$

These four inequalities are together equivalent to requiring that

$$a_i \in [(nx+2n-r-1)M/2, (nx+2n-r)M/2],$$

a closed interval of length $M/2$. Requiring instead that $w_i^{-1}(\widetilde{\lambda}(a_i)+\rho)-\rho$ is dominant leads to four similar inequalities, where the left-hand side and right-hand side differ from those in (5.4.9)–(5.4.12) by an integer of absolute value at most $2n-1$. If we choose a_i to be the unique integer in $[(nx+2n-r-1)M/2, (nx+2n-r)M/2]$ that is congruent to $M/8$ mod $M/2$, then $w_i^{-1}(\widetilde{\lambda}(a_i)+\rho)-\rho$ is dominant.

To complete the proof of the lemma, we just need to explain why $\widetilde{\lambda}_i = w_i^{-1}(\widetilde{\lambda}(a_i)+\rho)-\rho$ is CTG. It suffices to show that for any $\tau \in \mathrm{Hom}(F^+, E)$, and for any $w \in W_{\overline{v}}^P$ (where $\overline{v}$ is the place of F^+ induced by τ), the number

$$[w(\widetilde{\lambda}_i+\rho)-\rho]_{\tau,j} + [w(\widetilde{\lambda}_i+\rho)-\rho]_{\tau,2n+1-j} = w(\widetilde{\lambda}_i+\rho)_{\tau,j} + w(\widetilde{\lambda}_i+\rho)_{\tau,2n+1-j}$$

is not independent of j as j varies over integers $1 \le j \le n$. To show this, it suffices to show that the multiset

$$I = \{(\widetilde{\lambda}_i+\rho)_{\tau,j} + (\widetilde{\lambda}_i+\rho)_{\tau,k} \mid 1 \le j < k \le 2n\}$$

does not contain any element with multiplicity at least n. We first consider the multiset

$$I' = \{\widetilde{\lambda}_{i,\tau,j} + \widetilde{\lambda}_{i,\tau,k} \mid 1 \le j < k \le 2n\}.$$

It is a union of the three multisets

$$I_1' = \{(-n\alpha+\beta)M \mid 1 \le \alpha \le n, 0 \le \beta \le n-1\},$$
$$I_2' = \{-n(\alpha+\beta)M + 2a_i \mid 1 \le \alpha < \beta \le n\},$$

and

$$I_3' = \{(\alpha+\beta)M - 2a_i \mid 0 \le \alpha < \beta \le n-1\}.$$

Note that each element of I_1' has multiplicity 1. Each element of I_2' and I_3' has multiplicity at most $n/2$. Moreover, I_1', I_2', and I_3' are mutually disjoint (look modulo M). It follows that no element of I' has multiplicity at least n. To show that I has no element of multiplicity at least n, we use the analogous decomposition $I = I_1 \cup I_2 \cup I_3$. The sets I_1, I_2 and I_3 are disjoint (look modulo M, and use the fact that each entry of ρ has absolute value at most $(2n-1)/2$). Each element of I_1 appears with multiplicity 1, while each entry of I_2 and I_3 has multiplicity at most $n/2$. This completes the proof. □

Lemma 5.4.8 allows us to express certain cohomology groups of the spaces X_K in degrees divisible by $[F^+ : \mathbf{Q}]$ in terms of middle degree cohomology of the spaces $\partial\widetilde{X}_{\widetilde{K}}$ (and hence, using Theorem 4.3.3, of the spaces $\widetilde{X}_{\widetilde{K}}$). Indeed, combining the results so far of this section, we obtain the following result:

Proposition 5.4.13. *Suppose that $[F^+ : \mathbf{Q}] > 1$. Let $m \ge 1$ be an integer. Then there exists a dominant weight $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ such that a finite index subgroup of $\mathcal{O}_F^\times$ acts trivially on $\mathcal{V}_\lambda$ and for each $i = 0, \dots, n^2-1$, a dominant weight $\widetilde{\lambda}_i \in (\mathbf{Z}_+^{2n})^{\mathrm{Hom}(F^+,E)}$ which is CTG, an integer a_i divisible by $(p-1)$, and a Weyl element $w_i \in {}^rW^P$ such that the following conditions are satisfied: Let $\widetilde{K} \subset \widetilde{G}(\mathbf{A}_{F^+}^\infty)$ be a good subgroup which is decomposed with respect to P and such that for each $\overline{v} \in \overline{S}_p$, $\widetilde{K}_{\overline{v}} = \widetilde{\mathrm{Iw}}_{\overline{v}}$. Fix integers $c \ge b \ge 0$ with $c \ge 1$, and also an integer $m \ge 1$. Let $\mathfrak{m} \subset \mathbf{T}^S$ be a non-Eisenstein maximal ideal. Let $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m}) \subset \widetilde{\mathbf{T}}^S$, and suppose that $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic. Then*

(1) *there is an isomorphism* $\mathcal{O}(\lambda)/\varpi^m \cong \mathcal{O}/\varpi^m$ *of* $\mathcal{O}[T(F_p^+)]$*-modules*;
(2) *for each* $i = 0, \ldots, n^2-1$, *the map* $\mathcal{S}$ *descends to an algebra homomorphism*

$$\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}) \\ \to \mathbf{T}^{S,\mathrm{ord}}(\mathcal{O}(\alpha_{w_i}) \otimes_{\mathcal{O}} \tau_{w_i}^{-1} H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}}).$$

Proof. This follows on combining Theorems 4.3.3, 5.4.3 and Lemma 5.4.8. □

In order to access all degrees of cohomology, we use a trick based on the fact that the group G has a non-trivial center. This is the motivation behind the next few results.

If $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ is a good subgroup, then we define

$$A_K := F^\times \backslash \mathbf{A}_F^\times / \det(K) \det(K_\infty) \mathbf{R}_{>0}.$$

The quotient map

$$A_K \to F^\times \backslash \mathbf{A}_F^\times / \det(K) F_\infty^\times$$

identifies A_K with an extension of a ray class group by a real torus of dimension $[F^+ : \mathbf{Q}] - 1$ (with cocharacter lattice $F^\times \cap \det(K)$, a torsion-free congruence subgroup of $\mathcal{O}_F^\times$). We denote the identity component of A_K by A_K°. If $g \in \mathrm{GL}_n(\mathbf{A}_F^\infty)$, then we set $\Gamma_{g,K} = \mathrm{GL}_n(F) \cap gKg^{-1}$, or $\Gamma_g = \Gamma_{g,K}$ if the choice of K is fixed.

Lemma 5.4.14.

(1) *The maps* $x \mapsto (x, g)$ *induce a homeomorphism*

$$\coprod_{[g] \in \mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbf{A}_F^\infty)/K} \Gamma_g \backslash X \cong X_K.$$

(2) *The determinant gives a continuous map*

$$X_K \xrightarrow{\det} A_K$$

which induces a bijection on sets of connected components.

(3) *Suppose* $g \in \mathrm{GL}_n(\mathbf{A}_F^\infty)$ *and the two subgroups* $\det(\Gamma_g)$ *and* $\det(F^\times \cap K)$ *of* $F^\times$ *are equal. Let* $\Gamma_g^1 = \mathrm{SL}_n(F) \cap \Gamma_g$. *Then the product map*

$$\Gamma_g^1 \times (F^\times \cap K) \to \Gamma_g$$

is a group isomorphism. Decomposing X *similarly as*

$$X^1 \times \Big(\prod_{v|\infty} \mathbf{R}_{>0}\Big)/\mathbf{R}_{>0} = X,$$

where $X^1 = \mathrm{SL}_n(F_\infty)/\prod_{v|\infty} \mathrm{SU}(n)$, *we obtain a decomposition*

$$\Gamma_g \backslash X = \big(\Gamma_g^1 \backslash X^1\big) \times (F^\times \cap K) \backslash \Big(\prod_{v|\infty} \mathbf{R}_{>0}\Big)/\mathbf{R}_{>0}.$$

(4) *Still assuming that* $\det(\Gamma_g) = \det(F^\times \cap K)$, *the map* $\det : F^\times \cap K \to F^\times \cap \det(K)$ *is an isomorphism. The composition of maps*

$$\left(\Gamma_g^1 \backslash X^1\right) \times (F^\times \cap K) \backslash \Big(\prod_{v|\infty} \mathbf{R}_{>0}\Big) / \mathbf{R}_{>0} = \Gamma_g \backslash X \hookrightarrow X_K \to A_K$$

is given by $(x, z) \mapsto \det(g) z^n$, *and the map* $z \mapsto \det(g) z^n$ *is an isomorphism from* $(F^\times \cap K) \backslash (\prod_{v|\infty} \mathbf{R}_{>0}) / \mathbf{R}_{>0}$ *to the connected component* $A_K^{[\det(g)]}$ *of* A_K *containing* $[\det(g)]$.

Proof. The first part can be checked directly. The second part is equivalent to the statement that det induces a bijection

$$G(F^+) \backslash G(\mathbf{A}_{F^+}^\infty) / K \to F^\times \backslash (\mathbf{A}_F^\infty)^\times / \det(K).$$

This follows from strong approximation for the derived subgroup of G, which is isomorphic to $\mathrm{Res}_{F/F^+} \mathrm{SL}_n$. For the third part, injectivity of the natural map $\Gamma_g^1 \times (F^\times \cap K) \to \Gamma_g$ follows from neatness of K (since $F^\times \cap K$ contains no roots of unity, and hence no non-trivial elements of determinant 1). Surjectivity follows from the assumption that $\det(\Gamma_g) = \det(F^\times \cap K)$. The remainder of the third part (on the decomposition of $\Gamma_g \backslash X$) is an immediate consequence. Finally, for the fourth part, everything follows from the claim that $\det : F^\times \cap K \to F^\times \cap \det(K)$ is an isomorphism. Injectivity follows from neatness of K. Surjectivity follows from strong approximation for SL_n and the assumption that $\det(F^\times \cap K) = \det(\Gamma_g)$. Indeed, suppose we have $k \in K$ with $\det(k) \in F^\times$. We can find $\gamma \in \mathrm{GL}_n(F)$ such that $\det(\gamma) = \det(k)$, and strong approximation implies that we can find $\gamma' \in \mathrm{SL}_n(F)$ and $k' \in gKg^{-1} \cap \mathrm{SL}_n(\mathbf{A}_F^\infty)$ such that $\gamma(gkg^{-1})^{-1} = \gamma' k'$. We deduce that $(\gamma')^{-1}\gamma = k' gkg^{-1} \in gKg^{-1} \cap \mathrm{GL}_n(F)$ has the same determinant as k, which shows surjectivity. □

The following lemma shows how to choose K so that the conditions of Lemma 5.4.14 are satisfied.

Lemma 5.4.15. *Let* K *be a good subgroup of* $G(\mathbf{A}_{F^+}^\infty)$. *Fix a finite set* T *of finite places of* F. *There exists a good normal subgroup* $K' \subset K$ *with* $K'_T = K_T$ *such that* $\det(\Gamma_{g,K'}) = \det(F^\times \cap K')$ *for all* $g \in \mathrm{GL}_n(\mathbf{A}_F^\infty)$.

Proof. We begin by choosing an ideal $\mathfrak{a}$ of $\mathcal{O}_F$, prime to T, such that $\ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times)$ is torsion-free and is contained in $F^\times \cap K$. This is possible by Chevalley's theorem [Che51, Th. 1]. Similarly, we can choose another ideal $\mathfrak{b}$ of $\mathcal{O}_F$, prime to $\mathfrak{a}$ and T, such that $\ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a}\mathfrak{b})^\times)$ is contained in $(\ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times))^n$. We claim that

$$K' := \ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times) \cdot K(\mathfrak{a}\mathfrak{b})$$

has the desired properties, where $K(\mathfrak{a}\mathfrak{b})$ is the intersection of K with the principal congruence subgroup of level $\mathfrak{a}\mathfrak{b}$. Indeed, by construction we have

$\det(\mathrm{GL}_n(F) \cap gK'g^{-1}) = (\ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times))^n$ for all $g \in \mathrm{GL}_n(\mathbf{A}_F^\infty)$, whilst $F^\times \cap K' = \ker(\mathcal{O}_F^\times \to (\mathcal{O}_F/\mathfrak{a})^\times)$. □

The next lemma shows how to use Lemma 5.4.14 to understand all cohomology groups of a space X_K solely in terms of those in degrees divisible by $[F^+ : \mathbf{Q}]$. Note that $\dim(X_K) = d - 1 = [F^+ : \mathbf{Q}]n^2 - 1$ and $\dim(A_K) = [F^+ : \mathbf{Q}] - 1$.

LEMMA 5.4.16. *Let $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup, and let $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$. Suppose that the following conditions are satisfied*:

(1) $\det(\Gamma_g) = \det(F^\times \cap K)$ *for all* $g \in \mathrm{GL}_n(\mathbf{A}_F^\infty)$.
(2) $F^\times \cap K$ *acts trivially on* $\mathcal{V}_\lambda$.

Recall that we have defined a map $\det : X_K \to A_K$. *Then* $R\det_*(\mathcal{V}_\lambda)$ (*a complex of sheaves of* $\mathcal{O}$*-modules*) *is constant on each connected component of* A_K, *and we have* $R\det_*(\mathcal{V}_\lambda) = \bigoplus_{i=0}^{\dim(X^1)} R^i \det_*(\mathcal{V}_\lambda)[-i]$. *We obtain a* $\mathbf{T}^{S,\mathrm{ord}}$*-equivariant isomorphism of graded* $\mathcal{O}$*-modules*

$$\begin{aligned}(5.4.17)\qquad &\bigoplus_{i=0}^{\dim(X_K)} H^i(X_K, \mathcal{V}_\lambda) \\ &\cong \left(\bigoplus_{j=0}^{\dim(A_K^\circ)} H^j(A_K^\circ, \mathcal{O})\right) \otimes_{\mathcal{O}} \left(\bigoplus_{k=0}^{\dim(X^1)} H^0(A_K, R^k\det_*(\mathcal{V}_\lambda))\right),\end{aligned}$$

where the Hecke action on the first factor $\bigoplus_{j=0}^{\dim(A_K^\circ)} H^j(A_K^\circ, \mathcal{O})$ *is trivial.*

As a consequence, the image of $\mathbf{T}^{S,\mathrm{ord}}$ *in* $\mathrm{End}_{\mathcal{O}}(\bigoplus_{i=0}^{\dim(X_K)} H^i(X_K, \mathcal{V}_\lambda))$ *is equal to its image in* $\mathrm{End}_{\mathcal{O}}(\bigoplus_{i=0}^{n^2-1} H^{i[F^+:\mathbf{Q}]}(X_K, \mathcal{V}_\lambda))$.

Proof. It follows from our first assumption and Lemma 5.4.14 that every connected component of X_K decomposes as a product $(\Gamma_g^1 \backslash X^1) \times A_K^{[\det(g)]}$, with the map det given by the projection to the second factor. Our second assumption implies that the local system $\mathcal{V}_\lambda$ on this component is pulled back from a local system on $\Gamma_g^1 \backslash X^1$. We deduce that $R\det_*(\mathcal{V}_\lambda)$ is constant on $A_K^{[\det(g)]}$ (corresponding to $R\Gamma(\Gamma_g^1 \backslash X^1, \mathcal{V}_\lambda)$) and it decomposes as the direct sum of its shifted cohomology sheaves (since the same is true for any object in $\mathbf{D}(\mathcal{O})$, such as $R\Gamma(\Gamma_g^1 \backslash X^1, \mathcal{V}_\lambda)$). To save space, we now write $H^*(\cdots)$ for the graded cohomology module $\bigoplus_i H^i(\cdots)$.

Passing to global sections on A_K we get an isomorphism

$$\begin{aligned}H^*(X_K, \mathcal{V}_\lambda) &\cong \oplus_{i=0}^{\dim(X^1)} H^*(A_K, R^i\det_*(\mathcal{V}_\lambda)) \\ &\cong \oplus_{i=0}^{\dim(X^1)} \oplus_{[g]\in \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F^\infty)/K} H^*(A_K^{[\det g]}, R^i\det_*(\mathcal{V}_\lambda))\end{aligned}$$

$$\cong \oplus_{i=0}^{\dim(X^1)} \oplus_{[g]\in \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F^\infty)/K} H^*(A_K^{[\det g]}, \mathcal{O}) \otimes_{\mathcal{O}} H^0(A_K^{[\det g]}, R^i\det_*(\mathcal{V}_\lambda)).$$

Note that the cohomology groups $H^i(A_K^{[\det g]}, \mathcal{O})$ are torsion-free. We now use that the groups $H^*(A_K^{[\det g]}, \mathcal{O})$ are canonically independent of g, so they can all be identified with $H^*(A_K^\circ, \mathcal{O})$. We thus obtain an isomorphism of graded $\mathcal{O}$-modules

$$\begin{aligned} H^*(X_K, \mathcal{V}_\lambda) &\cong H^*(A_K^\circ, \mathcal{O}) \otimes_{\mathcal{O}} \oplus_{i=0}^{\dim(X^1)} \\ &\qquad \oplus_{[g]\in \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbf{A}_F^\infty)/K} H^0(A_K^{[\det g]}, R^i\det{}_*(\mathcal{V}_\lambda)) \\ &\cong H^*(A_K^\circ, \mathcal{O}) \otimes_{\mathcal{O}} \bigoplus_{i=0}^{\dim(X^1)} H^0(A_K, R^i\det{}_*(\mathcal{V}_\lambda)). \end{aligned}$$

We next need to understand the action of Hecke operators. If $g \in G^S$, then the action of the Hecke operator $[K^S g K^S]$ can be described with the aid of the diagram

$$\begin{tikzcd} X_K \arrow[d, "\det"] & X_{K\cap gKg^{-1}} \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "\det"] & X_K \arrow[d, "\det"] \\ A_K & A_{K\cap gKg^{-1}} \arrow[l, "q_1"] \arrow[r, "q_2"'] & A_K. \end{tikzcd}$$

Here p_1 and q_1 are induced by the action of g, while p_2 and q_2 are the natural projections; the action of $[K^S g K^S]$ on $R\Gamma(X_K, \mathcal{V}_\lambda)$ is given by the formula $p_{2,*} \circ p_1^*$. Pushing forward by det, we have a morphism $q_1^* R\det{}_* \mathcal{V}_\lambda \to q_2^* R\det{}_* \mathcal{V}_\lambda$, and the induced endomorphism of the complex $R\Gamma(A_K, R\det{}_* \mathcal{V}_\lambda)$ in $\mathbf{D}(\mathcal{O})$ agrees with $[K^S g K^S]$ under the natural identification $R\Gamma(A_K, R\det{}_* \mathcal{V}_\lambda) \cong R\Gamma(X_K, \mathcal{V}_\lambda)$. We see that the isomorphism (5.4.17) respects the action of $[K^S g K^S]$ if $[K^S g K^S]$ acts in the usual way on the left-hand side, as multiplication by $[F^\times \cap \det(K) : F^\times \cap \det(K \cap gKg^{-1})]^i$ on $H^i(A_K^\circ, \mathcal{O})$, and in the natural way on $H^0(A_K, R^i \det{}_* \mathcal{V}_\lambda)$. Our assumption $\det(\Gamma_g) = \det(F^\times \cap K)$ implies that $F^\times \cap \det(K) = F^\times \cap \det(K \cap gKg^{-1})$, giving the statement in the lemma.

It remains to check the final statement of the lemma. There is an isomorphism $H^*(A_K^\circ, \mathcal{O}) \cong \wedge_{\mathcal{O}}^* \mathrm{Hom}(F^\times \cap \det(K), \mathcal{O})$ of graded $\mathcal{O}$-modules. It follows that each cohomology group $H^0(A_K, R^i\det{}_*(\mathcal{V}_\lambda))$ appears as a direct summand of $H^*(X_K, \mathcal{V}_\lambda)$ in $[F^+ : \mathbf{Q}]$ consecutive degrees. In particular, it appears as a direct summand of a cohomology group in a degree divisible by $[F^+ : \mathbf{Q}]$. This completes the proof. $\square$

For the statement of the next proposition, we remind the reader that in Section 5.1 we have defined for each $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$, $v \in S_p$ and $i = 1, \ldots, n$, a character

$$\chi_{\lambda,v,i} : G_{F_v} \to \mathbf{T}^{S,\mathrm{ord}}(H^*(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}})^\times.$$

PROPOSITION 5.4.18. *Suppose that* $[F^+ : \mathbf{Q}] > 1$. *Let* $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$ *be a good subgroup such that for each* $v \in S_p$, $K_v = \mathrm{Iw}_v$. *Fix integers* $c \geq b \geq 0$ *with* $c \geq 1$, *and also an integer* $m \geq 1$. *Let* $\mathfrak{m} \subset \mathbf{T}^S$ *be a non-Eisenstein maximal ideal, and let* $\widetilde{\mathfrak{m}} = \mathcal{S}^*(\mathfrak{m})$. *Suppose that the following conditions are satisfied*:

(1) $\overline{\rho}_{\mathfrak{m}}$ *is decomposed generic.*
(2) *Let* v *be a finite place of* F *not contained in* S, *and let* l *be its residue characteristic. Then either* S *contains no* l*-adic places of* F *and* l *is unramified in* F, *or there exists an imaginary quadratic field* $F_0 \subset F$ *in which* l *splits.*

Then we can find $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,E)}$ *and an integer* $N \geq 1$ *depending only on* $[F^+ : \mathbf{Q}]$ *and* n *such that the following conditions are satisfied*:

(1) *There is an isomorphism* $\mathcal{O}(\lambda)/\varpi^m \cong \mathcal{O}/\varpi^m$ *of* $\mathcal{O}[T(F_p^+)]$*-modules.*
(2) *There exists a nilpotent ideal* J_i *of* $\mathbf{T}^{S,\mathrm{ord}}(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})$, *for each* $i = 0, \ldots, d-1$, *satisfying* $J_i^N = 0$ *and a continuous* n*-dimensional representation*
$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^{S,\mathrm{ord}}(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})/J_i)$$
such that the following conditions are satisfied:
   (a) *For each place* $v \notin S$ *of* F, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ *is equal to the image of* $P_v(X)$ *in* $(\mathbf{T}^{S,\mathrm{ord}}(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})/J_i)[X]$.
   (b) *For each place* $v|p$ *of* F *and for each* $g \in G_{F_v}$, *the characteristic polynomial of* $\rho_{\mathfrak{m}}(g)$ *equals* $\prod_{j=1}^n (X - \chi_{\lambda,v,j}(g))$.
   (c) *For each place* $v|p$ *of* F, *and for each sequence* $g_1, \ldots, g_n \in G_{F_v}$, *the image of the element*
$$(g_1 - \chi_{\lambda,v,1}(g_1))(g_2 - \chi_{\lambda,v,2}(g_2)) \ldots (g_n - \chi_{\lambda,v,n}(g_n))$$
of $\mathbf{T}^S(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})[G_{F_v}]$ *in* $M_n(\mathbf{T}^S(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})/J_i)$ *under* $\rho_{\mathfrak{m}}$ *is zero.*

Proof. We choose λ using Proposition 5.4.13. Note that for each cohomological degree i, by Theorem 2.3.7 we can find N, J_i and
$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^{S,\mathrm{ord}}(H^i(X_{K(b,c)}, \mathcal{V}_\lambda)_{\mathfrak{m}}^{\mathrm{ord}})/J_i)$$
satisfying condition (a) of the proposition. Indeed, this theorem and the discussion after Lemma 2.2.4 gives a representation with values in a quotient of $\mathbf{T}^S(K(0,c)/K(b,c), \mathcal{V}_\lambda)_{\mathfrak{m}}$ by a nilpotent ideal, which we compose with the canonical homomorphism to $\mathbf{T}^S(K(b,c), \lambda)_{\mathfrak{m}}^{\mathrm{ord}}$. Arguing with the Hochschild–Serre spectral sequence and twisting with characters as in the proof of Corollary 4.4.8, we are free to enlarge S and to shrink K at the prime-to-p places of S. We can therefore assume that the following conditions are satisfied:

(1) For each place $v \in S_p$, the two representations $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$, $(\overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})|_{G_{F_v}}$ have no Jordan–Hölder factors in common.

(2) $\overline{\rho}_{\widetilde{\mathfrak{m}}}$ is decomposed generic.
(3) K satisfies the conditions of Lemma 5.4.16.

After enlarging $\mathcal{O}$, we can assume that there exists a character $\chi : G_{F,S} \to k^\times$ satisfying the following conditions:

(1) For each place $v \in S_p$, $\chi|_{G_{F_v}}$ is unramified.
(2) For each place $v \in S_p$, the two representations $(\overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}) \otimes \chi|_{G_{F_v}}$ and $(\overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}) \otimes \chi^{c,\vee}|_{G_{F_v}}$ have no Jordan–Hölder factors in common.
(3) The representation $\overline{\rho}_{\mathfrak{m}} \otimes \chi \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \chi^{c,\vee}\epsilon^{1-2n}$ is decomposed generic.

It follows from Lemma 5.4.16 and Carayol's lemma (applied as in the proof of Corollary 4.4.8) that it suffices to establish conditions (b) and (c) for cohomological degrees $0, [F^+ : \mathbf{Q}], \dots, (n^2-1)[F^+ : \mathbf{Q}]$. (Carayol's lemma then gives us a Galois representation with coefficients in $\mathbf{T}^{S,\mathrm{ord}}(\oplus_{i=0}^{n^2-1} H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}})$ $= \mathbf{T}^{S,\mathrm{ord}}(\oplus_{i=0}^{d-1} H^i(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}})$ modulo a nilpotent ideal with the desired properties.)

We choose a good subgroup $\widetilde{K} \subset \widetilde{G}(\mathbf{A}^\infty_{F^+})$ which satisfies the conditions of Proposition 5.4.13 and such that $\widetilde{K} \cap G(\mathbf{A}^\infty_{F^+}) = K$. For each $i = 0, 1, \dots, n^2-1$, we let $\widetilde{\lambda}_i$, a_i and w_i be as in the statement of Proposition 5.4.13. Generalizing Proposition 2.2.23 slightly, we note that there is an isomorphism (cf. the proof of Theorem 4.5.1)

$$f : \mathbf{T}^{S,\mathrm{ord}}(H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}}) \cong \mathbf{T}^{S,\mathrm{ord}}(H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}}),$$

which carries $[K^S g K^S]$ to $\epsilon^{-a_i}(\mathrm{Art}_F(\det(g)))[K^S g K^S]$ and satisfies the identity $f \circ \chi_{\lambda(a_i),v,j} = \chi_{\lambda,v,j} \otimes \epsilon^{-a_i}$ $(v \in S_p)$. (Note that a_i is divisible by $p-1$, by construction, so we have $\mathfrak{m}(\epsilon^{-a_i}) = \mathfrak{m}$ in the notation of Section 2.2.20.) To prove the proposition, it will therefore suffice to prove the analogue of properties (b) and (c) for the representation $(f^{-1} \circ \rho_{\mathfrak{m}}) \otimes \epsilon^{-a_i}$ with coefficients in $\mathbf{T}^{S,\mathrm{ord}}(H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}})/f^{-1}J_{i[F^+:\mathbf{Q}]}$, which we already know satisfies the analogue of property (a). In order to simplify notation, we now write $\rho_{\mathfrak{m}}$ for this representation, J_i for the ideal $f^{-1}J_{i[F^+:\mathbf{Q}]}$, and $\chi_{v,j}$ for the character $\chi_{\lambda(a_i),v,j}$ valued in $\mathbf{T}^{S,\mathrm{ord}}(H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}})$.

We obtain from Proposition 5.4.13 a surjective algebra homomorphism

$$\begin{aligned}&\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})\\ &\qquad \to \mathbf{T}^{S,\mathrm{ord}}(\mathcal{O}(\alpha_{w_i}) \otimes_{\mathcal{O}} \tau_{w_i}^{-1} H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)}, \mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}}).\end{aligned}$$

Theorem 4.3.3 says that $H^d(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}$ is a torsion-free $\mathcal{O}$-module, and Theorem 2.4.11 (or rather its proof) shows how to compute $H^d(\widetilde{X}_{\widetilde{K}(b,c)}, \mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}$ $\otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$ in terms of cuspidal automorphic representations of $\widetilde{G}(\mathbf{A}^\infty_{F^+})$. Then [Ger19, Lem. 5.4] (which is stated for automorphic representations of GL_n, but which applies here, since $\widetilde{G}$ is split at the p-adic places of F^+) shows that

$\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})[1/p]$ is a semisimple E-algebra. By Theorem 2.3.3 and [Tho15, Th. 2.4], we can find a continuous representation

$$\widetilde{\rho}: G_{F,S} \to \mathrm{GL}_{2n}(\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p)$$

satisfying the following conditions:

(1) For each finite place $v \notin S$ of F, the characteristic polynomial of $\widetilde{\rho}(\mathrm{Frob}_v)$ is equal to the image of $\widetilde{P}_v(X)$.

(2) For each place $v|p$ of F, there is an isomorphism

$$\widetilde{\rho}|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \psi_{v,2n} \end{pmatrix}, \tag{5.4.19}$$

where for each $i = 1,\dots,2n$, $\psi_{v,i}: G_{F,v} \to \overline{\mathbf{Q}}_p^\times$ is the continuous character defined as follows. First, if $v \in \widetilde{S}_p$, then $\psi_{v,j}$ is the unique continuous character satisfying the following identities:

$$\begin{aligned} &\psi_{v,j} \circ \mathrm{Art}_{F_v}(u) \\ &= \epsilon^{1-j}(\mathrm{Art}_{F_v}(u))\left(\prod_\tau \tau(u)^{-(w_0^{\widetilde{G}}\widetilde{\lambda}_i)_{\tau|_{F^+},j}}\right)\langle \mathrm{diag}(1,\dots,1,u,1,\dots,1)\rangle \ (u \in \mathcal{O}_{F_v}^\times) \end{aligned}$$

(the product being over $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v,E)$) and

$$\psi_{v,j} \circ \mathrm{Art}_{F_v}(\varpi_v) = \epsilon^{1-j}(\mathrm{Art}_{F_v}(\varpi_v))\widetilde{U}_{v,j}/\widetilde{U}_{v,j-1}.$$

Second, if $v^c \in \widetilde{S}_p$, then $\psi_{v,j} = \psi^{c,\vee}_{v^c,2n+1-j}\epsilon^{1-2n}$.

We write $\widetilde{D}$ for the $2n$-dimensional determinant of $G_{F,S}$ associated to $\widetilde{\rho}$. By [Che14, Ex. 2.32], $\widetilde{D}$ is valued in $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}})$. To conserve notation, we now write

$$\widetilde{A}_0 = \widetilde{\mathbf{T}}^{S,\mathrm{ord}}(H^d(\widetilde{X}_{\widetilde{K}(b,c)},\mathcal{V}_{\widetilde{\lambda}_i})^{\mathrm{ord}}_{\widetilde{\mathfrak{m}}}).$$

$$A_0 = \mathbf{T}^{S,\mathrm{ord}}(H^{i[F^+:\mathbf{Q}]}(X_{K(b,c)},\mathcal{V}_{\lambda(a_i)})^{\mathrm{ord}}_{\mathfrak{m}}),$$

and $J = J_i$. By construction, we are given a homomorphism $\widetilde{A}_0 \to A_0$ which agrees with $\mathcal{S}$ on Hecke operators away from p, and such that for each $v \in S_p$, the image of the sequence

$$(\psi_{v,1},\dots,\psi_{v,2n})$$

of characters is the image of the sequence

$$(\chi^{c,\vee}_{v^c,n}\epsilon^{1-2n},\dots,\chi^{c,\vee}_{v^c,1}\epsilon^{1-2n},\chi_{v,1},\dots,\chi_{v,n})$$

under the permutation w_i^{-1}.

The rings $\widetilde{A}_0$ and A_0 are semi-local finite $\mathcal{O}$-algebras. Let A be a local direct factor of A_0, and let $\widetilde{A}$ be the corresponding local direct factor of $\widetilde{A}_0$. Thus there is a map $\widetilde{A} \to A$ such that $\widetilde{A} \to A/J$ is surjective. We will show that the properties (b) and (c) in the statement of the proposition hold in the ring A/J; since A_0/J is a direct product, this will give the desired result.

We first verify that for each place $v \in S_p$, we have $(\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}})^{\mathrm{ss}} \cong \oplus_{j=1}^n \overline{\chi}_{v,j}$, where the overline denotes reduction modulo the maximal ideal of A. By construction, we have

$$((\overline{\rho}_{\mathfrak{m}} \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})|_{G_{F_v}})^{\mathrm{ss}} \cong (\overline{\rho}_{\widetilde{\mathfrak{m}}}|_{G_{F_v}})^{\mathrm{ss}} \cong \oplus_{j=1}^n (\overline{\chi}_{v,j} \oplus \overline{\chi}_{v^c,j}^{c,\vee} \epsilon^{1-2n}).$$

Using the existence of the character χ and a character twisting argument as in the proof of Corollary 4.4.8, we see that we also have an isomorphism (over the residue field of A)

$$((\overline{\rho}_{\mathfrak{m}} \otimes \chi \oplus \overline{\rho}_{\mathfrak{m}}^{c,\vee} \otimes \chi^{c,\vee} \epsilon^{1-2n})|_{G_{F_v}})^{\mathrm{ss}} \cong \oplus_{j=1}^n (\overline{\chi}_{v,j} \chi \oplus \overline{\chi}_{v^c,j}^{c,\vee} \chi^{c,\vee} \epsilon^{1-2n}).$$

Our conditions on the character χ now force $(\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}})^{\mathrm{ss}} \cong \oplus_{j=1}^n \overline{\chi}_{v,j}$.

We can now argue in a similar way to the proof of Proposition 4.4.6. Let $\widetilde{D}_{A/J} = \widetilde{D} \otimes_A A/J$. Then $\widetilde{D}_{A/J} = \det(\rho_{\mathfrak{m}} \oplus \rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n})$. Just as in the proof of Proposition 4.4.6, we can identify $(A/J)[G_{F,S}]/\ker(\widetilde{D}_{A/J})$ with $M_n(A/J) \times M_n(A/J)$ (where the first projection gives $\rho_{\mathfrak{m}}^{c,\vee} \otimes \epsilon^{1-2n}$, and the second projection gives $\rho_{\mathfrak{m}}$).

On the other hand, the map $\widetilde{A}[G_{F,S}] \to (A/J)[G_{F,S}]/(\ker \widetilde{D}_{A/J})$ factors through the quotient $\widetilde{A}[G_{F,S}]/(\ker \widetilde{D})$. There is an algebra embedding

$$\widetilde{A}[G_{F,S}]/(\ker \widetilde{D}) \subset \widetilde{A}[G_{F,S}]/(\ker \widetilde{D}) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \subset M_{2n}(\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p).$$

The explicit form of $\widetilde{\rho}|_{G_{F_v}}$ shows that for each $v \in S_p$ and for each sequence of elements $Y, Y_1, \dots, Y_{2n}$ of elements of $\widetilde{A}[G_{F_v}]$, we have

$$\det(X - \widetilde{\rho}(Y)) = \prod_{j=1}^{2n} (X - \psi_{v,j}(Y)) \tag{5.4.20}$$

in $\widetilde{A}[X]$ and

$$(\widetilde{\rho}(Y_1) - \psi_{v,1}(Y_1))(\widetilde{\rho}(Y_2) - \psi_{v,2}(Y_2)) \dots (\widetilde{\rho}(Y_{2n}) - \psi_{v,2n}(Y_{2n})) = 0 \tag{5.4.21}$$

in $M_{2n}(\widetilde{A} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p)$. It follows that the same identities hold in $\widetilde{A}[G_{F,S}]/(\ker \widetilde{D})$, hence in

$$(A/J)[G_{F,S}]/(\ker \widetilde{D}_{A/J}) = M_n(A/J) \times M_n(A/J).$$

More precisely, for any sequence of elements $Y, Y_1, \dots, Y_{2n}$ of elements of $(A/J)[G_{F_v}]$, we have

$$\det(X - \rho_{\mathfrak{m}}(Y)) \det(X - \rho_{\mathfrak{m}}^{c,\vee} \epsilon^{1-2n}(Y)) = \prod_{j=1}^{2n} (X - \psi_{v,j}(Y)) \tag{5.4.22}$$

in $(A/J)[X]$ and

$$\left(\prod_{j=1}^{2n}(\rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n}(Y_j)-\psi_{v,j}(Y_j)),\prod_{j=1}^{2n}(\rho_{\mathfrak{m}}(Y_j)-\psi_{v,j}(Y_j))\right)=(0,0) \tag{5.4.23}$$

in $M_n(A/J)\times M_n(A/J)$. (Note that order matters in these products.) We need to show how to deduce our desired identities (b) and (c) from these ones. We now fix a choice of place $v\in S_p$ for the rest of the proof.

We can find an element $e\in(A/J)[G_{F_v}]$ which acts as 0 on $\overline{\rho}_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n}|_{G_{F_v}}$ and as the identity in $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ (because these two representations have no Jordan–Hölder factors in common). By [Bou61, Ch. III, §4, Exercise 5(b)] (lifting idempotents), we can assume that $\rho_{\mathfrak{m}}(e)=1$ and $\rho_{\mathfrak{m}}^{c,\vee}\otimes\epsilon^{1-2n}(e)=0$, and moreover that $\psi_{v,j}(e)=1$ if $\overline{\psi}_{v,j}$ appears in $\overline{\rho}_{\mathfrak{m}}|_{G_{F_v}}$ (in other words, if $\overline{\psi}_{v,j}=\overline{\chi}_{v,j'}$ for some $1\le j'\le n$, or equivalently if $j=w_i^{-1}(n+k)$ for some $1\le k\le n$), and $\psi_{v,j}(e)=0$ otherwise. Then applying the identity (5.4.22) to $ge\in(A/J)[G_{F_v}]$ gives

$$X^n\det(X-\rho_{\mathfrak{m}}(g))=\prod_{j=1}^{2n}(X-\psi_{v,j}(ge))=X^n\prod_{j=1}^{n}(X-\chi_{v,j}(g)),$$

which is the sought-after property (b) of the proposition. To get property (c), let $g_1,\dots,g_n\in G_{F_v}$, and let $Y_1,\dots,Y_{2n}\in(A/J)[G_{F_v}]$ be defined by $Y_j=e$ if $j\in w_i^{-1}(\{1,\dots,n\})$, and $Y_j=g_ke$ if $j=w_i^{-1}(n+k)$, $k\in\{1,\dots,n\}$. The identity (5.4.23) then becomes

$$(0,(\rho_{\mathfrak{m}}(g_1)-\chi_{v,1}(g_1))(\rho_{\mathfrak{m}}(g_2)-\chi_{v,2}(g_2))\dots(\rho_{\mathfrak{m}}(g_n)-\chi_{v,n}(g_n)))=(0,0)$$

in $M_n(A/J)\times M_n(A/J)$. This completes the proof. □

5.5. *The end of the proof.* We can now complete the proof of the main result of this chapter (Theorem 5.5.1). For the convenience of the reader, we repeat the statement here. We recall our standing hypothesis in this chapter that F contains an imaginary quadratic field in which p splits.

THEOREM 5.5.1. *Suppose that $[F^+:\mathbf{Q}]>1$. Let $K\subset\mathrm{GL}_n(\mathbf{A}_F^\infty)$ be a good subgroup such that for each place $v\in S_p$ of F, $K_v=\mathrm{Iw}_v$. Let $c\ge b\ge 0$ be integers with $c\ge 1$, let $\lambda\in(\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$, and let $\mathfrak{m}\subset\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}$ be a non-Eisenstein maximal ideal. Suppose that the following conditions are satisfied*:

(1) *Let v be a finite place of F not contained in S, and let l be its residue characteristic. Then either S contains no l-adic places of F and l is unramified in F, or there exists an imaginary quadratic field $F_0\subset F$ in which l splits.*
(2) *$\overline{\rho}_{\mathfrak{m}}$ is decomposed generic.*

Then we can find an integer $N \geq 1$, which depends only on $[F^+ : \mathbf{Q}]$ and n, an ideal $J \subset \mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}$ such that $J^N = 0$, and a continuous representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}/J)$$

satisfying the following conditions:

(a) *For each finite place $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $(\mathbf{T}^S(K(b,c),\lambda)^{\mathrm{ord}}_{\mathfrak{m}}/J)[X]$.*

(b) *For each $v \in S_p$, and for each $g \in G_{F_v}$, the characteristic polynomial of $\rho_{\mathfrak{m}}(g)$ equals $\prod_{i=1}^n (X - \chi_{\lambda,v,i}(g))$.*

(c) *For each $v \in S_p$, and for each $g_1, \dots, g_n \in G_{F_v}$, we have*

$$(\rho_{\mathfrak{m}}(g_1) - \chi_{\lambda,v,1}(g_1))(\rho_{\mathfrak{m}}(g_2) - \chi_{\lambda,v,2}(g_2)) \dots (\rho_{\mathfrak{m}}(g_n) - \chi_{\lambda,v,n}(g_n)) = 0.$$

Proof. Let $0 \leq q \leq d-1$, $m \geq 1$ be integers, and define

$$A(K,\lambda,q) = \mathbf{T}^{S,\mathrm{ord}}(H^q(X_{K(b,c)},\mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}})$$

and

$$A(K,\lambda,q,m) = \mathbf{T}^{S,\mathrm{ord}}(H^q(X_{K(b,c)},\mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}}_{\mathfrak{m}}).$$

By the same sequence of reductions as in the proof of Theorem 4.5.1, it is enough to show the existence of an ideal $J \subset A(K,\lambda,q,m)$ satisfying $J^N = 0$ and a continuous representation $\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(A(K,\lambda,q,m)/J)$ satisfying conditions (a), (b) and (c) of the theorem. After an application of the Hochschild–Serre spectral sequence and Corollary 5.2.16, we can assume that $c = b \geq m$. Corollary 5.2.18 allows us to assume that λ is the weight whose existence is asserted by Proposition 5.4.18. The existence of a Galois representation valued in (quotients by nilpotent ideals of) the Hecke algebras $A(K,\lambda,q)$ and $A(K,\lambda,q+1)$ is then a consequence of Proposition 5.4.18. The existence of the short exact sequence of $\mathbf{T}^{S,\mathrm{ord}}$-modules

$$0 \to H^q(X_{K(b,c)},\mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}}/\varpi^m \to H^q(X_{K(b,c)},\mathcal{V}_\lambda/\varpi^m)^{\mathrm{ord}}_{\mathfrak{m}} \\ \to H^{q+1}(X_{K(b,c)},\mathcal{V}_\lambda)^{\mathrm{ord}}_{\mathfrak{m}}[\varpi^m] \to 0$$

then implies the existence of a Galois representation $\rho_{\mathfrak{m}}$ over a quotient of $A(K,\lambda,q,m)$ by a nilpotent ideal with the required properties. □

As suggested by a referee, we finish this section by recording a local-global compatibility result for a single automorphic representation. This is a partial generalization of [Ger19, Prop. 5.10] and [Tho15, Th. 2.4], although we must impose an assumption on the residual Galois representation. In this result, we drop the standing hypothesis that F contains an imaginary quadratic field in which p splits.

Corollary 5.5.2. *Let F be an imaginary CM field, let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, regular algebraic of weight $\iota\lambda$ for $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$. Suppose that*

(1) *For every $v \in S_p$, π is ι-ordinary at v (in the sense of* [Ger19, Def. 5.3]).
(2) *The residual representation $\overline{r_\iota(\pi)}$ is decomposed generic and irreducible.*

Then $r_\iota(\pi)|_{G_{F_v}}$ is ordinary of weight λ, in the sense of [Ger19, §5.2], *for every $v \in S_p$. More precisely, for each place $v \in S_p$, there is an isomorphism*

$$r_\iota(\pi)|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \psi_{v,n} \end{pmatrix},$$

where for each $i = 1, \dots, n$, $\psi_{v,i} : G_{F,v} \to \overline{\mathbf{Q}}_p^\times$ is the unique continuous character satisfying the identities (cf. the definition of $\chi_{\lambda,v,i}$ in Section 5.1):

$$\psi_{\lambda,v,i} \circ \mathrm{Art}_{F_v}(u) = \epsilon^{1-i}(\mathrm{Art}_{F_v}(u)) \left(\prod_\tau \tau(u)^{-(w_0^G\lambda)_{\tau,i}} \right) \langle u \rangle_{\iota,i} \ (u \in \mathcal{O}_{F_v}^\times)$$

(the product being over $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)$) and, with fixed choices of uniformizers ϖ_v for $v \in S_p$,

$$\psi_{v,i} \circ \mathrm{Art}_{F_v}(\varpi_v) = \epsilon^{1-i}(\mathrm{Art}_{F_v}(\varpi_v)) \frac{u_{\lambda,\varpi_v}^{(i)}}{u_{\lambda,\varpi_v}^{(i-1)}},$$

with $\langle u \rangle_{\iota,i}$ and $u_{\lambda,\varpi_v}^{(i)}$ denoting Hecke eigenvalues on $(\iota^{-1}\pi_v)^{\mathrm{ord}}$ defined in [Ger19, Def. 5.5].

Proof. We make a solvable Galois base change to a CM field extension F'/F which is disjoint over F from the fixed field $\overline{F}^{\ker \overline{r_\iota(\pi)}}$, contains an imaginary quadratic field in which p splits, and in which all the places in S_p split completely. We will also assume that $[F' : \mathbf{Q}] > 2$. Using [Ger19, Lem. 5.7], we see that $\pi_{F'}$ is ι-ordinary at w for every place $w|p$ of F' and it suffices to prove the corollary under the additional assumptions that $[F^+ : \mathbf{Q}] > 1$ and F contains an imaginary quadratic field in which p splits. Now the result follows from Theorem 5.5.1 and Lemma 6.2.11. □

6. **Automorphy lifting theorems**

6.1. *Statements.* In this chapter, we will prove two automorphy lifting theorems (Theorems 6.1.1 and 6.1.2) for n-dimensional Galois representations of CM fields without imposing a self-duality condition. The first is for Galois representations which satisfy a Fontaine–Laffaille condition.

THEOREM 6.1.1. *Let F be an imaginary CM or totally real field, let $c \in \mathrm{Aut}(F)$ be complex conjugation, and let p be a prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) *ρ is unramified almost everywhere.*
(2) *For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is crystalline. The prime p is unramified in F.*
(3) *$\overline{\rho}$ is absolutely irreducible and decomposed generic (Definition 4.3.1). The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is enormous (Definition 6.2.29).*
(4) *There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n^2$.*
(5) *There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:*
   (a) *π is regular algebraic of weight λ, this weight satisfying*
   $$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n$$
   for all τ.
   (b) *There exists an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{\rho} \cong \overline{r_\iota(\pi)}$ and the Hodge–Tate weights of ρ satisfy the formula for each $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$:*
   $$HT_\tau(\rho) = \{\lambda_{\iota\tau,1} + n - 1, \lambda_{\iota\tau,2} + n - 2, \ldots, \lambda_{\iota\tau,n}\}.$$
   (c) *If $v|p$ is a place of F, then π_v is unramified.*

Then ρ is automorphic: there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ such that $\rho \cong r_\iota(\Pi)$. Moreover, if v is a finite place of F and either $v|p$ or both ρ and π are unramified at v, then Π_v is unramified.

The second main theorem is for Galois representations which satisfy an ordinariness condition.

THEOREM 6.1.2. *Let F be an imaginary CM or totally real field, let $c \in \mathrm{Aut}(F)$ be complex conjugation, and let p be a prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) *ρ is unramified almost everywhere.*
(2) *For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is potentially semi-stable, ordinary with regular Hodge–Tate weights. In other words, there exists a weight $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ such that for each place $v|p$, there is an isomorphism*
   $$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \psi_{v,n} \end{pmatrix},$$

where for each $i = 1, \dots, n$, the character $\psi_{v,i} : G_{F_v} \to \overline{\mathbf{Q}}_p^\times$ agrees with the character

$$\sigma \in I_{F_v} \mapsto \prod_{\tau \in \mathrm{Hom}(F_v, \overline{\mathbf{Q}}_p)} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$

on an open subgroup of the inertia group I_{F_v}.

(3) *$\overline{\rho}$ is absolutely irreducible and decomposed generic (Definition 4.3.1). The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is enormous (Definition 6.2.29).*

(4) *There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n$.*

(5) *There exist a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that π is ι-ordinary and $\overline{r_\iota(\pi)} \cong \overline{\rho}$.*

Then ρ is ordinarily automorphic of weight $\iota\lambda$: there exists an ι-ordinary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$ such that $\rho \cong r_\iota(\Pi)$. Moreover, if $v \nmid p$ is a finite place of F and both ρ and π are unramified at v, then Π_v is unramified.

Remark 6.1.3. It follows from the existence of Π that the weight λ is conjugate self-dual up to twist: there is an integer $w \in \mathbf{Z}$ such that for all $\tau : F \hookrightarrow \mathbf{C}$ and for each $i = 1, \dots, n$, we have $\lambda_{\tau,i} + \lambda_{\tau c, n+1-i} = w$. (This in turn is a consequence of the purity lemma of [Clo90, Lem. 4.9].) However, we do not need to assume this at the outset. What we in fact prove is that ρ contributes to the ordinary part of the completed cohomology; we then deduce the existence of Π by an argument of "independence of weight."

Remark 6.1.4. The image of the projective representation $\mathrm{P}\overline{\rho}$ coincides with the image of the adjoint representation $\mathrm{ad}\,\overline{\rho}$. Hence the first part of conditions Theorems 6.1.1(4) and 6.1.2(4) are equivalent to $\zeta_p \notin \overline{F}^{\ker \mathrm{ad}\,\overline{\rho}}$. If p is unramified in F (as in condition (2) of Theorem 6.1.1), it is implied by the non-existence of a surjection $(\mathrm{ad}\,\overline{\rho})(G_F) \twoheadrightarrow (\mathbf{Z}/p\mathbf{Z})^\times$. It may be possible to remove the requirement of such a σ by using arguments similar to those of [Tho12], in particular, by adding Iwahori level structure at a prime which is not $1 \mod p$ and then using [Tho12, Prop. 3.17]. However, this would (at least) necessitate some modifications to the Ihara avoidance arguments of Section 6.3, and so we have not attempted to do this, especially because condition (4) is usually easy to verify in practice.

The proof of these two theorems will occupy the rest of this chapter. Since this chapter is quite long, we now discuss the structure of the proof. We recall that the authors of [CG18] implemented a generalization of the Taylor–Wiles method in situations where the "numerical coincidence" fails to hold, assuming the existence of Galois representations associated to torsion classes in the

cohomology of arithmetic locally symmetric spaces, and an appropriate form of local-global compatibility for these Galois representations. They also had to assume that the cohomology groups vanish in degrees outside a given range, after localization at a non-Eisenstein maximal ideal. (This range is the same range in which cohomological cuspidal automorphic representations of GL_n contribute non-trivially.) Under these assumptions, they proved rather general automorphy lifting theorems; in particular, they were able to implement the "Ihara avoidance" trick of [Tay08] to obtain lifting results at non-minimal level.

There are a few innovations that allow us to obtain unconditional results here, building on the techniques of [CG18]. The first is the proof (in the preceding sections) of a sufficiently strong version of local-global compatibility for the torsion Galois representations constructed in [Sch15]. The second is the observation that one can carry out a version of the "Ihara avoidance" trick under somewhat weaker assumptions than those used in [CG18]. Indeed, in [KT17], it was shown that one can prove some kind of automorphy lifting results using only that the rational cohomology is concentrated in the expected range — and this is known unconditionally, by Matsushima's formula and its generalizations (in particular, Theorem 2.4.10). Here we show that the "Ihara avoidance" technique is robust enough to give a general automorphy lifting result using only the assumption that the rational cohomology is concentrated in the expected range.

We now describe the organization of this chapter. As the above discussion may suggest, our arguments are rather intricate, and we have broken them into several parts in the hope that this will make the individual steps easier to digest. We begin in Section 6.2 by giving a set-up for Galois deformation theory. This is mostly standard, although there are some differences to other works: we do not fix the determinant of our n-dimensional Galois representations, and we must prove slightly stronger versions of our auxiliary results (e.g., existence of Taylor–Wiles primes) because of the hypotheses required elsewhere to be able to prove local-global compatibility.

In Sections 6.3 and 6.4, we carry out the main technical steps. First, in Section 6.3, we give an axiomatic approach to the "Ihara avoidance" technique that applies in our particular set-up. Second, in Section 6.4, we describe an abstract patching argument that gives as output the objects required in Section 6.3. We find it convenient to use the language of ultrafilters here, following [Sch18] and [GN22]. Finally, in Section 6.5, we combine these arguments to prove Theorems 6.1.1 and 6.1.2

6.2. *Galois deformation theory.* Let $E \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$, with valuation ring $\mathcal{O}$, uniformizer ϖ, and residue field k. Given a complete Noetherian local $\mathcal{O}$-algebra Λ with residue field k, we let CNL_Λ denote the

category of complete Noetherian local Λ-algebras with residue field k. We refer to an object in CNL_Λ as a CNL_Λ-algebra.

We fix a number field F and let S_p be the set of places of F above p. We assume that E contains the images of all embeddings of F in $\overline{\mathbf{Q}}_p$. We also fix a continuous absolutely irreducible homomorphism $\overline{\rho}\colon G_F \to \mathrm{GL}_n(k)$. We assume throughout that $p \nmid 2n$.

6.2.1. *Deformation problems.* Let S be a finite set of finite places of F containing S_p and all places at which $\overline{\rho}$ is ramified. We write F_S for the maximal subextension of $\overline{F}/F$ which is unramified outside S. For each $v \in S$, we fix $\Lambda_v \in \mathrm{CNL}_{\mathcal{O}}$ and set $\Lambda = \widehat{\otimes}_{v\in S}\Lambda_v$, where the completed tensor product is taken over $\mathcal{O}$. There is a forgetful functor $\mathrm{CNL}_\Lambda \to \mathrm{CNL}_{\Lambda_v}$ for each $v \in S$ via the canonical map $\Lambda_v \to \Lambda$. A *lift* (also called a *lifting*) of $\overline{\rho}|_{G_{F_v}}$ is a continuous homomorphism $\rho\colon G_{F_v} \to \mathrm{GL}_n(A)$ to a CNL_{Λ_v}-algebra A such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}|_{G_{F_v}}$.

We let $\mathcal{D}_v^\square$ denote the set valued functor on CNL_{Λ_v} that sends A to the set of all lifts of $\overline{\rho}|_{G_{F_v}}$ to A. This functor is representable, and we denote the representing object by $R_v^\square$.

A *local deformation problem* for $\overline{\rho}|_{G_{F_v}}$ is a subfunctor $\mathcal{D}_v$ of $\mathcal{D}_v^\square$ satisfying the following:

- $\mathcal{D}_v$ is represented by a quotient R_v of $R_v^\square$.
- For all $A \in \mathrm{CNL}_{\Lambda_v}$, $\rho \in \mathcal{D}_v(A)$, and $a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$, we have $a\rho a^{-1} \in \mathcal{D}_v(A)$.

The notion of global deformation problem that we use in this paper is the following:

Definition 6.2.2. A *global deformation problem* is a tuple

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}),$$

where

- $\overline{\rho}$, S, and $\{\Lambda_v\}_{v\in S}$ are as above;
- for each $v \in S$, $\mathcal{D}_v$ is a local deformation problem for $\overline{\rho}|_{G_{F_v}}$.

This differs from that of [CG18, §8.5.2] and [KT17, Def. 4.2] in that we do not fix the determinant.

As in the local case, a *lift* (or *lifting*) of $\overline{\rho}$ is a continuous homomorphism $\rho\colon G_F \to \mathrm{GL}_n(A)$ to a CNL_Λ-algebra A, such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}$. We say that two lifts $\rho_1, \rho_2\colon G_F \to \mathrm{GL}_n(A)$ are *strictly equivalent* if there is $a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ such that $\rho_2 = a\rho_1 a^{-1}$. A *deformation of* $\overline{\rho}$ is a strict equivalence class of lifts of $\overline{\rho}$.

For a global deformation problem

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}),$$

we say that a lift $\rho\colon G_F \to \mathrm{GL}_n(A)$ is of *type* $\mathcal{S}$ if $\rho|_{G_{F_v}} \in \mathcal{D}_v(A)$ for each $v \in S$. Note that if ρ_1 and ρ_2 are strictly equivalent lifts of $\overline{\rho}$, and ρ_1 is of type $\mathcal{S}$, then so is ρ_2. A *deformation of type* $\mathcal{S}$ is then a strict equivalence class of lifts of type $\mathcal{S}$, and we denote by $\mathcal{D}_{\mathcal{S}}$ the set-valued functor that takes a CNL_Λ-algebra A to the set of deformations $\rho\colon G_F \to \mathrm{GL}_n(A)$ of type $\mathcal{S}$.

Given a subset $T \subseteq S$, a *T-framed lift of type* $\mathcal{S}$ is a tuple $(\rho, \{\alpha_v\}_{v\in T})$, where $\rho\colon G_F \to \mathrm{GL}_n(A)$ is a lift of $\overline{\rho}$ of type $\mathcal{S}$ and $\alpha_v \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ for each $v \in T$. We say that two T-framed lifts $(\rho_1, \{\alpha_v\}_{v\in T})$ and $(\rho_2, \{\beta_v\}_{v\in T})$ to a CNL_Λ-algebra A are strictly equivalent if there is $a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ such that $\rho_2 = a\rho_1 a^{-1}$, and $\beta_v = a\alpha_v$ for each $v \in T$. A strict equivalence class of T-framed lifts of type $\mathcal{S}$ is called a *T-framed deformation of type* $\mathcal{S}$. We denote by $\mathcal{D}_{\mathcal{S}}^T$ the set valued functor that sends a CNL_Λ-algebra A to the set of T-framed deformations to A of type $\mathcal{S}$.

THEOREM 6.2.3. *Let $\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and let T be a subset of S. The functors $\mathcal{D}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}^T$ are representable; we denote their representing objects by $R_{\mathcal{S}}$ and $R_{\mathcal{S}}^T$, respectively.*

Proof. This is well known. See [Gou01, App. 1] for a proof of the representability of $\mathcal{D}_{\mathcal{S}}$. The representability of $\mathcal{D}_{\mathcal{S}}^T$ can be deduced from this. □

If $T = \emptyset$, then tautologically $R_{\mathcal{S}} = R_{\mathcal{S}}^T$. Otherwise, the relation between these two deformation rings is given by the following lemma.

LEMMA 6.2.4. *Let $\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and let T be a nonempty subset of S. Fix some $v_0 \in T$, and define $\mathcal{T} = \mathcal{O}[\![\{X_{v,i,j}\}_{v\in T, 1\le i,j\le n}]\!]/(X_{v_0,1,1})$. The choice of a representative $\rho_{\mathcal{S}}\colon G_F \to \mathrm{GL}_n(R_{\mathcal{S}})$ for the universal type $\mathcal{S}$ deformation determines a canonical isomorphism $R_{\mathcal{S}}^T \cong R_{\mathcal{S}}\widehat{\otimes}_{\mathcal{O}}\mathcal{T}$.*

Proof. This can be proved in the same way as the second part of [CHT08], using Schur's lemma. A representative for the universal T-framed deformation over $R_{\mathcal{S}}\widehat{\otimes}_{\mathcal{O}}\mathcal{T}$ is $(\rho_{\mathcal{S}}, \{1 + (X_{v,i,j})\}_{v\in T})$. □

6.2.5. *Some local deformation problems.* We now fix some finite place v of F and introduce the local deformation rings that we will use in the proofs of our automorphy lifting theorems.

6.2.6. *Ordinary deformations.* Assume that $v|p$, and that there is an increasing filtration

$$0 = \overline{\mathrm{Fil}}_v^0 \subset \overline{\mathrm{Fil}}_v^1 \subset \cdots \subset \overline{\mathrm{Fil}}_v^n = k^n$$

that is G_{F_v}-stable under $\overline{\rho}|_{G_{F_v}}$ with one-dimensional graded pieces. We will construct and study a local deformation ring $R_v^{\mathrm{det,ord}}$ whose corresponding

local deformation problem $\mathcal{D}_v^{\mathrm{det,ord}}$ will be used in the proof of our ordinary automorphy lifting theorem.

Consider the completed group algebra $\mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$, where $\mathcal{O}_{F_v}^\times(p)$ denotes the pro-p completion of $\mathcal{O}_{F_v}^\times$. There is an isomorphism $\mathrm{Art}_{F_v} : \mathcal{O}_{F_v}^\times(p) \to I_{F_v^{ab}/F_v}(p)$. Fix a non-empty set of minimal prime ideals of $\mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$, and let $\mathfrak{a}$ be their intersection. We then set $\Lambda_v = \mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]/\mathfrak{a}$.

For each $1 \le i \le n$, let $\overline{\widetilde{\chi}}_i \colon G_{F_v} \to k^\times$ denote the character given by $\overline{\rho}|_{G_{F_v}}$ on $\overline{\mathrm{Fil}}_v^i/\overline{\mathrm{Fil}}_v^{i-1}$, and let $\overline{\chi}_i = \overline{\widetilde{\chi}}_i|_{I_{F_v}}$. For each $1 \le i \le n$, we have a canonical character $\chi_i^{\mathrm{univ}} \colon I_{F_v} \to \Lambda_v^\times$ that is the product of the Teichmüller lift of $\overline{\chi}_i$ with the map that sends I_{F_v} to the ith copy of $\mathcal{O}_{F_v}^\times(p)$ in $\mathcal{O}_{F_v}^\times(p)^n$ via $\mathrm{Art}_{F_v}^{-1}$. The ideal $\mathfrak{a}$ corresponds to a fixed collection of ordered tuples of characters of the torsion subgroup of $I_{F_v^{ab}/F_v}(p)$.

We recall some constructions from [Ger19, §3.1]. We recall that $R_v^\square \in \mathrm{CNL}_{\Lambda_v}$ denotes the universal lifting ring of $\overline{\rho}|_{G_{F_v}}$. Let $\mathcal{F}$ denote the flag variety over $\mathcal{O}$ classifying complete flags $0 = \mathrm{Fil}^0 \subset \cdots \subset \mathrm{Fil}^n = \mathcal{O}^n$, and let $\mathcal{G}_v \subset \mathcal{F} \times_{\mathrm{Spec}\,\mathcal{O}} \mathrm{Spec}\, R_v^\square$ denote the closed subscheme whose A-points for an $R_v^\square$-algebra A consist of those filtrations $\mathrm{Fil} \in \mathcal{F}(A)$ such that for each $i = 1, \dots, n$, Fil^i is preserved by the specialization of the universal lifting to A and such that the induced action of $I_{F_v} \subset G_{F_v}$ on $\mathrm{Fil}^i / \mathrm{Fil}^{i-1}$ is by the pushforward of the character χ_i^{univ}.

We now define two ordinary deformation rings:

- We define $R_v^\triangle$ to be the image of the homomorphism
$$R_v^\square \to H^0(\mathcal{G}_v, \mathcal{O}_{\mathcal{G}_v}).$$
- Let $\widetilde{\Lambda}_v = \mathcal{O}[\![F_v^\times(p)^n]\!] \otimes_{\mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]} \Lambda_v$, and let $\widetilde{R}_v^\square = R_v^\square \otimes_{\Lambda_v} \widetilde{\Lambda}_v$. The characters χ_i^{univ} naturally extend to characters $\widetilde{\chi}_i^{\mathrm{univ}} : G_{F_v} \to \widetilde{\Lambda}_v^\times$ lifting $\overline{\widetilde{\chi}}_i$. Let $\widetilde{R}_v^{\mathrm{det,ord}}$ denote the maximal quotient of $\widetilde{R}_v^\square$ over which the relations
$$\det(X - \rho^\square(g)) = \prod_{i=1}^n (X - \widetilde{\chi}_i^{\mathrm{univ}}(g)) \tag{6.2.7}$$
and
$$(\rho^\square(g_1) - \widetilde{\chi}_1^{\mathrm{univ}}(g_1))(\rho^\square(g_2) - \widetilde{\chi}_2^{\mathrm{univ}}(g_2)) \dots (\rho^\square(g_n) - \widetilde{\chi}_n^{\mathrm{univ}}(g_n)) = 0 \tag{6.2.8}$$
hold for all $g, g_1, \dots, g_n \in G_{F_v}$. We define $R_v^{\mathrm{det,ord}}$ to be the image of the homomorphism
$$R_v^\square \to \widetilde{R}_v^{\mathrm{det,ord}}.$$

(A ring similar to $R_v^{\mathrm{det,ord}}$ was also defined in [CS19].)

Lemma 6.2.9. *Let $\widetilde{R}_v^{\mathrm{det,ord}}$ be a finite $R_v^{\mathrm{det,ord}}$-algebra.*

Proof. It is enough to show that $\widetilde{R}_v^{\det,\mathrm{ord}}$ is a finite $R_v^{\square}$-algebra or, by the completed version of Nakayama's lemma, that $\widetilde{R}_v^{\det,\mathrm{ord}}/\mathfrak{m}_{R_v^{\square}}$ is an Artinian k-algebra. This follows from the relation (6.2.7) applied with $g = \mathrm{Art}_{F_v}(\varpi_v)$. $\square$

For a domain $R \in \mathrm{CNL}_{\Lambda_v}$ and K an algebraic closure of the fraction field of R, an R-point of $\operatorname{Spec} R_v^{\square}$ factors through $\operatorname{Spec} R_v^{\triangle}$ if and only if the following condition is satisfied:

- Let $\rho\colon G_{F_v} \to \mathrm{GL}_n(R)$ be the pushforward of the universal lifting to R. Then there is a filtration $0 = \mathrm{Fil}^0 \subset \cdots \subset \mathrm{Fil}^n = K^n$ on $\rho \otimes_R K$ which is preserved by G_{F_v}, and such that the action of I_{F_v} on $\mathrm{Fil}^i / \mathrm{Fil}^{i-1}$ $(i = 1, \ldots, n)$ is given by the push-forward of the universal character χ_j^{univ} to R.

On the other hand, suppose that $R \to S$ is an injective morphism of $R_v^{\square}$-algebras, and suppose that there exist characters $\psi_1, \ldots, \psi_n : G_{F_v} \to S^\times$ such that for each $i = 1, \ldots, n$, $\psi_i|_{I_{F_v}}$ equals the pushforward of χ_i^{univ} to S, and that for each $g, g_1, \ldots, g_n \in G_{F_v}$, the analogues of the relations (6.2.7) and (6.2.8) for the characters ψ_i and the pushforward of the universal lifting hold in S. Then $R_v^{\square} \to R$ factors through $R_v^{\det,\mathrm{ord}}$. We see, in particular, that there is an inclusion of topological spaces $\operatorname{Spec} R_v^{\triangle} \subset \operatorname{Spec} R_v^{\det,\mathrm{ord}}$. Indeed, applying the above for $R = R_v^{\triangle}/\mathfrak{p}$, where $\mathfrak{p}$ is a minimal prime of $R_v^{\triangle}$, and S its integral closure in a sufficiently large finite extension of its fraction field, we see that $R_v^{\square} \to R_v^{\triangle}/\mathfrak{p}$ factors through $R_v^{\det,\mathrm{ord}}$. Since the maximal reduced quotient $(R_v^{\triangle})_{\mathrm{red}}$ of $R_v^{\triangle}$ is the image of the map $R_v^{\square} \to \prod_{\mathfrak{p}} R_v^{\triangle}/\mathfrak{p}$, we deduce that there is a surjection of $R_v^{\square}$-algebras $R_v^{\det,\mathrm{ord}} \twoheadrightarrow (R_v^{\triangle})_{\mathrm{red}}$.

The ring $R_v^{\triangle}$ was introduced in [Ger19]. Its properties in an important special case are summarized in the following proposition.

Proposition 6.2.10. *If $[F_v : \mathbf{Q}_p] > \frac{n(n-1)}{2} + 1$ and $\overline{\rho}|_{G_{F_v}}$ is trivial, then $R_v^{\triangle}$ is $\mathcal{O}$-flat, reduced and equidimensional of dimension $1 + n^2 + \frac{n(n+1)}{2}[F_v : \mathbf{Q}_p]$. Moreover, the map $\operatorname{Spec} R_v^{\triangle} \to \operatorname{Spec} \Lambda_v$ is bijective on the level of generic points, hence on the level of irreducible components.*

Proof. This is essentially contained in [Tho15, Prop. 3.14]. More precisely, that reference proves the proposition under the assumption that $\Lambda_v = \mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$, but also shows that minimal prime ideals of Λ_v generate minimal prime ideals of $R_v^{\triangle}$. The more general case where Λ_v is allowed to be a quotient of $\mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$ by the intersection of an arbitrary collection of minimal prime ideals follows from this. $\square$

Our analysis of the ring $R_v^{\det,\mathrm{ord}}$ will be coarser. It begins with the following lemma:

LEMMA 6.2.11. *Let K be a field, G a group and $\rho : G \to \mathrm{GL}_n(K)$ a representation. Suppose that there exist pairwise distinct characters $\chi_1, \dots, \chi_n : G \to K^\times$ satisfying the following conditions*:

(1) *For all $g \in G$,*

$$\det(X - \rho(g)) = \prod_{i=1}^{n} (X - \chi_i(g)).$$

(2) *For all $g_1, \dots, g_n \in G$,*

$$(\rho(g_1) - \chi_1(g_1))(\rho(g_2) - \chi_2(g_2)) \cdots (\rho(g_n) - \chi_n(g_n)) = 0.$$

Then there is a filtration $0 = \mathrm{Fil}^0 \subset \cdots \subset \mathrm{Fil}^n = K^n$ by G-stable subspaces such that for each $i = 1, \dots, n$, $\mathrm{Fil}^i / \mathrm{Fil}^{i-1} \cong K(\chi_i)$.

Proof. We define subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset V = K^n$ be declaring that for each $i = 1, \dots, n$, V_i/V_{i-1} is the maximal subspace of V/V_{i-1} where G acts by the character χ_i. Each V_i is G-stable, and the second condition of the lemma implies that $V_n = V$. On the other hand, each V_i/V_{i-1} is isomorphic to $K(\chi_i)^{\dim_K V_i/V_{i-1}}$. The first condition of the lemma implies that we must therefore have $\dim_K V_i/V_{i-1} = 1$ for each $i = 1, \dots, n$. The proof is complete on taking $\mathrm{Fil}^i = V_i$. □

Let $U \subset \operatorname{Spec} \Lambda_v$ be the open subscheme where the characters $\chi_1^{\mathrm{univ}}, \dots, \chi_n^{\mathrm{univ}}$ are pairwise distinct, and let Z denote its complement.

PROPOSITION 6.2.12. *Let $f : \operatorname{Spec} R_v^{\triangle} \to \operatorname{Spec} \Lambda_v$, $g : \operatorname{Spec} R_v^{\det,\mathrm{ord}} \to \operatorname{Spec} \Lambda_v$ be the structural maps. Suppose that $\overline{\rho}|_{G_{F_v}}$ is trivial and that $[F_v : \mathbf{Q}_p] > \frac{n(n+1)}{2} + 1$.*

(1) *We have $f^{-1}(U) = g^{-1}(U)$ as subspaces of $\operatorname{Spec} R_v^{\square}$. Consequently, for each irreducible component C of $\operatorname{Spec} \Lambda_v$, there is a unique irreducible component C' of $\operatorname{Spec} R_v^{\det,\mathrm{ord}}$ which dominates C. It has dimension $n^2 + 1 + \frac{n(n+1)}{2}[F_v : \mathbf{Q}_p]$.*

(2) *Let C' be an irreducible component of $R_v^{\det,\mathrm{ord}}$ which does not dominate an irreducible component of $\operatorname{Spec} \Lambda_v$. Then $C' \subset g^{-1}(Z)$ and C' has dimension at most $n^2 - 1 + \frac{n(n+1)}{2}[F_v : \mathbf{Q}_p]$.*

Proof. We have already observed that there is an inclusion $\operatorname{Spec} R_v^{\triangle} \subset \operatorname{Spec} R_v^{\det,\mathrm{ord}}$. We must first show that if $s : \operatorname{Spec} K \to g^{-1}(U) \subset \operatorname{Spec} R_v^{\det,\mathrm{ord}}$ is a geometric point, then s factors through $\operatorname{Spec} R_v^{\triangle}$. By Lemma 6.2.9, s lifts to a point $s' : \operatorname{Spec} K \to \operatorname{Spec} \widetilde{R}_v^{\det,\mathrm{ord}}$. Then Lemma 6.2.11 shows that s factors through $R_v^{\triangle}$. The first part of the proposition now follows from Proposition 6.2.10, which says that $f|_U$ induces a bijection on generic points, hence on irreducible components.

For the second part, let C' be an irreducible component of $R_v^{\mathrm{det,ord}}$ which does not dominate an irreducible component of $\operatorname{Spec}\Lambda_v$. It follows from the first part that we must have $g(C') \subset Z$. To bound the dimension of C', we claim that there is a permutation $\sigma \in S_n$ such that C' is contained in the closed subspace $h^{-1}(Z)$ of $\operatorname{Spec} R_v^{\triangle,\sigma}$, where $h : \operatorname{Spec} R_v^{\triangle,\sigma} \to \operatorname{Spec}\Lambda_v$ is the quotient of $R_v^{\square}$ which is defined in the same way as $R_v^{\triangle}$, except that we require the action of I_{F_v} on the ith graded piece of the filtration to be by the character $\chi_{\sigma(i)}^{\mathrm{univ}}$. There is a corresponding surjective morphism $\mathcal{G}_v^{\sigma} \to \operatorname{Spec} R_v^{\triangle,\sigma}$. To show the claim, it suffices to check that there is a σ such that a geometric generic point of C' is contained in $\operatorname{Spec} R_v^{\triangle,\sigma}$. To see this, we observe that the Galois representation corresponding to a geometric generic point of C' has semisimplification a direct sum of characters whose restriction to I_v is the push-forward of $\oplus_{i=1}^n \chi_i^{\mathrm{univ}}$. It follows that this representation has a filtration with the Galois action on its graded pieces given by the universal characters in some order.

We thus have

$$\dim C' \leq \dim h^{-1}(Z) \leq \dim \mathcal{G}_v^{\sigma} \times_{\operatorname{Spec}\Lambda_v} Z.$$

We can bound $\dim \mathcal{G}_v^{\sigma} \times_{\operatorname{Spec}\Lambda_v} Z$ by bounding the dimension of the completed local rings at its closed points, using essentially the same tangent space calculation as in [Ger19, Lem. 3.7] (although over a finite field). This yields

$$\begin{aligned}\dim \mathcal{G}_v^{\sigma} \times_{\operatorname{Spec}\Lambda_v} Z &\leq 1 + n^2 + n(n+1)/2 + n(n+1)[F_v : \mathbf{Q}_p]/2 - [F_v : \mathbf{Q}_p] \\ &\leq n^2 - 1 + n(n+1)[F_v : \mathbf{Q}_p]/2,\end{aligned}$$

using our assumption $[F_v : \mathbf{Q}_p] > \frac{n(n+1)}{2} + 1$. This completes the proof. $\square$

6.2.13. *Fontaine–Laffaille deformations.* We again suppose $v|p$, but take $\Lambda_v = \mathcal{O}$. We assume that $F_v/\mathbf{Q}_p$ is unramified. Recall that in Section 4.1 we defined a category $\mathcal{MF}_{\mathcal{O}}$ and a functor $\mathbf{G}$ on $\mathcal{MF}_{\mathcal{O}}$ that take values in the category of finite $\mathcal{O}$-modules with continuous $\mathcal{O}$-linear G_{F_v}-action.

For each embedding $\tau\colon F_v \hookrightarrow E$, let $\lambda_\tau = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})$ be a tuple of integers satisfying

$$\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \cdots \geq \lambda_{\tau,n}$$

and

$$\lambda_{\tau,1} - \lambda_{\tau,n} < p - n.$$

We say a representation of G_{F_v} on a finite $\mathcal{O}$-module W is *Fontaine–Laffaille of type* $(\lambda_\tau)_{\tau \in \operatorname{Hom}(F_v,E)}$ if there exists an $M \in \mathcal{MF}_{\mathcal{O}}$ with $W \cong \mathbf{G}(M)$ such that

$$\mathrm{FL}_\tau(M \otimes_{\mathcal{O}} k) = \{\lambda_{\tau,1} + n - 1, \lambda_{\tau,2} + n - 2, \ldots, \lambda_{\tau,n}\}$$

for each $\tau\colon F_v \hookrightarrow E$. The following proposition follows from [CHT08, §2.4.1] and a twisting argument; see Section 4.1.

PROPOSITION 6.2.14. *Assume that* $\overline{\rho}|_{G_{F_v}}$ *is Fontaine–Laffaille of type* $(\lambda_\tau)_{\tau\in\mathrm{Hom}(F_v,E)}$. *Then there is a quotient* R_v^{FL} *of* $R_v^{\square}$ *satisfying the following*:

(1) R_v^{FL} *represents a local deformation problem* $\mathcal{D}_v^{\mathrm{FL}}$.
(2) *For a* $\mathrm{CNL}_{\mathcal{O}}$*-algebra* A *that is finite over* $\mathcal{O}$, *a lift* $\rho\in\mathcal{D}_v^{\square}(A)$ *lies in* $\mathcal{D}_v^{\mathrm{FL}}$ *if and only if* ρ *is Fontaine–Laffaille of type* $(\lambda_\tau)_{\tau\in\mathrm{Hom}(F_v,E)}$.
(3) R_v^{FL} *is a formally smooth over* $\mathcal{O}$ *of dimension* $1+n^2+\frac{n(n-1)}{2}[F_v:\mathbf{Q}_p]$.

6.2.15. *Level raising deformations.* Assume that $q_v\equiv 1 \bmod p$, that $\overline{\rho}|_{G_{F_v}}$ is trivial, and that $p>n$. We take $\Lambda_v=\mathcal{O}$.

Let $\chi=(\chi_1,\dots,\chi_n)$ be a tuple of continuous characters $\chi_i\colon \mathcal{O}_{F_v}^\times\to\mathcal{O}^\times$ that are trivial modulo ϖ. We let $\mathcal{D}_v^\chi$ be the functor of lifts $\rho\colon G_{F_v}\to\mathrm{GL}_n(A)$ such that

$$\mathrm{char}_{\rho(\sigma)}(X)=\prod_{i=1}^n(X-\chi_i(\mathrm{Art}_{F_v}^{-1}(\sigma)))$$

for all $\sigma\in I_{F_v}$. Then $\mathcal{D}_v^\chi$ is a local deformation problem, and we denote its representing object by R_v^χ. The following two propositions are contained in [Tay08, Prop. 3.1].

PROPOSITION 6.2.16. *Assume that* $\chi_i=1$ *for all* $1\le i\le n$. *Then* R_v^1 *satisfies the following properties*:

(1) $\mathrm{Spec}\,R_v^1$ *is equidimensional of dimension* $1+n^2$ *and every generic point has characteristic zero.*
(2) *Every generic point of* $\mathrm{Spec}\,R_v^1/\varpi$ *is the specialization of a unique generic point of* $\mathrm{Spec}\,R_v^1$.

PROPOSITION 6.2.17. *Assume that the* χ_i *are pairwise distinct. Then* $\mathrm{Spec}\,R_v^\chi$ *is irreducible of dimension* $1+n^2$, *and its generic point has characteristic zero.*

6.2.18. *Taylor–Wiles deformations.* Assume that $q_v\equiv 1 \bmod p$, and that $\overline{\rho}|_{G_{F_v}}$ is unramified. We take $\Lambda_v=\mathcal{O}$. We assume that $\overline{\rho}|_{G_{F_v}}$ has n-distinct eigenvalues $\alpha_1,\dots,\alpha_n\in k$. For each $1\le i\le n$, let $\overline{\gamma}_i\colon G_{F_v}\to k^\times$ be the unramified character that sends Frob_v to α_i.

LEMMA 6.2.19. *Let* $\rho\colon G_{F_v}\to\mathrm{GL}_n(A)$ *be any lift of* $\overline{\rho}$. *There are unique continuous characters* $\gamma_i\colon G_{F_v}\to A^\times$, *for* $1\le i\le n$, *such that* ρ *is* $\mathrm{GL}_n(A)$*-conjugate to a lift of the form* $\gamma_1\oplus\cdots\oplus\gamma_n$, *where* $\gamma_i \bmod \mathfrak{m}_A=\overline{\gamma}_i$ *for each* $1\le i\le n$.

Proof. This is similar to [DDT97, Lem. 2.44]. The details are left to the reader. □

Let $\Delta_v = k(v)^\times(p)^n$, where $k(v)^\times(p)$ is the maximal p-power quotient of $k(v)^\times$. Let $\rho\colon G_{F_v} \to \mathrm{GL}_n(R_v^\square)$ denote the universal lift. Then ρ is $\mathrm{GL}_n(R_v^\square)$-conjugate to a lift of the form $\gamma_1 \oplus \cdots \oplus \gamma_n$, with $\gamma_i \bmod \mathfrak{m}_{R^\square} = \overline{\gamma}_i$. For each $1 \le i \le n$, the character $\gamma_i \circ \mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}^\times}$ factors through $k(v)^\times(p)$, so we obtain a canonical local $\mathcal{O}$-algebra morphism $\mathcal{O}[\Delta_v] \to R_v^\square$. Note that this depends on the choice of ordering $\alpha_1, \ldots, \alpha_n$. It is straightforward to check that this morphism is formally smooth of relative dimension n^2.

6.2.20. *Formally smooth deformations.* Assume that $v \nmid p$. The following is a standard argument in obstruction theory, and the proof is left to the reader.

Proposition 6.2.21. *If $H^2(F_v, \mathrm{ad}\,\overline{\rho}) = 0$, then $R_v^\square$ is isomorphic to a power series ring over $\mathcal{O}$ in n^2 variables.*

6.2.22. *Presentations.* Fix a global deformation problem

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}),$$

and for each $v \in S$, let R_v denote the object representing $\mathcal{D}_v$. Let T be a (possibly empty) subset of S such that $\Lambda_v = \mathcal{O}$ for all $v \in S \smallsetminus T$, and define $R_{\mathcal{S}}^{T,\mathrm{loc}} = \widehat{\otimes}_{v\in T} R_v$, with the completed tensor product being taken over $\mathcal{O}$. It is canonically a Λ-algebra, via the canonical isomorphism $\widehat{\otimes}_{v\in T}\Lambda_v \cong \widehat{\otimes}_{v\in S}\Lambda_v$. For each $v \in T$, the morphism $\mathcal{D}_{\mathcal{S}}^T \to \mathcal{D}_v$ given by $(\rho, \{\alpha_v\}_{v\in T}) \mapsto \alpha_v^{-1}\rho|_{G_{F_v}}\alpha_v$ induces a local Λ_v-algebra morphism $R_v \to R_{\mathcal{S}}^T$. We thus have a local Λ-algebra morphism $R_{\mathcal{S}}^{T,\mathrm{loc}} \to R_{\mathcal{S}}^T$. To understand the relative tangent space of this map, we use a Galois cohomology complex following [CHT08, §2] (cf. [KT17, §4.2]).

We let $\mathrm{ad}\,\overline{\rho}$ denote the space of $n \times n$ matrices $\mathrm{M}_{n\times n}(k)$ over k with adjoint G_F-action via $\overline{\rho}$. For each $v \in S$, we let $Z^1(F_v, \mathrm{ad}\,\overline{\rho})$ denote the k-vector space of continuous 1-cocycles of G_{F_v} with coefficients in $\mathrm{ad}\,\overline{\rho}$. The map $c \mapsto (1+\varepsilon c)\overline{\rho}$ gives an isomorphism

$$Z^1(F_v, \mathrm{ad}\,\overline{\rho}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CNL}_{\Lambda_v}}(R_v^\square, k[\varepsilon]/(\varepsilon^2)).$$

We denote by $\mathcal{L}_v^1$ the pre-image of

$$\mathrm{Hom}_{\mathrm{CNL}_{\Lambda_v}}(R_v, k[\varepsilon]/(\varepsilon^2)) \subseteq \mathrm{Hom}_{\mathrm{CNL}_{\Lambda_v}}(R_v^\square, k[\varepsilon]/(\varepsilon^2))$$

under this isomorphism. Note that $\mathcal{L}_v^1$ contains the subspace of coboundaries. We then let $\mathcal{L}_v$ be the image of $\mathcal{L}_v^1$ in $H^1(F_v, \mathrm{ad}\,\overline{\rho})$.

We define a complex $C^\bullet_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$ by

$$C^i_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = \begin{cases} C^0(F_S/F, \operatorname{ad}\overline{\rho}) & \text{if } i = 0, \\ C^1(F_S/F, \operatorname{ad}\overline{\rho}) \oplus \bigoplus_{v\in T} C^0(F_v, \operatorname{ad}\overline{\rho}) & \text{if } i = 1, \\ C^2(F_S/F, \operatorname{ad}\overline{\rho}) \oplus \bigoplus_{v\in T} C^1(F_v, \operatorname{ad}\overline{\rho}) \oplus_{v\in S\smallsetminus T} C^1(F_v, \operatorname{ad}\overline{\rho})/\mathcal{L}^1_v & \text{if } i = 2, \\ C^i(F_S/F, \operatorname{ad}\overline{\rho}) \oplus \bigoplus_{v\in S} C^{i-1}(F_v, \operatorname{ad}\overline{\rho}) & \text{otherwise,} \end{cases}$$

with boundary map $C^i_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to C^{i+1}_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$ given by

$$(\phi, (\psi_v)_v) \mapsto (\partial\phi, (\phi|_{G_{F_v}} - \partial\psi_v)_v).$$

We denote the cohomology groups of this complex by $H^i_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$ and denote their k-dimension by $h^i_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$. (We use similar notation for the k-dimension of local and global Galois cohomology groups.)

There is a long exact sequence in cohomology

$$\begin{aligned} 0 &\to H^0_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to H^0(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in T} H^0(F_v, \operatorname{ad}\overline{\rho}) \\ &\to H^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to H^1(F_S/F, \operatorname{ad}\overline{\rho}) \\ &\to \oplus_{v\in T} H^1(F_v, \operatorname{ad}\overline{\rho}) \oplus_{v\in S\smallsetminus T} H^1(F_v, \operatorname{ad}\overline{\rho})/\mathcal{L}_v \\ &\to H^2_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to H^2(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in S} H^2(F_v, \operatorname{ad}\overline{\rho}) \to \cdots . \end{aligned} \tag{6.2.23}$$

Since we are assuming that $p > 2$, the groups $H^i(F_S/F, \operatorname{ad}\overline{\rho})$ vanish for $i \geq 3$, as do the groups $H^i(F_v, \operatorname{ad}\overline{\rho})$. So $H^i_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = 0$ for $i > 3$, and we have a relation among Euler characteristics

$$\chi_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = \chi(F_S/F, \operatorname{ad}\overline{\rho}) - \sum_{v\in S} \chi(F_v, \operatorname{ad}\overline{\rho}) - \sum_{v\in S\smallsetminus T} (\dim_k \mathcal{L}_v - h^0(F_v, \operatorname{ad}\overline{\rho})). \tag{6.2.24}$$

The trace pairing $(X, Y) \mapsto \operatorname{tr}(XY)$ on $\operatorname{ad}\overline{\rho}$ is perfect and G_F-equivariant, so $\operatorname{ad}\overline{\rho}(1)$ is isomorphic to the Tate dual of $\operatorname{ad}\overline{\rho}$. For each $v \in S$, we let $\mathcal{L}_v^\perp \subseteq H^1(F_v, \operatorname{ad}\overline{\rho}(1))$ be the exact annihilator of $\mathcal{L}_v$ under local Tate duality. We then define

$$H^1_{\mathcal{S}^\perp,T}(\operatorname{ad}\overline{\rho}(1)) = \ker\Big(H^1(F_S/F, \operatorname{ad}\overline{\rho}(1)) \to \prod_{v\in S\smallsetminus T} H^1(F_v, \operatorname{ad}\overline{\rho}(1))/\mathcal{L}_v^\perp\Big).$$

The following is proved in the same way as [KT17, Prop. 4.7], based on ideas of Kisin [Kis07, Prop. 4.1.5, Rem. 4.1.7].

PROPOSITION 6.2.25. *Let the notation and assumptions be as in the beginning of Section 6.2.22, and assume further that T is nonempty. Then*

there is a local Λ-algebra surjection $R_{\mathcal{S}}^{T,\mathrm{loc}}[\![X_1,\dots,X_g]\!] \to R_{\mathcal{S}}^T$, with

$$g = h^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = h^1_{\mathcal{S}^\perp,T}(\operatorname{ad}\overline{\rho}(1)) - h^0(F_S/F, \operatorname{ad}\overline{\rho}(1)) \\ - \sum_{v|\infty} h^0(F_v, \operatorname{ad}\overline{\rho}) + \sum_{v\in S\smallsetminus T} (\dim_k \mathcal{L}_v - h^0(F_v, \operatorname{ad}\overline{\rho})).$$

Proof. The first claim with $g = h^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$ follows from showing

$$H^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \cong \operatorname{Hom}_{\mathrm{CNL}_\Lambda}(R_{\mathcal{S}}^T/(\mathfrak{m}_{R_{\mathcal{S}}^{T,\mathrm{loc}}}), k[\varepsilon]/(\varepsilon^2)) \\ \cong \operatorname{Hom}_k(\mathfrak{m}_{R_{\mathcal{S}}^T}/(\mathfrak{m}^2_{R_{\mathcal{S}}^T}, \mathfrak{m}_{R_{\mathcal{S}}^{T,\mathrm{loc}}}), k).$$

To see this, note that any T-framed lifting of $\overline{\rho}$ to $k[\varepsilon]/(\varepsilon^2)$ can be written as $((1+\varepsilon\kappa)\overline{\rho}, (1+\varepsilon\alpha_v)_{v\in T})$, with $\kappa \in Z^1(F_S/F, \operatorname{ad}\overline{\rho})$, and $\alpha_v \in \operatorname{ad}\overline{\rho}$. It is the trivial lift at $v \in T$ if and only if

$$(1-\varepsilon\alpha_v)(1+\varepsilon\kappa|_{G_{F_v}})\overline{\rho}|_{G_{F_v}}(1+\varepsilon\alpha_v) = \overline{\rho}|_{G_{F_v}},$$

equivalently,

$$\kappa|_{G_{F_v}} = (\operatorname{ad}\overline{\rho}|_{G_{F_v}} - 1)\alpha_v.$$

Such a lift is further of type $\mathcal{S}$ if and only if $\kappa|_{G_{F_v}} \in \mathcal{L}^1_v$ for all $v \in S \smallsetminus T$. This sets up a bijection between the set of 1-cocycles of the complex $C^\bullet_{\mathcal{S},T}(\operatorname{ad}\overline{\rho})$ and the set of T-framed lifts of type $\mathcal{S}$ that are trivial at $v \in T$. Two cocycles $(\kappa, \{\alpha_v\}_{v\in T})$ and $(\kappa', \{\alpha'\}_{v\in T})$ define strictly equivalent T-framed lifts if and only if there is $\beta \in \operatorname{ad}\overline{\rho}$ such that

$$\kappa' = \kappa + (\operatorname{ad}\overline{\rho} - 1)\beta \quad \text{and} \quad \alpha'_v = \alpha_v + \beta$$

for all $v \in T$, i.e., if and only if they differ by a coboundary. This induces the desired isomorphism

$$H^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \cong \operatorname{Hom}_{\mathrm{CNL}_\Lambda}(R_{\mathcal{S}}^T/(\mathfrak{m}_{R_{\mathcal{S}}^{T,\mathrm{loc}}}), k[\varepsilon]/(\varepsilon^2)).$$

Since T is nonempty, $h^0_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = 0$. Then (6.2.24) together with the local and global Euler characteristic formulas imply

$$h^1_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = h^2_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) - h^3_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \\ - \sum_{v|\infty} h^0(F_v, \operatorname{ad}\overline{\rho}) + \sum_{v\in S\smallsetminus T} (\dim_k \mathcal{L}_v - h^0(F_v, \operatorname{ad}\overline{\rho})).$$

To finish the proof, we deduce equalities $h^2_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = h^1_{\mathcal{S}^\perp,T}(\operatorname{ad}\overline{\rho}(1))$ and $h^3_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) = h^0(F_S/F, \operatorname{ad}\overline{\rho}(1))$ by comparing the exact sequence

$$\to H^1(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in T} H^1(F_v, \operatorname{ad}\overline{\rho}) \oplus_{v\in S\smallsetminus T} H^1(F_v, \operatorname{ad}\overline{\rho})/\mathcal{L}_v \\ \to H^2_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to H^2(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in S} H^2(F_v, \operatorname{ad}\overline{\rho}) \\ \to H^3_{\mathcal{S},T}(\operatorname{ad}\overline{\rho}) \to 0,$$

which is part of (6.2.23), with the exact sequence

$$\begin{aligned}&\to H^1(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in T} H^1(F_v, \operatorname{ad}\overline{\rho}) \oplus_{v\in S\smallsetminus T} H^1(F_v, \operatorname{ad}\overline{\rho})/\mathcal{L}_v\\ &\to H^1_{\mathcal{S}^\perp,T}(\operatorname{ad}\overline{\rho}(1))^\vee \to H^2(F_S/F, \operatorname{ad}\overline{\rho}) \to \oplus_{v\in S} H^2(F_v, \operatorname{ad}\overline{\rho})\\ &\to H^0(F_S/F, \operatorname{ad}\overline{\rho}(1))^\vee \to 0,\end{aligned}$$

which is part of the Poitou–Tate long exact sequence. □

We will apply this with our choices of local deformation rings as in Section 6.2.5. By applying Propositions 6.2.14, 6.2.16, 6.2.17, 6.2.21, and [BLGHT11, Lem. 3.3], we obtain the following:

LEMMA 6.2.26. *We assume that our deformation problem $\mathcal{S}$ and $T \subseteq S$ satisfy the following*:

- *T is a disjoint union $S_p \sqcup R \sqcup S_a$.*
- *For each $v \in S_p$, we assume that $F_v/\mathbf{Q}_p$ is unramified and that $\overline{\rho}|_{G_{F_v}}$ is as in Proposition 6.2.14. We take $\mathcal{D}_v = \mathcal{D}_v^{\mathrm{FL}}$.*
- *For each $v \in R$, we assume that $q_v \equiv 1 \bmod p$ and that $\overline{\rho}|_{G_{F_v}}$ is trivial. We take $\mathcal{D}_v = \mathcal{D}_v^{\chi_v}$ for some tuple $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$ of characters $\chi_{v,i}\colon \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ that are trivial modulo ϖ.*
- *For each $v \in S_a$, we assume that $H^2(F_v, \operatorname{ad}\overline{\rho}) = 0$, and we take $\mathcal{D}_v = \mathcal{D}_v^\square$.*

Then $R_{\mathcal{S}}^{T,\mathrm{loc}}$ satisfies the following properties:

(1) *Assume that $\chi_{v,i} = 1$ for each $v \in R$ and $1 \le i \le n$. Then $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$ is equidimensional of dimension $1 + n^2|T| + \frac{n(n-1)}{2}[F:\mathbf{Q}]$, and every generic point has characteristic 0. Further, every generic point of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}/\varpi$ is the specialization of a unique generic point of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$.*

(2) *Assume that $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct for each $v \in R$. Then $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$ is irreducible of dimension $1 + n^2|T| + \frac{n(n-1)}{2}[F:\mathbf{Q}]$ and its generic point has characteristic 0.*

In the ordinary case, we will use the following:

LEMMA 6.2.27. *We assume that our deformation problem $\mathcal{S}$ and $T \subseteq S$ satisfy the following*:

- *T is a disjoint union $S_p \sqcup R \sqcup S_a$.*
- *For each $v \in S_p$, we assume that $[F_v : \mathbf{Q}_p] > \frac{n(n+1)}{2} + 1$ and that $\overline{\rho}|_{G_{F_v}}$ is trivial. We take Λ_v to be the quotient of $\mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$ by a minimal prime ideal $\wp_v$ and take $\mathcal{D}_v = \mathcal{D}_v^{\mathrm{det,ord}}$ to be the local deformation problem classified by $R_v^{\mathrm{det,ord}}$.*
- *For each $v \in R$, we assume that $q_v \equiv 1 \bmod p$ and assume that $\overline{\rho}|_{G_{F_v}}$ is trivial. We take $\mathcal{D}_v = \mathcal{D}_v^{\chi_v}$ for some tuple $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$ of characters $\chi_{v,i}\colon \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ that are trivial modulo ϖ.*

- *For each $v \in S_a$, we assume that $H^2(F_v, \operatorname{ad}\overline{\rho}) = 0$ and we take $\mathcal{D}_v = \mathcal{D}_v^{\square}$.*

Then $R_{\mathcal{S}}^{T,\mathrm{loc}}$ satisfies the following properties:

(1) *Assume that $\chi_{v,i} = 1$ for each $v \in R$ and $1 \leq i \leq n$. Then $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$ has dimension $1 + n^2|T| + \frac{n(n+1)}{2}[F : \mathbf{Q}]$, any irreducible component of maximum dimension has a characteristic 0 generic point, and any irreducible component that does not have maximum dimension has dimension $\leq n^2|T| - 1 + \frac{n(n+1)}{2}[F : \mathbf{Q}]$. Further, any irreducible component of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}/(\lambda)$ of maximum dimension is the specialization of a unique generic point of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$.*

(2) *Assume that $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct for each $v \in R$. Then $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$ has dimension $1 + n^2|T| + \frac{n(n+1)}{2}[F : \mathbf{Q}]$, it has a unique irreducible component of maximum dimension, and the generic point of this irreducible component has characteristic 0. Any other irreducible component has dimension $\leq n^2|T| - 1 + \frac{n(n+1)}{2}[F : \mathbf{Q}]$.*

(3) *If x is a point of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$ lying in an irreducible component of non-maximum dimension, then there is some $v \in S_p$ such that the image of x in $\operatorname{Spec} \Lambda_v$ lies in the closed locus defined by $\chi_i^{\mathrm{univ}} = \chi_j^{\mathrm{univ}}$ for some $i \neq j$.*

Proof. For each $v \in S_p$, Proposition 6.2.12 implies that $\operatorname{Spec} R_v$ has a unique irreducible component of dimension $\dim R_v = 1 + n^2 + \frac{n(n+1)}{2}[F_v : \mathbf{Q}_p]$, and this irreducible component has characteristic 0. Let $\mathfrak{q}_v$ be the minimal prime of R_v corresponding to this irreducible component. Then we can apply [BLGHT11, Lem. 3.3] to

$$R' = \widehat{\otimes}_{v \in S_p} R_v / \mathfrak{q}_v \widehat{\otimes}_{v \in R \cup S_a} R_v$$

together with Propositions 6.2.16, 6.2.17, and 6.2.21 to obtain the following:

(1) If $\chi_{v,i} = 1$ for each $v \in R$ and $1 \leq i \leq n$, then $\operatorname{Spec} R'$ is equidimensional of dimension $1 + n^2|T| + \frac{n(n+1)}{2}[F : \mathbf{Q}]$, and every generic point has characteristic 0. Further, every generic point of $\operatorname{Spec} R'/\varpi$ is the specialization of a unique generic point of $\operatorname{Spec} R'$.

(2) If $\chi_{v,1}, \ldots, \chi_{v,n}$ are pairwise distinct for each $v \in R$, then $\operatorname{Spec} R'$ is irreducible of dimension $1 + n^2|T| + \frac{n(n+1)}{2}[F : \mathbf{Q}]$ and its generic point has characteristic 0.

Since any minimal prime $\mathfrak{p}$ of $R_{\mathcal{S}}^{T,\mathrm{loc}}$ pulls back to minimal prime ideals $\mathfrak{p}_v$ of R_v for each $v \in T$ and induces a surjection

$$\widehat{\otimes}_{v \in T} R_v / \mathfrak{p}_v \to R_{\mathcal{S}}^{T,\mathrm{loc}} / \mathfrak{p},$$

we see that $\operatorname{Spec} R'$ is a union of irreducible components of $\operatorname{Spec} R_{\mathcal{S}}^{T,\mathrm{loc}}$. To finish the proof of the lemma, it suffice to note that if $\mathfrak{p}_v \neq \mathfrak{q}_v$ for some $v \in S_p$, then by Proposition 6.2.12, $\dim R_v/\mathfrak{p}_v \leq n^2 - 1 + \frac{n(n+1)}{2}[F_v : \mathbf{Q}_p]$ and the image

of $\mathfrak{p}_v$ in Λ_v lies in the closed locus defined by $\chi_i^{\mathrm{univ}} = \chi_j^{\mathrm{univ}}$ for some $i \neq j$. In this case, $\dim R_{\mathcal{S}}^{T,\mathrm{loc}}/\mathfrak{p} \leq n^2|T| - 1 + \frac{n(n+1)}{2}[F : \mathbf{Q}]$. □

6.2.28. *Taylor–Wiles primes.* In this section we show how to generate Taylor–Wiles data. We first need to introduce a definition, essentially equivalent to that of [KT17, Def. 4.10] and [CG18, §9.2] (see Remark 6.2.31 below). For the moment, let k be any algebraic extension of $\mathbf{F}_p$.

Definition 6.2.29. Let ad^0 denote the space of trace zero matrices in $\mathrm{M}_{n\times n}(k)$ with the adjoint $\mathrm{GL}_n(k)$-action. An absolutely irreducible subgroup $H \subseteq \mathrm{GL}_n(k)$ is called *enormous* over k if it satisfies the following:

(1) H has no nontrivial p-power order quotient.
(2) $H^0(H, \mathrm{ad}^0) = H^1(H, \mathrm{ad}^0) = 0$.
(3) For any simple $k[H]$-submodule $W \subseteq \mathrm{ad}^0$, there is a regular semisimple $h \in H$ such that $W^h \neq 0$.

Note that this only depends on the image of H in $\mathrm{PGL}_n(k)$. If p divides n, then no subgroup of $\mathrm{GL}_n(k)$ is enormous (because ad^0 contains the scalar matrices).

Lemma 6.2.30. *Let k'/k be an algebraic extension, and let $H \subset \mathrm{GL}_n(k)$ be a subgroup. Then H is enormous over k if and only if it is enormous over k'.*

Proof. It suffices to address condition (3), which is equivalent to the following statement: for all non-zero $k[H]$-submodules $W \subseteq \mathrm{ad}^0$, there is a regular semisimple element $h \in H$ such that $W^h \neq 0$. This makes it clear that if H is enormous over k', then it is enormous over k.

Suppose therefore that H is enormous over k. The property that a $k'[H]$-module V satisfies $V^h = 0$ is closed under taking direct sums and taking quotients. (The latter is true because $V^h \neq 0$ if and only if $V_h \neq 0$.) If $V \subset \mathrm{ad}^0 \otimes_k k'$, then $\sigma h v = h \sigma v$ for all $\sigma \in \mathrm{Gal}(k'/k)$ (since $H \subset \mathrm{GL}_n(k)$) and so $V^h \neq 0$ if and only if $(\sigma V)^h \neq 0$. In particular, if W' is a simple $k'[H]$-submodule of $\mathrm{ad}^0 \otimes_k k'$ with no invariants by $h \in H$, the same is true for $\sigma W'$ for all $\sigma \in \mathrm{Gal}(k'/k)$, as well as the submodule of $\mathrm{ad}^0 \otimes_k k'$ generated by the sum of all such $\sigma W'$. But the latter is stable under both H and $\mathrm{Gal}(k'/k)$, and thus (by descent) has the form $W \otimes_k k'$ for some $k[H]$-submodule of ad^0. But now applying condition (3) to any $k[H]$-simple submodule of W, we deduce that $W^h \neq 0$ for some regular semisimple h, from which it follows that the same holds for W'. □

Henceforth we drop the "over k" and refer simply to enormous subgroups of $\mathrm{GL}_n(k)$.

Remark 6.2.31. Assuming that k is sufficiently large to contain all eigenvalues of the elements of H, it can be checked that Definition 6.2.29 is equivalent to [KT17, Def. 4.10].

We now return to the assumptions described at the beginning of Section 6.2, assuming further that k contains all eigenvalues of the elements of $\overline{\rho}(G_F)$. We again fix a global deformation problem

$$\mathcal{S} = (\overline{\rho}, S, \{\Lambda_v\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S}).$$

We define a *Taylor–Wiles datum* to be a tuple $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q})$ consisting of

- a finite set of finite places Q of F, disjoint from S, such that $q_v \equiv 1 \bmod p$ for each $v \in Q$;
- for each $v \in Q$, an ordering $\alpha_{v,1}, \dots, \alpha_{v,n}$ of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_v)$, which are assumed to be k-rational and distinct.

Given a Taylor–Wiles datum $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q})$, we define the augmented global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, S\cup Q, \{\Lambda_v\}_{v\in S}\cup\{\mathcal{O}\}_{v\in Q}, \{\mathcal{D}_v\}_{v\in S}\cup\{\mathcal{D}_v^{\square}\}_{v\in Q}).$$

Set $\Delta_Q = \prod_{v\in Q} k(v)^\times(p)^n$. By Section 6.2.18, the fixed ordering $\alpha_{v,1}, \dots, \alpha_{v,n}$ for each $v \in Q$ determines a $\Lambda[\Delta_Q]$-algebra structure on $R^T_{\mathcal{S}_Q}$ for any subset T of S. Letting $\mathfrak{a}_Q = \ker(\Lambda[\Delta_Q] \to \Lambda)$ be the augmentation ideal, the natural surjection $R^T_{\mathcal{S}_Q} \to R^T_{\mathcal{S}}$ has kernel $\mathfrak{a}_Q R^T_{\mathcal{S}_Q}$.

Lemma 6.2.32. *Let $T \subseteq S$. Assume that F is CM with maximal totally real subfield F^+, that $\zeta_p \notin F$, and that $\overline{\rho}(G_{F(\zeta_p)})$ is enormous. Let $q \geq h^1_{\mathcal{S}^\perp,T}(\mathrm{ad}\,\overline{\rho}(1))$. Then for every $N \geq 1$, there is a choice of Taylor–Wiles datum $(Q_N, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v\in Q_N})$ satisfying the following*:

(1) $\#Q_N = q$.
(2) *For each $v \in Q_N$, $q_v \equiv 1 \bmod p^N$, and v has degree one over $\mathbf{Q}$.*
(3) $h^1_{\mathcal{S}^\perp_{Q_N},T}(\mathrm{ad}\,\overline{\rho}(1)) = 0$.

Proof. Since the augmented deformation datum $\mathcal{S}_{Q_N}$ has $\mathcal{D}_v = \mathcal{D}_v^{\square}$ for $v \in Q_N$, we have $\mathcal{L}_v = H^1(G_v, \mathrm{ad}\,\overline{\rho})$ and

$$H^1_{\mathcal{S}^\perp_{Q_N},T}(\mathrm{ad}\,\overline{\rho}(1)) = \ker\Big(H^1_{S^\perp,T}(\mathrm{ad}\,\overline{\rho}(1)) \to \prod_{v\in Q_N} H^1(F_v, \mathrm{ad}\,\overline{\rho}(1))\Big).$$

So by induction, it suffices to show that given any cocycle κ representing a nonzero element of $H^1_{\mathcal{S}^\perp,T}(\mathrm{ad}\,\overline{\rho}(1))$, there are infinitely many finite places v of F such that

- v has degree one over $\mathbf{Q}$ and splits in $F(\zeta_{p^N})$;
- $\overline{\rho}(\mathrm{Frob}_v)$ has n-distinct eigenvalues $\alpha_{v,1}, \dots, \alpha_{v,n}$ in k;
- the image of κ in $H^1(F_v, \mathrm{ad}\,\overline{\rho}(1))$ is nonzero.

The set of places of F that have degree one over $\mathbf{Q}$ has density one, so it suffices to show that the remaining properties are satisfied by a positive density set of places of F. Then by Chebotarev density, we are reduced to showing that given any cocycle κ representing a nonzero element of $H^1_{S^\perp,T}(\operatorname{ad}\overline{\rho}(1))$, there is $\sigma \in G_{F(\zeta_{p^N})}$ such that

- $\overline{\rho}(\sigma)$ has distinct k-rational eigenvalues;
- $p_\sigma \kappa(\sigma) \neq 0$, where $p_\sigma\colon \operatorname{ad}\overline{\rho} \to (\operatorname{ad}\overline{\rho})^\sigma$ is the σ-equivariant projection.

(The second condition guarantees that the restriction of κ will not be a coboundary.) Since $p \nmid n$, we have a G_F-equivariant decomposition $\operatorname{ad}\overline{\rho} = k \oplus \operatorname{ad}^0\overline{\rho}$, and we treat separately the cases where κ represents a cohomology class in $H^1(F_S/F, \operatorname{ad}^0\overline{\rho}(1))$ and in $H^1(F_S/F, k(1))$.

First assume that κ represents a cohomology class in $H^1(F_S/F, \operatorname{ad}^0\overline{\rho}(1))$. Let L/F be the splitting field of $\overline{\rho}$. The definition of enormous implies that the restriction map

$$H^1(F_S/F, \operatorname{ad}^0\overline{\rho}(1)) \to H^1(F_S/L(\zeta_{p^N}), \operatorname{ad}^0\overline{\rho}(1))^{G_F}$$

is injective. Indeed, letting $H = \overline{\rho}(G_{F(\zeta_p)})$, since H has no p-power order quotients, $H = \overline{\rho}(G_{F(\zeta_{p^N})})$ and $H^0(H, \operatorname{ad}^0\overline{\rho}) = 0$ implies that the restriction to $H^1(F_S/F(\zeta_{p^N}), \operatorname{ad}^0\overline{\rho})$ is injective. Then the condition $H^1(H, \operatorname{ad}^0\overline{\rho}) = 0$ implies that the further restriction to $H^1(F_S/L(\zeta_{p^N}), \operatorname{ad}^0\overline{\rho}(1))$ is injective. So the restriction of κ defines a nonzero $G_{F(\zeta_{p^N})}$-equivariant homomorphism $\operatorname{Gal}(F_S/L(\zeta_{p^N})) \to \operatorname{ad}^0\overline{\rho}$.

Let W be a nonzero irreducible subrepresentation in the k-span of

$$\kappa(\operatorname{Gal}(F_S/L(\zeta_{p^N})).$$

The enormous assumption implies that there is $\sigma_0 \in G_{F(\zeta_{p^N})}$ such that $\overline{\rho}(\sigma_0)$ has distinct k-rational eigenvalues and such that $W^{\sigma_0} \neq 0$. This implies that $\kappa(\operatorname{Gal}(F_S/L(\zeta_{p^N}))$ is not contained in the kernel of the σ_0-equivariant projection $p_{\sigma_0}\colon \operatorname{ad}^0\overline{\rho} \to (\operatorname{ad}^0\overline{\rho})^{\sigma_0}$. If $p_{\sigma_0}\kappa(\sigma_0) \neq 0$, then we take $\sigma = \sigma_0$. Otherwise, we choose $\tau \in G_{L(\zeta_{p^N})}$ such that $p_{\sigma_0}\kappa(\tau) \neq 0$, and we take $\sigma = \tau\sigma_0$. This does the job since $\overline{\rho}(\sigma) = \overline{\rho}(\sigma_0)$ and $\kappa(\sigma) = \kappa(\sigma_0) + \kappa(\tau)$.

Now assume that κ represents a cohomology class in $H^1(F_S/F, k(1))$. The cohomology class of κ corresponds to a Kummer extension $F(\zeta_p, y)$ with $y^p \in F(\zeta_p)$. Since κ is nontrivial and $\zeta_p \notin F$, this extension $F(\zeta_p, y)$ is not abelian over F. It follows that $y^p \notin F(\zeta_{p^N})$ for any $N \geq 1$, and the restriction of κ to $G_{F(\zeta_{p^N})}$ is nontrivial. Since the extension $F(\zeta_{p^N}, y)/F(\zeta_{p^N})$ has degree p, it is disjoint from the extension cut out by the restriction of $\overline{\rho}$ to $G_{F(\zeta_{p^N})}$ by the enormous assumption. It follows that we can find $\sigma \in G_{F(\zeta_{p^N})}$ such that $\overline{\rho}(\sigma)$ has distinct eigenvalues and such that $\kappa(\sigma) \neq 0 \in k$. This completes the proof. $\square$

PROPOSITION 6.2.33. *Take $T = S$, and let $q \geq h^1_{\mathcal{S}^\perp,T}(\operatorname{ad}\overline{\rho}(1))$. Assume that $F = F^+F_0$ with F^+ totally real and F_0 an imaginary quadratic field, that $\zeta_p \notin F$, and that $\overline{\rho}(G_{F(\zeta_p)})$ is enormous. Then for every $N \geq 1$, there is a choice of Taylor–Wiles datum $(Q_N, (\alpha_{v,1}, \ldots, \alpha_{v,n})_{v \in Q_N})$ satisfying the following*:

(1) $\#Q_N = q$.
(2) *For each $v \in Q_N$, $q_v \equiv 1 \bmod p^N$ and the rational prime below v splits in F_0.*
(3) *There is a local Λ-algebra surjection $R^{T,\mathrm{loc}}_{\mathcal{S}}[\![X_1, \ldots, X_g]\!] \to R^T_{\mathcal{S}_{Q_N}}$, with*

$$g = qn - n^2[F^+ : \mathbf{Q}].$$

Proof. If v is a finite place of F that is degree one over $\mathbf{Q}$, then the rational prime below it must split in F_0. So Proposition 6.2.25 and Lemma 6.2.32 imply that the proposition holds with

$$g = -h^0(F_S/F, \operatorname{ad}\overline{\rho}(1)) - n^2[F^+ : \mathbf{Q}] + \sum_{v \in Q}(\dim \mathcal{L}_v - \dim h^0(F_v, \operatorname{ad}\overline{\rho})).$$

The assumptions that $\overline{\rho}(G_{F(\zeta_p)})$ is enormous and that $\zeta_p \notin F$ imply that $H^0(F_S/F, \operatorname{ad}\overline{\rho}(1))$ is trivial. For each $v \in Q$, we have $\mathcal{L}_v = H^1(F_v, \operatorname{ad}\overline{\rho})$, so

$$\dim \mathcal{L}_v - \dim h^0(F_v, \operatorname{ad}\overline{\rho}) = h^0(F_v, \operatorname{ad}\overline{\rho}(1)) = n,$$

where the first equality follows from local Tate duality and the local Euler characteristic, and the second from the fact that $q_v \equiv 1 \bmod p$ and $\overline{\rho}(\mathrm{Frob}_v)$ has distinct eigenvalues. □

6.3. *Avoiding Ihara's lemma.* In this section we will axiomatically explain how to deduce a patched automorphy theorem from the result of the patching process. See Section 6.3.5 and particularly Proposition 6.3.8. We begin, however, with a little commutative algebra.

6.3.1. *Some commutative algebra.*

LEMMA 6.3.2. *Suppose that T is an excellent local ring with $\operatorname{Spec} T$ irreducible, that $f \in \mathfrak{m}_T$, and that $T/(f)$ has Krull dimension 0.*

If T has dimension 0, then for every finitely generated T module M, we have

$$\lg_T(M/fM) - \lg_T(M[f]) = 0$$

(*and these lengths are both finite*).

Otherwise T has dimension 1 and a unique prime ideal $\mathfrak{p}$ other than $\mathfrak{m}_T$. In this case there is a constant $a \in \mathbf{Z}_{>0}$ such that for any finitely generated T-module M, we have

$$\lg_T(M/fM) - \lg_T(M[f]) = a \lg_{T_\mathfrak{p}}(M_\mathfrak{p})$$

(*and all these lengths are finite*).

Proof. If T has dimension 0, then it is Artinian and every finitely generated T-module has finite length. If the desired length equality holds for two modules in a short exact sequence, it also holds for the third (by the snake lemma). Thus we are reduced to checking the lemma in the case $M = T/\mathfrak{m}_T$, in which case it is obvious.

Now suppose that T has dimension 1. Note that $T/(f)$ is Artinian and so any finitely generated $T/(f)$-module has finite length over T. Let $\widetilde{T}$ denote the normalization of $T/\mathfrak{p}$. As T is excellent, $\widetilde{T}$ is a finitely generated T-module. We will take $a = \lg_T(\widetilde{T}/(f))$. (This is positive because f is not a unit in $\widetilde{T}$. In fact it lies in every maximal ideal.)

Note that the conclusion of the lemma is true for $M = T/\mathfrak{m}$ and for $M = \widetilde{T}$. Also note that if the conclusion of the lemma holds for two modules in a short exact sequence, then it holds for the third (again by the snake lemma). In particular, the lemma holds for all finite length T-modules.

Filtering M by the submodules $\mathfrak{p}^i M$ we reduce to checking the lemma for M a $T/\mathfrak{p}$-module. Write Q for the quotient $\widetilde{T}/(T/\mathfrak{p})$. It has support $\{\mathfrak{m}_T\} \subset \operatorname{Spec} T$. If M is any finitely generated $T/\mathfrak{p}$-module, we have an exact sequence

$$\operatorname{Tor}_1^T(M, Q) \longrightarrow M \longrightarrow M \otimes_T \widetilde{T} \longrightarrow M \otimes_T Q \longrightarrow (0).$$

Both $M \otimes_T Q$ and $\operatorname{Tor}_1^T(M, Q)$ are finitely generated T-modules with support contained in $\{\mathfrak{m}_T\}$, and hence of finite length. Thus we are reduced to proving the lemma for M a finitely generated $\widetilde{T}$-module.

Note that $\widetilde{T}$ is a Dedekind domain with only finitely many maximal ideals, and hence a PID. By the structure theorem for finitely generated modules over a PID, it suffices to check the conclusion of the lemma in the following two cases: M is a finite length $\widetilde{T}$-module, and $M = \widetilde{T}$. However we have already treated both these cases. $\square$

We will actually make use of a derived version of this lemma. Suppose that S is a ring, that T is a noetherian S-algebra, and that $C \in D^b(S)$ is equipped with a map $T \to \operatorname{End}_{D^b(S)}(C)$ over S such that the cohomology of C has finite length over T. Then we define

$$\lg_T(C) = \sum_i (-1)^i \lg_T(H^i(C)).$$

Note that if

$$C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow$$

is an exact triangle in $D^b(S)$ with compatible actions of T, and if two of the C_i have cohomology of finite length over T, then so does the third and we have

$$\lg_T(C_2) = \lg_T(C_1) + \lg_T(C_3).$$

Note also that if $f \in S$ and the cohomology of C is finitely generated over T, then the cohomology of $C \otimes_S^{\mathbf{L}} S/(f)$ is also finitely generated over T; look at the long exact sequence in cohomology coming from the exact triangle

$$(C \xrightarrow{f} C \longrightarrow C \otimes_S^{\mathbf{L}} S/(f) \longrightarrow) \;=\; C \otimes_S^{\mathbf{L}} (S \xrightarrow{f} S \longrightarrow S/(f) \longrightarrow).$$

Before stating the derived version we need one other remark:

LEMMA 6.3.3. *Suppose that A is a noetherian ring and that $\mathfrak{m}$ is a maximal ideal of A that is simultaneously a minimal prime ideal. Then $A \xrightarrow{\sim} A_{\mathfrak{m}} \times B$ for some ring B.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ denote the other minimal prime ideals of A, and set $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$. If $\mathfrak{m} \supset I$, then $\mathfrak{m} \supset \mathfrak{p}_i$ for some i, a contradiction. Thus $\mathfrak{m} + I = A$ and $A/(\mathfrak{m} \cap I) \xrightarrow{\sim} A/\mathfrak{m} \times A/I$. However, $\mathfrak{m} \cap I$ is nilpotent so we can lift the idempotent $(1,0) \in A/\mathfrak{m} \times A/I$ to an idempotent $e \in A$. Then $1 - e \in \mathfrak{m}$, and e will lie in every prime ideal of A other than $\mathfrak{m}$. Thus $\mathfrak{m} = e\mathfrak{m} \times (1-e)A \subset eA \times (1-e)A$, and every other prime ideal of A contains e and so has the form $eA \times \mathfrak{q}$. In particular, eA is Artinian local and $A_{\mathfrak{m}} = (eA)_{e\mathfrak{m}} = eA$. □

Suppose that S is a noetherian ring, that T is a finite S-algebra, that $\mathfrak{p}$ is a minimal prime ideal of T, and that $C \in D^b(S)$ is equipped with a map $T \to \mathrm{End}_{D^b(S)}(C)$ over S. Let $\mathfrak{q}$ denote the contraction of $\mathfrak{p}$ to S. As T is finite over S, we see that $\mathfrak{p}$ is also maximal ideal in $T_{\mathfrak{q}}$, and so by the above lemma we can write $T_{\mathfrak{q}} \cong T_{\mathfrak{p}} \times B$ for some S-algebra B. Let $e_{\mathfrak{p}} \in T_{\mathfrak{q}}$ be the idempotent corresponding $(1,0) \in T_{\mathfrak{p}} \times B$. Then, perhaps by an abuse of notation, we will write

$$C_{\mathfrak{p}} = e_{\mathfrak{p}}(C \otimes_S S_{\mathfrak{q}}).$$

It is an object of $D^b(S_{\mathfrak{q}})$ with an action of $T_{\mathfrak{p}}$. It is not literally a localization over T, but if C (with its action of T) happens to be represented by a complex of T-modules C^i, then $C_{\mathfrak{p}}$ is represented by the complex $C^i_{\mathfrak{p}}$. Moreover if the cohomology of C is finitely generated over S, then the cohomology of $C_{\mathfrak{p}}$ has finite length over $T_{\mathfrak{p}}$ (being finitely generated over the Artinian ring $T_{\mathfrak{p}}$).

LEMMA 6.3.4. *Suppose that S is an excellent local ring and that $f \in \mathfrak{m}_S$ is a non-zero divisor. Suppose also that T is a finite S-algebra with a maximal ideal $\mathfrak{m}$ such that $\operatorname{Spec} T_{\mathfrak{m}}$ is irreducible and $T_{\mathfrak{m}}/(f)$ has Krull dimension 0. Note that $T_{\mathfrak{m}}$ has dimension at most 1.*

If $T_{\mathfrak{m}}$ has dimension 0, then for every $C \in D^b(S)$ such that $T \to \mathrm{End}_{D^b(S)}(C)$ over S and C has finitely generated cohomology, we have

$$\lg_{T_{\mathfrak{m}}}((C \otimes_S^{\mathbf{L}} S/(f))_{\mathfrak{m}}) = 0.$$

If not, then $T_{\mathfrak{m}}$ has a unique prime ideal $\mathfrak{p}$ other than $\mathfrak{m}$. In this case there is $a \in \mathbf{Z}_{>0}$ with the following property:

Suppose that $C \in D^b(S)$ such that C has finitely generated cohomology, and that $T \to \mathrm{End}_{D^b(S)}(C)$ over S. Then

$$\lg_{T_\mathfrak{m}}((C \otimes^{\mathbf{L}}_S S/(f))_\mathfrak{m}) = a \lg_{T_\mathfrak{p}}(C_\mathfrak{p}).$$

Proof. We will take the a as in Lemma 6.3.2 for the ring $T_\mathfrak{m}$. If the lemma holds for two terms in an exact triangle, it holds for the third term too. Thus one may inductively reduce to the case that C is quasi-isomorphic to $M[i]$ for a finitely generated S-module M with a compatible action of T. In this case $C \otimes^{\mathbf{L}}_S S/(f)$ is quasi-isomorphic to $(M \xrightarrow{f} M)[-i]$. Moreover,

$$\lg_{T_\mathfrak{m}}((C \otimes^{\mathbf{L}}_S S/(f))_\mathfrak{m}) = (-1)^i(\lg_{T_\mathfrak{m}}((M)/fM)_\mathfrak{m}) - \lg_{T_\mathfrak{m}}(M[f]_\mathfrak{m}))$$

and

$$\lg_{T_\mathfrak{p}}(C_\mathfrak{p}) = (-1)^i \lg_{T_\mathfrak{p}}(M_\mathfrak{p}).$$

Thus the present lemma follows from Lemma 6.3.2. □

We remark that if C is perfect, then the cohomology of C and $C \otimes_S S/(f)$ will automatically be finitely generated (over S and hence over T). In this case, if T is an S subalgebra of $\mathrm{End}_{D^b(S)}(C)$, then it will automatically be finite over S.

6.3.5. *Application.* Let Λ be a ring that is isomorphic to a power series ring over $\mathcal{O}$. We assume given the following objects:

(1) A power series ring $S_\infty = \Lambda[[X_1, \cdots, X_r]]$ with augmentation ideal $\mathfrak{a}_\infty = (X_1, \ldots, X_r)$.

(2) Perfect complexes C_∞, C'_∞ of S_∞-modules, and a fixed isomorphism

$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi \cong C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi$$

in $\mathbf{D}(S_\infty/\varpi)$.

(3) Two S_∞-subalgebras

$$T_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(C_\infty)$$

and

$$T'_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(C'_\infty),$$

which have the same image in

$$\mathrm{End}_{\mathbf{D}(S_\infty/\varpi)}(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi) = \mathrm{End}_{\mathbf{D}(S_\infty/\varpi)}(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi),$$

where these endomorphism algebras are identified using the fixed isomorphism. Call this common image $\overline{T}_\infty$. Note that T_∞ and T'_∞ are finite S_∞-algebras.

(4) Two Noetherian complete local S_∞-algebras R_∞ and R'_∞ and surjections $R_\infty \twoheadrightarrow T_\infty/I_\infty$, $R'_\infty \twoheadrightarrow T'_\infty/I'_\infty$, where I_∞ and I'_∞ are nilpotent ideals. We write $\overline{I}_\infty$ and $\overline{I}'_\infty$ for the image of these ideals in $\overline{T}_\infty$. Note that it then makes sense to talk about the support of $H^*(C_\infty)$ and $H^*(C'_\infty)$ over R_∞, R'_∞, even though they are not genuine modules over these rings. These

supports actually belong to the closed subsets of $\operatorname{Spec} R_\infty$, $\operatorname{Spec} R'_\infty$ given by $\operatorname{Spec} T_\infty$, $\operatorname{Spec} T'_\infty$, and hence are finite over $\operatorname{Spec} S_\infty$.

(5) An isomorphism $R_\infty/\varpi \cong R'_\infty/\varpi$ compatible with the S_∞-algebra structure and the actions (induced from T_∞ and T'_∞) on

$$H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty) = H^*(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty),$$

where these cohomology groups are identified using the fixed isomorphism.

(6) Integers $q_0 \in \mathbf{Z}$ and $l_0 \in \mathbf{Z}_{\geq 0}$.

Assumption 6.3.6. Our set-up is assumed to satisfy the following:

(1) $\dim R_\infty = \dim R'_\infty = \dim S_\infty - l_0$, and $\dim R_\infty/\varpi = \dim R'_\infty/\varpi = \dim S_\infty - l_0 - 1$.

(2) (Behavior of components). Assume that each generic point of $\operatorname{Spec} R_\infty/\varpi$ of maximal dimension (i.e., of dimension $\dim R_\infty - 1$) is the specialization of a unique generic point of $\operatorname{Spec} R_\infty$ of dimension $\dim R_\infty$, and $\operatorname{Spec} R'_\infty$ has a unique generic point x' of dimension $\dim R_\infty$. Assume also that any generic points of $\operatorname{Spec} R_\infty$, $\operatorname{Spec} R'_\infty$, $\operatorname{Spec} R_\infty/\varpi$ which are not of maximal dimension have dimension $< \dim S_\infty - l_0 - 1$.

These hypotheses imply every generic point of $\operatorname{Spec} R_\infty$ and $\operatorname{Spec} R'_\infty$ of dimension $\dim R_\infty$ has characteristic 0.

(3) (Generic concentration). There exists a dimension 1 characteristic 0 prime $\mathfrak{p}$ of S_∞ containing $\mathfrak{a}_\infty$ such that

$$H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{p})[\frac{1}{p}] \neq 0,$$

and these groups are non-zero only for degrees in the interval $[q_0, q_0 + l_0]$.

Note that $\operatorname{Supp}_{R_\infty}(H^*(C_\infty)) = \operatorname{Spec} T_\infty$ and $\operatorname{Supp}_{R'_\infty}(H^*(C'_\infty)) = \operatorname{Spec} T'_\infty$ — the only point being that the kernel of $T_\infty \to \operatorname{End}_{S_\infty}(H^*(C_\infty))$ is nilpotent.

The following result is an immediate corollary of Lemma 6.3.4.

LEMMA 6.3.7. *Suppose that $\overline{x}$ is a minimal prime of $R_\infty/(\varpi)$ of dimension $\dim S_\infty - l_0 - 1$ containing a minimal prime x of R_∞ of dimension $\dim S_\infty - l_0$. Note that $R_{\infty,\overline{x}}$ has a unique minimal prime ideal x, and that $R_{\infty,\overline{x}}/(\varpi)$ has Krull dimension 0. Moreover,*

(1) *If $\overline{x} \in \operatorname{Spec} T_\infty$ and if $\lg_{T_{\infty,\overline{x}}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}}) \neq 0$, then $x \in \operatorname{Spec} T_\infty$ and $\lg_{T_{\infty,x}}(C_{\infty,x}) \neq 0$.*

(2) *If $x \in \operatorname{Spec} T_\infty$, then $\overline{x} \in \operatorname{Spec} T_\infty$. If, moreover, $\lg_{T_{\infty,x}}(C_{\infty,x}) \neq 0$, then $\lg_{T_{\infty,\overline{x}}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}}) \neq 0$.*

The same is true with R'_∞, T'_∞, C'_∞ replacing R_∞, T_∞ and C_∞.

We now come to the principal result of this section.

PROPOSITION 6.3.8. *With the notation and assumptions just established,*

$$\mathrm{Supp}_{R_\infty}(H^*(C_\infty)) = \mathrm{Spec}\, T_\infty \subset \mathrm{Spec}\, R_\infty$$

contains every irreducible component of $\mathrm{Spec}\, R_\infty$ *of maximal dimension.*

Proof. As T_∞ and $H^*(C_\infty)$ are finite over S_∞ we see that $\mathrm{Supp}_{S_\infty}(H^*(C_\infty))$ is the image of $\mathrm{Spec}\, T_\infty \to \mathrm{Spec}\, S_\infty$. Thus $\mathrm{Supp}_{S_\infty}(H^*(C_\infty))$ has dimension at most $\dim R_\infty = \dim S_\infty - l_0$, and any prime in $\mathrm{Supp}_{S_{\infty,\mathfrak{p}}}(H^*(C_{\infty,\mathfrak{p}}))$ has codimension at least l_0 in $\mathrm{Spec}\, S_{\infty,\mathfrak{p}}$.

Since $S_{\infty,\mathfrak{p}}/\mathfrak{p} \cong (S_\infty/\mathfrak{p})[\frac{1}{p}]$, our assumptions imply that $C_{\infty,\mathfrak{p}} \otimes^{\mathbf{L}} S_{\infty,\mathfrak{p}}/\mathfrak{p}$ is non-zero and has cohomology concentrated in degrees $[q_0, q_0 + l_0]$. Thus $C_{\infty,\mathfrak{p}}$ is quasi-isomorphic to a perfect complex of $S_{\infty,\mathfrak{p}}$-modules concentrated in degrees $[q_0, q_0 + l_0]$. (See, for instance, [KT17, Lem. 2.3].) From the key [CG18, Lem. 6.2] we deduce that $H^*(C_{\infty,\mathfrak{p}})$ is non-zero exactly in degree $q_0 + l_0$ and that $\mathrm{Supp}_{S_{\infty,\mathfrak{p}}}(H^{q_0+l_0}(C_{\infty,\mathfrak{p}}))$ contains a prime of codimension at most l_0 in $\mathrm{Spec}\, S_{\infty,\mathfrak{p}}$. Thus $\mathrm{Supp}_{S_\infty}(H^*(C_\infty))$ contains a prime of dimension $\dim S_\infty - l_0$. Let x_1 denote a pre-image of this prime in $\mathrm{Spec}\, T_\infty$, so that x_1 must be a generic point of $\mathrm{Spec}\, R_\infty - \mathrm{Spec}\, R_\infty/(\varpi)$ of dimension $\dim S_\infty - l_0$. Moreover $\mathrm{lg}_{T_{\infty,x_1}}(C_{\infty,x_1}) \neq (0)$. Choose a generic point $\overline{x}_1$ of $\mathrm{Spec}\, R_\infty/(x_1, \varpi)$, which must have dimension $\dim S_\infty - l_0 - 1$ and be a generic point of $\mathrm{Spec}\, R_\infty/(\varpi)$ in the image of $\mathrm{Spec}\, T_\infty$. Let $\overline{x}'_1$ denote the corresponding point of $\mathrm{Spec}\, R'_\infty/(\varpi)$. It cannot be generic in $\mathrm{Spec}\, R'_\infty$ and so must generalize to x'.

Now let x_2 be any other generic point of $\mathrm{Spec}\, R_\infty$ of dimension $\dim S_\infty - l_0$. We wish to show that it lies in $\mathrm{Spec}\, T_\infty$. Choose a generic point $\overline{x}_2$ of $\mathrm{Spec}\, R_\infty/(x_2, \varpi)$, which must have dimension $\dim S_\infty - l_0 - 1$ and be a generic point of $\mathrm{Spec}\, R_\infty/(\varpi)$. Let $\overline{x}'_2$ denote the corresponding point of $\mathrm{Spec}\, R'_\infty/(\varpi)$. It cannot be generic in $\mathrm{Spec}\, R'_\infty$ and so must generalize to x'.

We now repeatedly use Lemma 6.3.7. As $\mathrm{lg}_{T_{\infty,x_1}}(C_{\infty,x_1}) \neq (0)$, we deduce that $\overline{x}_1 \in \mathrm{Spec}\, T_\infty$ and $\mathrm{lg}_{T_{\infty,\overline{x}_1}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}_1}) \neq 0$. Thus $\overline{x}'_1 \in \mathrm{Spec}\, T'_\infty$ and $\mathrm{lg}_{T'_{\infty,\overline{x}'_1}}((C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}'_1}) \neq 0$, from which we deduce that $x' \in \mathrm{Spec}\, T'_\infty$ and $\mathrm{lg}_{T'_{\infty,x'}}(C'_{\infty,x'}) \neq 0$. We further deduce that $\overline{x}'_2 \in \mathrm{Spec}\, T'_\infty$ and $\mathrm{lg}_{T'_{\infty,\overline{x}'_2}}((C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}'_2}) \neq 0$. Thus $\overline{x}_2 \in \mathrm{Spec}\, T_\infty$ and $\mathrm{lg}_{T_{\infty,\overline{x}_2}}((C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/(\varpi))_{\overline{x}_2}) \neq 0$, from which we finally deduce that $x_2 \in \mathrm{Spec}\, T_\infty$ (and $\mathrm{lg}_{T_{\infty,x_2}}(C_{\infty,x_2}) \neq (0)$). $\square$

COROLLARY 6.3.9. *Let x be a prime of R_∞ lying in an irreducible component of* $\mathrm{Spec}\, R_\infty$ *of maximal dimension. Let y be the contraction of x in S_∞. Then the support of $H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/y)_y$ over* $\mathrm{Spec}\, R_\infty$ *contains x. If y is one dimensional of characteristic* 0, *this says that x is in the support of $H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/y)[1/p]$.*

Proof. It follows from Proposition 6.3.8 that x is contained in $\operatorname{Spec} T_\infty$ and occurs in the support of $H^*(C_\infty)$. It also occurs in the support of $H^*(C_{\infty,y}) = H^*(C_\infty)_y$. Let r be maximal such that $H^r(C_{\infty,y})_x$ is non-zero. From the Tor spectral sequence

$$\operatorname{Tor}_{-i}^{S_{\infty,y}}(H^j(C_{\infty,y}), S_{\infty,y}/y) \Rightarrow H^{i+j}(C_{\infty,y} \otimes^{\mathbf{L}}_{S_{\infty,y}} S_{\infty,y}/y)$$

we see that $H^r(C_{\infty,y} \otimes^{\mathbf{L}}_{S_{\infty,y}} S_{\infty,y}/y)_x$ surjects onto $H^r(C_{\infty,y})_x/y \neq (0)$, so that x lies in the support of $H^r(C_{\infty,y} \otimes^{\mathbf{L}}_{S_{\infty,y}} S_{\infty,y}/y) = H^r(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/y)_y$, as desired. □

6.4. *Ultrapatching.*

6.4.1. *Set-up for patching.* We begin by fixing a non-principal ultrafilter $\mathfrak{F}$ on the set $\mathbf{N} = \{N \geq 1\}$. We fix a ring Λ that is isomorphic to a power series ring over $\mathcal{O}$.

Let δ, g, q be positive integers, and set $\Delta_\infty = \mathbf{Z}_p^{nq}$. We let $\mathcal{T}$ be a formal power series ring over Λ (it will come from framing variables in our application) and let $S_\infty = \mathcal{T}[[\Delta_\infty]]$. We view S_∞ as an augmented Λ-algebra and denote the augmentation ideal by $\mathfrak{a}_\infty$. We also suppose we have two rings $R^{\mathrm{loc}}, R'^{\mathrm{loc}}$ in CNL_Λ with a fixed isomorphism $R^{\mathrm{loc}}/\varpi \cong R'^{\mathrm{loc}}/\varpi$ and denote by R_∞ and R'_∞ the formal power series rings in g variables over R^{loc} and R'^{loc}.

Our input for patching is the following data for each $N \in \mathbf{N} \cup \{0\}$:

(1) A quotient Δ_N of Δ_∞ such that the kernel of $\Delta_\infty \to \Delta_N$ is contained in $(p^N\mathbf{Z}_p)^{nq} \subset \Delta_\infty$. If $N = 0$, we let Δ_0 be the trivial group, thought of as a quotient of Δ_∞. We set $S_N = \mathcal{T}[\Delta_N]$.

(2) A pair of perfect complexes $\mathcal{C}_N$, $\mathcal{C}'_N$ in $\mathbf{D}(\Lambda[\Delta_N])$, together with an isomorphism $\mathcal{C}_N \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \Lambda/\varpi[\Delta_N] \cong \mathcal{C}'_N \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \Lambda/\varpi[\Delta_N]$ in $\mathbf{D}(\Lambda/\varpi[\Delta_N])$. We denote these complexes by $\mathcal{C}_N/\varpi$ and $\mathcal{C}'_N/\varpi$ for short. We moreover assume that we have commutative $\Lambda[\Delta_N]$-subalgebras $T_N \subset \operatorname{End}_{\mathbf{D}(\Lambda[\Delta_N])}(\mathcal{C}_N)$, $T'_N \subset \operatorname{End}_{\mathbf{D}(\Lambda[\Delta_N])}(\mathcal{C}'_N)$ that map to the same subalgebra

$$\overline{T}_N \subset \operatorname{End}_{\mathbf{D}(\Lambda/\varpi[\Delta_N])}(\mathcal{C}_N/\varpi) = \operatorname{End}_{\mathbf{D}(\Lambda/\varpi[\Delta_N])}(\mathcal{C}'_N/\varpi),$$

where these endomorphism algebras are identified using our fixed quasi-isomorphism $\mathcal{C}_N/\varpi \cong \mathcal{C}'_N/\varpi$.

(3) A pair of rings R_N, R'_N in $\mathrm{CNL}_{\Lambda[\Delta_N]}$ with an isomorphism $R_N/\varpi \cong R'_N/\varpi$ together with R^{loc}- and R'^{loc}-algebra structures on $\mathcal{T}\widehat{\otimes}_\Lambda R_N$ and $\mathcal{T}\widehat{\otimes}_\Lambda R'_N$ respectively that are compatible modulo ϖ with the isomorphisms $R_N/\varpi \cong R'_N/\varpi$ and $R^{\mathrm{loc}}/\varpi \cong R'^{\mathrm{loc}}/\varpi$.

(4) Surjective R^{loc}- and R'^{loc}-algebra maps $R_\infty \to \mathcal{T}\widehat{\otimes}_\Lambda R_N$ and $R'_\infty \to \mathcal{T}\widehat{\otimes}_\Lambda R_N$, which are compatible modulo ϖ.

(5) Nilpotent ideals I_N of T_N and I'_N of T'_N with nilpotence degree $\leq \delta$, and continuous surjections $R_N \to T_N/I_N$, $R'_N \to T_N/I'_N$. We demand that

these maps are also compatible modulo ϖ, in the following sense: denote by $\overline{I}_N$ and $\overline{I}'_N$ the images of I_N and I'_N in $\overline{T}_N$. Then the induced maps $R_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ and $R'_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ are equal when we identify R_N/ϖ and R'_N/ϖ via the fixed isomorphism between them.

We moreover assume that for each $N \geq 1$, we have isomorphisms $\pi_N : \mathcal{C}_N \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \Lambda \cong \mathcal{C}_0$ and $\pi'_N : \mathcal{C}'_N \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \Lambda \cong \mathcal{C}'_0$ in $\mathbf{D}(\Lambda)$ that are compatible mod ϖ. We obtain induced maps $T_N \otimes_{\Lambda[\Delta_N]} \Lambda \to \mathrm{End}_{\mathbf{D}(\Lambda)}(\mathcal{C}_0)$ and $T'_N \otimes_{\Lambda[\Delta_N]} \Lambda \to \mathrm{End}_{\mathbf{D}(\Lambda)}(\mathcal{C}'_0)$ that we assume factor through maps $T_N \otimes_{\Lambda[\Delta_N]} \Lambda \to T_0$ and $T'_N \otimes_{\Lambda[\Delta_N]} \Lambda \to T'_0$ that are surjective when composed with the projections to T_0/I_0 and T'_0/I'_0.

Finally, we assume that we have isomorphisms $R_N \otimes_{\Lambda[\Delta_N]} \Lambda \cong R_0$ and $R'_N \otimes_{\Lambda[\Delta_N]} \Lambda \cong R'_0$ that are compatible mod ϖ and with the maps from R_∞ in part (4). We also also demand compatibility with the maps $T_N \otimes_{\Lambda[\Delta_N]} \Lambda \to T_0$ and $T'_N \otimes_{\Lambda[\Delta_N]} \Lambda \to T'_0$ above. More precisely, we denote by $I_{N,0}$ and $I'_{N,0}$ the images of I_N and I'_N in T_0/I_0 and T'_0/I'_0, and we demand that the surjective maps $R_N \otimes_{\Lambda[\Delta_N]} \Lambda \to (T_0/I_0)/I_{N,0}$ and $R'_N \otimes_{\Lambda[\Delta_N]} \Lambda \to (T'_0/I'_0)/I'_{N,0}$ are identified with the maps $R_0 \to (T_0/I_0)/I_{N,0}$ and $R'_0 \to (T'_0/I'_0)/I'_{N,0}$ via the isomorphisms $R_N \otimes_{\Lambda[\Delta_N]} \Lambda \cong R_0$ and $R'_N \otimes_{\Lambda[\Delta_N]} \Lambda \cong R'_0$.

6.4.2. *Patched complexes.* Apart from Remark 6.4.14 and Proposition 6.4.17, results and definitions in this subsection will be stated just for the complexes $\mathcal{C}_N$ and the associated objects and structures, but they also apply to the complexes $\mathcal{C}'_N$.

Definition 6.4.3. Let J be an open ideal in S_∞. Let I_J be the (cofinite) subset of $N \in \mathbf{N}$ such that J contains the kernel of $S_\infty \to S_N$. For $N \in I_J$, we define

$$\mathcal{C}(J,N) = S_\infty/J \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \mathcal{C}_N \in \mathbf{D}(S_\infty/J),$$

let $T(J,N)$ denote the image of $S_\infty/J \otimes_{\Lambda[\Delta_N]} T_N$ in $\mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J,N))$, and denote by $I(J,N)$ the ideal generated by the image of I_N in $T(J,N)$. We have $I(J,N)^\delta = 0$.

Additionally, for $d \geq 1$, we define

$$R(d,J,N) = R_N/\mathfrak{m}^d_{R_N} \otimes_{\Lambda[\Delta_N]} S_\infty/J.$$

For every d, J and N, we have a surjective R^{loc}-algebra map $R_\infty \to R(d,J,N)$, which factors through a finite quotient $R_\infty/\mathfrak{m}^{e(d,J)}_{R_\infty}$ for some $e(d,J)$ that is independent of N.

For each pair (J,N) such that $\mathcal{C}(J,N)$ is defined, we fix a choice $\mathcal{F}(J,N)$ of minimal complex of finite free S_∞/J-modules that is quasi-isomorphic to $\mathcal{C}(J,N)$ (cf. [KT17, Lem. 2.3]). Then for any $i \in \mathbf{Z}$, we have

$$\mathrm{rk}_{S_\infty/J}(\mathcal{F}(J,N)^i) = \dim_k H^i(\mathcal{C}_0 \otimes^{\mathbf{L}}_\Lambda k).$$

Remark 6.4.4. Recall that we have a surjective map $R_N \to T_N/I_N$. We therefore obtain a surjective map $R_N \otimes_{\Lambda[\Delta_N]} S_\infty/J \to T(J,N)/I(J,N)$. For d sufficiently large depending on J (but *not* depending on N), this map factors through a surjective map $R(d,J,N) \to T(J,N)/I(J,N)$. Indeed, it suffices to show that there is an integer $d_0(J)$ such that for any $d \geq d_0(J)$, and for any $x \in \mathfrak{m}_{T_N}$, the image of x^d in $\mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J,N))$ (and therefore the image of x^d in $T(J,N)$) is zero. Since

$$H^*(\mathcal{C}(J,N) \otimes^{\mathbf{L}}_{S_\infty/J} k) \cong H^*(\mathcal{C}_0 \otimes^{\mathbf{L}}_{\Lambda} k)$$

is a vector space of finite dimension independent of N and J, we can find an integer d_1 such that $x^{d_1} H^*(\mathcal{C}(J,N) \otimes^{\mathbf{L}}_{S_\infty/J} k) = 0$ (because x acts through a nilpotent endomorphism). The existence of the spectral sequence of a filtered complex implies that there is an integer d_2 such that $x^{d_2} H^*(\mathcal{C}(J,N)) = 0$. (Here we are using the fact that S_∞/J has finite length as a module over itself). Finally, the fact that $\mathcal{C}(J,N)$ is a perfect complex, with cohomology bounded in a range that depends only on $\mathcal{C}_0$, implies the existence of the integer $d_0(J)$ (use [KT17, Lem. 2.5].) (A similar argument appears at the start of the proof of [KT17, Prop. 3.1].)

Remark 6.4.5. If J contains $\mathfrak{a}_\infty$, then we can identify S_∞/J with $\Lambda/s(J)$, where $s(J)$ is an open ideal of Λ. For each $N \in I_J$, the isomorphism $\pi_N : \mathcal{C}_N \otimes^{\mathbf{L}}_{\Lambda[\Delta_N]} \Lambda \cong \mathcal{C}_0$ induces an isomorphism $\pi_{J,N} : \mathcal{C}(J,N) \cong \mathcal{C}_0 \otimes^{\mathbf{L}}_{\Lambda} \Lambda/s(J)$.

Remark 6.4.6. Suppose we have open ideals $J_1 \subset J_2$ of S_∞ and $N \in I_{J_1}$. Then we have a natural map $\mathcal{C}(J_1,N) \to \mathcal{C}(J_2,N)$ that induces a quasi-isomorphism

$$S_\infty/J_2 \otimes^{\mathbf{L}}_{S_\infty/J_1} \mathcal{C}(J_1,N) \cong \mathcal{C}(J_2,N).$$

We obtain a surjective map $T(J_1,N) \to T(J_2,N)$, and the image of $I(J_1,N)$ under this map is equal to $I(J_2,N)$. So we also obtain a surjective map $T(J_1,N)/I(J_1,N) \to T(J_2,N)/I(J_2,N)$.

For J an open ideal in S_∞, $\mathfrak{F}$ restricts to give a non-principal ultrafilter on I_J, which we again denote by $\mathfrak{F}$. This corresponds to a point $x_{\mathfrak{F}} \in \mathrm{Spec}(\prod_{N \in I_J} S_\infty/J)$ by [GN22, Lem. 2.2.2], with localization $(\prod_{N \in I_J} S_\infty/J)_{x_{\mathfrak{F}}}$ canonically isomorphic to S_∞/J.

Definition 6.4.7. We make the following definitions:

$$\mathcal{C}(J,\infty) = \Big(\prod_{N \in I_J} S_\infty/J\Big)_{x_{\mathfrak{F}}} \otimes_{\prod_{N \in I_J} S_\infty/J} \prod_{N \in I_J} \mathcal{C}(J,N) \in \mathbf{D}(S_\infty/J),$$

$$R(d,J,\infty) = \Big(\prod_{N \in I_J} S_\infty/J\Big)_{x_{\mathfrak{F}}} \otimes_{\prod_{N \in I_J} S_\infty/J} \prod_{N \in I_J} R(d,J,N),$$

and $T(J,\infty)$ is defined to be the image of

$$\Big(\prod_{N\in I_J} S_\infty/J\Big)_{x_{\mathfrak{F}}} \otimes_{\prod_{N\in I_J} S_\infty/J} \prod_{N\in I_J} T(J,N)$$

in $\mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J,\infty))$, and the ideal $I(J,\infty)\subset T(J,\infty)$ is defined to be the image of $(\prod_{N\in I_J} S_\infty/J)_{x_{\mathfrak{F}}} \otimes_{\prod_{N\in I_J} S_\infty/J} \prod_{N\in I_J} I(J,N)$ in $T(J,\infty)$.

Remark 6.4.8. Since the rings $R(d,J,N)$ are all quotients of $R_\infty/\mathfrak{m}_{R_\infty}^{e(d,J)}$ (and are, in particular, finite of bounded cardinality), the ultraproduct $R(d,J,\infty)$ is itself a quotient of $R_\infty/\mathfrak{m}_{R_\infty}^{e(d,J)}$.

Lemma 6.4.9.

(1) *$I(J,\infty)$ is a nilpotent ideal of $T(J,\infty)$, with $I(J,\infty)^\delta = 0$.*
(2) *The maps $R(d,J,N)\to T(J,N)/I(J,N)$ for d sufficiently large depending on J (see Remark 6.4.4) induce a surjective S_∞/J-algebra map $R(d,J,\infty)\to T(J,\infty)/I(J,\infty)$.*

Proof. The first part follows from the fact that $\prod_{N\in I_J} I(J,N)$ is a nilpotent ideal of $\prod_{N\in I_J} T(J,N)$ with nilpotence degree $\le\delta$. The second part follows by first considering the map $\prod_{N\in I_J} R(d,J,N)\to\prod_{N\in I_J}(T(J,N)/I(J,N)) = (\prod_{N\in I_J} T(J,N))/(\prod_{N\in I_J} I(J,N))$, localizing at $x_{\mathfrak{F}}$ and finally passing to the image in $T(J,\infty)/I(J,\infty)$. □

Proposition 6.4.10.

(1) *$\mathcal{C}(J,\infty)$ is a perfect complex of S_∞/J-modules.*
(2) *The maps $R_\infty\to T(J,N)/I(J,N)$ induce a surjection $R_\infty\to T(J,\infty)/I(J,\infty)$.*
(3) *If J contains $\mathfrak{a}_\infty$, then the isomorphisms $\pi_{J,N}$ induce an isomorphism $\pi_{J,\infty}:\mathcal{C}(J,\infty)\cong\mathcal{C}_0\otimes^{\mathbf{L}}_\Lambda \Lambda/s(J)$.*
(4) *Suppose we have open ideals $J_1\subset J_2$ of S_∞. Then the maps $\mathcal{C}(J_1,N)\to\mathcal{C}(J_2,N)$ in $\mathbf{D}(S_\infty/J_1)$ for $N\in I_{J_1}$ induce an isomorphism*

$$S_\infty/J_2\otimes^{\mathbf{L}}_{S_\infty/J_1}\mathcal{C}(J_1,\infty)\cong\mathcal{C}(J_2,\infty).$$

(5) *Let J_1,J_2 be as in the previous part. The map $\mathcal{C}(J_1,\infty)\to\mathcal{C}(J_2,\infty)$ induces a surjective map $T(J_1,\infty)\to T(J_2,\infty)$, and the image of $I(J_1,\infty)$ under this map is equal to $I(J_2,\infty)$.*

Proof.

(1) Perfectness of $\mathcal{C}(J,\infty)$ follows from [GN22, Cor. 2.2.7] — to apply this corollary we need to show that there are constants D,a,b (independent of N) such that the complexes $\mathcal{C}(J,N)$ are each quasi-isomorphic to complexes of finite free S_∞/J-modules of rank $\le D$ concentrated in degrees $[a,b]$. This follows from the theory of minimal resolutions, which we have already applied in order to assert the existence of the complexes $\mathcal{F}(J,N)$ above.

(2) By the previous part of the proof, the complexes $\mathcal{F}(J,N)$ ($N \in I_J$) fall into finitely many isomorphism classes. Therefore there is an element Σ_0 of the ultrafilter $\mathfrak{F}$ on I_J such that the $\mathcal{F}(J,N)$ are isomorphic for all $N \in \Sigma_0$. We fix isomorphisms (of complexes) between the $\mathcal{F}(J,N)$ for $N \in \Sigma_0$ and a single complex $\mathcal{F}(J,\infty)$. Then for all $N \in \Sigma_0$, we can identify all the finite endomorphism algebras $\mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J,N))$ with each other. We deduce that there is a subset $\Sigma_1 \subset \Sigma_0$ with $\Sigma_1 \in \mathfrak{F}$ such that, under this identification, the finite Hecke rings $T(J,N)$ and their ideals $I(J,N)$ are also identified. So $T(J,\infty) \cong T(J,N)$ and $I(J,\infty)$ corresponds to $I(J,N)$ for $N \in \Sigma_1$. Since each map $R_\infty \to T(J,N)/I(J,N)$ is surjective, the map $R_\infty \to T(J,\infty)/I(J,\infty)$ is also surjective.

(3) The third part follows immediately from the exactness of products and localization.

(4) First we consider the map of complexes

$$\prod_{N \in I_{J_1}} \mathcal{C}(J_1,N) \to \prod_{N \in I_{J_1}} \mathcal{C}(J_2,N).$$

Since $\prod_{N \in I_{J_1}} S_\infty/J_2$ is a finitely presented $\prod_{N \in I_{J_1}} S_\infty/J_1$-module (as direct products are exact), the tensor product $(\prod_{N \in I_{J_1}} S_\infty/J_2) \otimes_{\prod S_\infty/J_1}$ commutes with direct products ([Stacks, Tag 059K]). We deduce (using Remark 6.4.6) that

$$\Big(\prod_{N \in I_{J_1}} S_\infty/J_2\Big) \otimes^{\mathbf{L}}_{\prod S_\infty/J_1} \prod_{N \in I_{J_1}} \mathcal{C}(J_1,N)$$
$$= \prod_{N \in I_{J_1}} S_\infty/J_2 \otimes^{\mathbf{L}}_{S_\infty/J_1} \mathcal{C}(J_1,N) \cong \prod_{N \in I_{J_1}} \mathcal{C}(J_2,N).$$

Localizing at $x_{\mathfrak{F}}$ gives the desired statement; since I_{J_1} is cofinite in I_{J_2}, we can naturally identify the localization of $\prod_{N \in I_{J_1}} \mathcal{C}(J_2,N)$ with the localization of $\prod_{N \in I_{J_2}} \mathcal{C}(J_2,N)$.

(5) The final statement follows from the proof of part (2): there is a $\Sigma \subset I_{J_1}$ with $\Sigma \in \mathfrak{F}$ such that $T(J_i,\infty) \cong T(J_i,N)$ and $I(J_i,\infty)$ corresponds to $I(J_i,N)$ under these isomorphisms for all $N \in \Sigma$. Now the desired statement is a consequence of Remark 6.4.6. □

We write $\mathcal{F}(J,\infty)$ for the minimal complex isomorphic to $\mathcal{C}(J,\infty)$ in $\mathbf{D}(S_\infty/J)$ constructed in the proof of the previous proposition.

Definition 6.4.11. We define a complex of S_∞-modules

$$\mathcal{C}_\infty = \varprojlim_r \mathcal{F}(\mathfrak{m}^r_{S_\infty},\infty),$$

where the transition maps in the inverse limit are given by making a choice for each $r \geq 1$ of a map of complexes lifting the natural maps $\mathcal{C}(\mathfrak{m}^{r+1}_{S_\infty},\infty) \to$

$\mathcal{C}(\mathfrak{m}_{S_\infty}^r, \infty)$ in $\mathbf{D}(S_\infty/\mathfrak{m}_{S_\infty}^{r+1})$. To see that such a map of complexes exists, note that since $\mathcal{F}(\mathfrak{m}_{S_\infty}^{r+1}, \infty)$ is a bounded complex of free $S_\infty/\mathfrak{m}_{S_\infty}^{r+1}$-modules, viewed as an element of the homotopy category $\mathbf{K}(S_\infty/\mathfrak{m}_{S_\infty}^{r+1})$ of chain complexes of $S_\infty/\mathfrak{m}_{S_\infty}^{r+1}$-modules, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{K}(S_\infty/\mathfrak{m}_{S_\infty}^{r+1})}(\mathcal{F}(\mathfrak{m}_{S_\infty}^{r+1}, \infty), \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty))\\ &\quad = \mathrm{Hom}_{\mathbf{D}(S_\infty/\mathfrak{m}_{S_\infty}^{r+1})}(\mathcal{C}(\mathfrak{m}_{S_\infty}^{r+1}, \infty), \mathcal{C}(\mathfrak{m}_{S_\infty}^r, \infty)).\end{aligned}$$

Similarly, we let $T_\infty = \varprojlim_J T(J, \infty)$, where the transition maps in the inverse limit are described in Proposition 6.4.10(5). The inverse system of ideals $I(J, \infty)$ defines an ideal I_∞ of T_∞ that satisfies $I_\infty^\delta = 0$.

Proposition 6.4.12.

(1) *$\mathcal{C}_\infty$ is a bounded complex of finite free S_∞-modules, and for each open ideal J of S_∞, there is an isomorphism $\mathcal{C}_\infty \otimes_{S_\infty} S_\infty/J \cong \mathcal{C}(J, \infty)$ in $\mathbf{D}(S_\infty/J)$.*

(2) *The natural map $\mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty) \to \varprojlim_J \mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J, \infty))$ is an isomorphism, and we therefore obtain an injective map $T_\infty \to \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty)$.*

(3) *The surjective Λ-algebra maps $R_\infty \to T(J, \infty)/I(J, \infty)$ induce a surjection $R_\infty \to T_\infty/I_\infty$, which factors as a composition of the map $R_\infty \to \varprojlim_{d,J} R(d, J, \infty)$ and the S_∞-algebra map $\varprojlim_{d,J} R(d, J, \infty) \to T_\infty/I_\infty$ defined by taking the inverse limit of the maps in Lemma 6.4.9(2).*

Proof.

(1) It follows from the proof of Proposition 6.4.10(1) that

$$\mathrm{rk}_{S_\infty/\mathfrak{m}_{S_\infty}^r}(\mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)^i) = \dim_k H^i(\mathcal{C}_0 \otimes_\Lambda^{\mathbf{L}} k)$$

for all r. Moreover, it follows from Proposition 6.4.10(4) and the fact that any quasi-isomorphism of minimal complexes is an isomorphism that the transition map $\mathcal{F}(\mathfrak{m}_{S_\infty}^{r+1}, \infty) \to \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)$ induces an isomorphism

$$S_\infty/\mathfrak{m}_{S_\infty}^r \otimes_{S_\infty/\mathfrak{m}_{S_\infty}^{r+1}} \mathcal{F}(\mathfrak{m}_{S_\infty}^{r+1}, \infty) \cong \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty).$$

It is now clear that $\mathcal{C}_\infty$ is a bounded complex of finite free S_∞-modules. If J is an open ideal of S_∞, then for r sufficiently large so that $\mathfrak{m}_{S_\infty}^r \subset J$, $\mathcal{C}_\infty \otimes_{S_\infty} S_\infty/J$ is isomorphic to $S_\infty/J \otimes_{S_\infty/\mathfrak{m}_{S_\infty}^r} \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)$, which is quasi-isomorphic to $\mathcal{C}(J, \infty)$ by Proposition 6.4.10(4).

(2) For the second part, we first note that T_∞ injects into the inverse limit $\varprojlim_J \mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J, \infty))$, since inverse limits are left exact. The natural map $\mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty) \to \varprojlim_J \mathrm{End}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J, \infty))$ is an isomorphism, by the first part of this proposition and (the proof of) [KT17, Lem. 2.13(3)].

(3) Since the $T(J, \infty)$ are finite rings, it follows that the inverse system $I(J, \infty)_J$ satisfies the Mittag-Leffler condition and the natural map $T_\infty/I_\infty \to$

$\varprojlim_J T(J,\infty)/I(J,\infty)$ is an isomorphism. For each J, the surjective map $R_\infty \to T(J,\infty)/I(J,\infty)$ factors through a finite quotient $R_\infty/\mathfrak{m}_{R_\infty}^{d(J)}$ of R_∞. Again, finiteness implies that the Mittag-Leffler condition holds, so taking the inverse limit over J gives a surjective map $R_\infty = \varprojlim_J R_\infty/\mathfrak{m}_{R_\infty}^{d(J)} \to T_\infty/I_\infty = \varprojlim_J T(J,\infty)/I(J,\infty)$. The desired factorization of the map $R_\infty \to T_\infty/I_\infty$ follows from the fact that the maps $R_\infty \to T(J,\infty)/I(J,\infty)$ factor through $R(d,J,\infty)$ for d sufficiently large. □

Remark 6.4.13. There is a natural isomorphism $H^*(\mathcal{C}_\infty) \cong \varprojlim_J H^*(\mathcal{C}_\infty/J) = \varprojlim_J H^*(\mathcal{C}(J,\infty))$, so the cohomology of $\mathcal{C}_\infty$ is independent of the choices of transition maps made to construct $\mathcal{C}_\infty$. Moreover, if we denote by $\mathcal{D}_\infty$ the complex constructed with a different choice of transition maps, we have $\mathrm{Hom}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty,\mathcal{D}_\infty) = \varprojlim_J \mathrm{Hom}_{\mathbf{D}(S_\infty/J)}(\mathcal{C}(J,\infty),\mathcal{C}(J,\infty))$ by the argument of Proposition 6.4.12(2), so there is a canonical isomorphism between $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$ in $\mathbf{D}(S_\infty)$.

Remark 6.4.14. Note that the map $\alpha : R_\infty \to \varprojlim_{d,J} R(d,J,\infty)$ is surjective, and $\varprojlim_{d,J} R(d,J,\infty)$ is an S_∞-algebra. As S_∞ is formally smooth over Λ, we can choose a lift of the map $S_\infty \to \alpha(R_\infty)$ to a map $S_\infty \to R_\infty$. In fact, we can and do make such a choice for R_∞ and R'_∞ compatibly mod ϖ since

$$(\varprojlim_{d,J} R(d,J,\infty))/\varpi = \varprojlim_{d,J}(R(d,J,\infty)/\varpi) \cong \varprojlim_{d,J}(R'(d,J,\infty)/\varpi)$$

and since the sequence

$$R_\infty \xrightarrow{x\mapsto(x \bmod \varpi,\alpha(x))} R_\infty/\varpi \times \varprojlim_{d,J} R(d,J,\infty) \xrightarrow{(y,z)\mapsto\alpha(y)-z \bmod \varpi} R(d,J,\infty)/\varpi$$

(and the analogous one for R'_∞) is exact. We regard R_∞ as an S_∞-algebra from now on. The map $R_\infty \to T_\infty/I_\infty$ is an S_∞-algebra map.

Lemma 6.4.15. *The isomorphisms* $R(d,J,N) \otimes_{S_\infty/J} S_\infty/(J+\mathfrak{a}_\infty) \cong R_0/(\mathfrak{m}_{R_0}^d, s(J+\mathfrak{a}_\infty))$ *induce a surjective map* $R_\infty/\mathfrak{a}_\infty \to R_0$.

Proof. First we note that, following the proof of 6.4.10(4), the isomorphisms

$$R(d,J,N) \otimes_{S_\infty/J} S_\infty/(J+\mathfrak{a}_\infty) \cong R_0/(\mathfrak{m}_{R_0}^d, s(J+\mathfrak{a}_\infty))$$

induce an isomorphism

$$R(d,J,\infty) \otimes_{S_\infty/J} S_\infty/(J+\mathfrak{a}_\infty) \cong R_0/(\mathfrak{m}_{R_0}^d, s(J+\mathfrak{a}_\infty)).$$

In particular, the map

$$R_\infty/\mathfrak{a}_\infty \to R(d,J,\infty) \otimes_{S_\infty/J} S_\infty/(J+\mathfrak{a}_\infty) = R_0/(\mathfrak{m}_{R_0}^d, s(J+\mathfrak{a}_\infty))$$

is surjective and factors through $R_\infty/(\mathfrak{m}_{R_\infty}^{e(d,J)} + \mathfrak{a}_\infty)$ for some $e(d,J)$. Taking the inverse limit, we obtain a surjective map $R_\infty/\mathfrak{a}_\infty \to R_0$. □

PROPOSITION 6.4.16. *There is an isomorphism $\mathcal{C}_\infty/\mathfrak{a}_\infty \to \mathcal{C}_0$ in $\mathbf{D}(\Lambda)$ that induces a map $T_\infty \to T_0$ that becomes surjective when composed with the projection $T_0 \to T_0/I_0$. Denoting the image of I_∞ under this surjective map by $I_{\infty,0}$, we obtain a surjective map $R_\infty/\mathfrak{a}_\infty \to (T_0/I_0)/I_{\infty,0}$. This map is the composition of the map $R_\infty/\mathfrak{a}_\infty \to R_0$ in Lemma 6.4.15 with the map $R_0 \to (T_0/I_0)/I_{\infty,0}$ coming from our original set-up.*

Proof. We have $\mathcal{C}_\infty/\mathfrak{a}_\infty = \varprojlim_r \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\mathfrak{a}_\infty$, and $\mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\mathfrak{a}_\infty$ is a minimal resolution of $\mathcal{C}_\infty/(\mathfrak{m}_{S_\infty}^r + \mathfrak{a}_\infty) \cong \mathcal{C}(\mathfrak{m}_{S_\infty}^r + \mathfrak{a}_\infty, \infty)$. By Proposition 6.4.10(3), this is quasi-isomorphic to $\mathcal{C}_0 \otimes_\Lambda^{\mathbf{L}} \Lambda/s(\mathfrak{m}_{S_\infty}^r + \mathfrak{a}_\infty)$. Replacing $\mathcal{C}_0$ by a quasi-isomorphic bounded complex of finite projective Λ-modules and applying [KT17, Lem. 2.13], we see that the quasi-isomorphisms $\mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\mathfrak{a}_\infty \cong \mathcal{C}_0 \otimes_\Lambda^{\mathbf{L}} \Lambda/s(\mathfrak{m}_{S_\infty}^r + \mathfrak{a}_\infty)$ induce a quasi-isomorphism $\varprojlim_r \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\mathfrak{a}_\infty \cong \mathcal{C}_0$.

The induced map $T_\infty \to \mathrm{End}_{\mathbf{D}(\Lambda)}(\mathcal{C}_0)$ is the composite of the surjective map $T_\infty \to \varprojlim_{\mathfrak{a}_\infty \subset J} T(J, \infty)$ and an inverse limit of maps $T(J, \infty) \to \mathrm{End}_{\mathbf{D}(\Lambda/s(J))}(\mathcal{C}_0 \otimes_\Lambda^{\mathbf{L}} \Lambda/s(J))$. Each of these maps factors through T_0, and if we denote the image of T_0 in $\mathrm{End}_{\mathbf{D}(\Lambda/s(J))}(\mathcal{C}_0 \otimes_\Lambda^{\mathbf{L}} \Lambda/s(J))$ by T_0^J, then $T(J, \infty)$ surjects onto T_0^J/I_0. Passing to the inverse limit gives the desired map $T_\infty \to T_0$.

The compatibility with the map $R_\infty \to R_0$ follows from the compatibility between the maps $T_N \to T_0$ and $R_N \to R_0$ in our original set-up. □

We now separate out the primed and unprimed situations; so we have two perfect complexes of S_∞-modules, $\mathcal{C}_\infty$ and $\mathcal{C}'_\infty$.

PROPOSITION 6.4.17.

(1) *The quasi-isomorphisms $\mathcal{C}_N/\varpi \cong \mathcal{C}'_N/\varpi$ induce a quasi-isomorphism $\mathcal{C}_\infty/\varpi \cong \mathcal{C}'_\infty/\varpi$.*
(2) *Via the identification $\mathcal{C}_\infty/\varpi \cong \mathcal{C}'_\infty/\varpi$ of the previous part, T_∞ and T'_∞ have the same image in $\mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty/\varpi)$ and $\mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}'_\infty/\varpi)$. Call this common image $\overline{T}_\infty$.*
(3) *Write $\overline{I}_\infty$ and $\overline{I}'_\infty$ for the images of I_∞ and I'_∞ in $\overline{T}_\infty$ The actions of $R_\infty/\varpi \cong R'_\infty/\varpi$ (induced from T_∞ and T'_∞ respectively) on $H^*(\mathcal{C}_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty)$ and $H^*(\mathcal{C}'_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty)$ are identified via $\mathcal{C}_\infty/\varpi \cong \mathcal{C}'_\infty/\varpi$.*

Proof.

(1) The isomorphisms $\mathcal{C}_N/\varpi \cong \mathcal{C}'_N/\varpi$ in $\mathbf{D}(\Lambda[\Delta_N])$ induce compatible isomorphisms $\mathcal{C}(J + \varpi, \infty) \cong \mathcal{C}'(J + \varpi, \infty)$ for all J. Since
$$\mathcal{C}_\infty/\varpi = \varprojlim_r \mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\varpi$$
and $\mathcal{F}(\mathfrak{m}_{S_\infty}^r, \infty)/\varpi$ is a minimal resolution of $\mathcal{C}(\mathfrak{m}_{S_\infty}^r + \varpi, \infty)$, we have
$$\begin{aligned}&\mathrm{Hom}_{\mathbf{D}(S_\infty/\varpi)}(\mathcal{C}_\infty/\varpi, \mathcal{C}'_\infty/\varpi)\\ &\quad = \varprojlim_J \mathrm{Hom}_{\mathbf{D}(S_\infty/(J+\varpi))}(\mathcal{C}(J + \varpi, \infty), \mathcal{C}'(J + \varpi, \infty)).\end{aligned}$$

We therefore deduce the first part of the proposition.

(2) By the proof of the previous part, it suffices to show that the images of T_∞ and T'_∞ in $\mathrm{End}_{\mathbf{D}(S_\infty/(J+\varpi))}(\mathcal{C}(J+\varpi,\infty))$ and $\mathrm{End}_{\mathbf{D}(S_\infty/(J+\varpi))}(\mathcal{C}'(J+\varpi,\infty))$ respectively (which are $T(J+\varpi,\infty)$ and $T'(J+\varpi,\infty)$) are identified via the quasi-isomorphisms $\mathcal{C}(J+\varpi,\infty)\cong\mathcal{C}'(J+\varpi,\infty)$. This follows from the fact that for every $N\in I_{J+\varpi}$, $T(J+\varpi,N)$ and $T'(J+\varpi,N)$ are identified via the quasi-isomorphism $\mathcal{C}(J+\varpi,N)\cong\mathcal{C}'(J+\varpi,N)$, which is a consequence of our original assumptions; see point (2) in Section 6.4.1.

(3) It suffices to show that the maps $R_\infty/\varpi\to T(J+\varpi,\infty)/I(J+\varpi,\infty)$ and $R'_\infty/\varpi\to T'(J+\varpi,\infty)/I'(J+\varpi,\infty)$ are equal when we identify R_∞/ϖ with R'_∞/ϖ, $T(J+\varpi,\infty)$ with $T'(J+\varpi,\infty)$ and pass to the quotient by $I(J+\varpi,\infty)+I'(J+\varpi,\infty)$. This follows from the compatibility in point (5) of Section 6.4.1. □

6.5. *The proof of Theorem* 6.1.1. We are now in a position to prove the first main theorem of this chapter (Theorem 6.1.1). We first establish the result under additional conditions in Section 6.5.1, and then reduce to this case using soluble base change in Section 6.5.12.

6.5.1. *Application of the patching argument* (*Fontaine–Laffaille case*). We take F to be an imaginary CM number field and fix the following data:

(1) an integer $n\geq 2$ and a prime $p>n^2$;
(2) a finite set S of finite places of F, including the places above p;
(3) a (possibly empty) subset $R\subset S$ of places prime to p;
(4) a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, regular algebraic of some weight λ;
(5) a choice of isomorphism $\iota:\overline{\mathbf{Q}}_p\cong\mathbf{C}$.

We assume that the following conditions are satisfied:

(6) If l is a prime lying below an element of S, or which is ramified in F, then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.
(7) The prime p is unramified in F.
(8) For each embedding $\tau:F\hookrightarrow\mathbf{C}$, we have
$$\lambda_{\tau,1}+\lambda_{\tau c,1}-\lambda_{\tau,n}-\lambda_{\tau c,n}<p-2n.$$
(9) For each $v\in S_p$, let $\overline{v}$ denote the place of F^+ lying below v. Then there exists a place $\overline{v}'\neq\overline{v}$ of F^+ such that $\overline{v}'|p$ and
$$\sum_{\overline{v}''\neq\overline{v},\overline{v}'}[F^+_{\overline{v}''}:\mathbf{Q}_p]>\frac{1}{2}[F^+:\mathbf{Q}].$$
(10) The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible.

(11) If v is a place of F lying above p, then π_v is unramified.
(12) If $v \in R$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$.
(13) If $v \in S-(R\cup S_p)$, then π_v is unramified, $v \notin R^c$, and $H^2(F_v, \mathrm{ad}\,\overline{r_\iota(\pi)}) = 0$.
(14) $S-(R\cup S_p)$ contains at least two places with distinct residue characteristics.
(15) If $v \notin S$ is a finite place of F, then π_v is unramified.
(16) If $v \in R$, then $q_v \equiv 1 \bmod p$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.
(17) The representation $\overline{r_\iota(\pi)}$ is decomposed generic in the sense of Definition 4.3.1, and the image of $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ is enormous in the sense of Definition 6.2.29.

We define an open compact subgroup $K = \prod_v K_v$ of $\mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ as follows:

- If $v \notin S$, or $v \in S_p$, then $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.
- If $v \in R$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (R \cup S_p)$, then $K_v = \mathrm{Iw}_{v,1}$ is the pro-v Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$.

The following lemma shows that K is neat and hence is a good subgroup of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$.

Lemma 6.5.2. *Suppose that $K = \prod_v K_v \subset \mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ is an open compact subgroup and that there exist two places v, v' of F such that v, v' have distinct residue characteristics q, q' and $K_v = \mathrm{Iw}_{v,1}$, $K_{v'} = \mathrm{Iw}_{v',1}$. Then K is neat.*

Proof. We show that if $(g_v, g_{v'}) \in \mathrm{Iw}_{v,1} \times \mathrm{Iw}_{v',1}$; then the group $\Gamma_v \cap \Gamma_{v'}$ (see the definition of neat in Section 2.1.1) is trivial. Suppose this is not the case, then it contains a root of unity ζ of some prime order q''.

If α is an eigenvalue of g_v in $\overline{F}_v$, then $\alpha - 1$ is in the maximal ideal of $\mathcal{O}_{\overline{F}_v}$. The same is then true for ζ, thus $q'' = q$. However, running the above for v' instead of v also shows $q'' = q'$, so $q' = q$, a contradiction. $\square$

By Theorem 2.4.10, we can find a coefficient field $E \subset \overline{\mathbf{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K, \mathcal{V}_\lambda)$ such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{r_\iota(\pi)}$. After possibly enlarging E, we can and do assume that the residue field of $\mathfrak{m}$ is equal to k. For each tuple $(\chi_{v,i})_{v\in R, i=1,\dots,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ that are trivial modulo ϖ, we define a global deformation problem by the formula

$$\mathcal{S}_\chi = (\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v^{\mathrm{FL}}\}_{v\in S_p} \cup \{\mathcal{D}_v^{\chi}\}_{v\in R} \cup \{\mathcal{D}_v^{\square}\}_{v\in S-(R\cup S_p)}).$$

We fix representatives $\rho_{\mathcal{S}_\chi}$ of the universal deformations that are identified modulo ϖ (via the identifications $R_{\mathcal{S}_\chi}/\varpi \cong R_{\mathcal{S}_1}/\varpi$). We observe that the local deformation problems defining $\mathcal{S}_\chi$ are formally smooth away from the places in R. We define an $\mathcal{O}[K_S]$-module $\mathcal{V}_\lambda(\chi^{-1}) = \mathcal{V}_\lambda \otimes_{\mathcal{O}} \mathcal{O}(\chi^{-1})$, where K_S acts on $\mathcal{V}_\lambda$ by projection to K_p and on $\mathcal{O}(\chi^{-1})$ by the projection $K_S \to K_R = \prod_{v\in R} \mathrm{Iw}_v \to \prod_{v\in R}(k(v)^\times)^n$.

PROPOSITION 6.5.3. *There exist an integer $\delta \geq 1$, depending only on n and $[F : \mathbf{Q}]$, an ideal $J \subset \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1})))_\mathfrak{m}$ such that $J^\delta = 0$, and a continuous surjective homomorphism*

$$f_{\mathcal{S}_\chi} : R_{\mathcal{S}_\chi} \to \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1})))_\mathfrak{m}/J$$

such that for each finite place $v \notin S$ of F, the characteristic polynomial of $f_{\mathcal{S}_\chi} \circ \rho_{\mathcal{S}_\chi}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $\mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1})))_\mathfrak{m}/J$.

Proof. This is a matter of combining the various local-global compatibility results we have proved so far. The existence of a Galois representation $\rho_\mathfrak{m} : G_{F,S\cup S^c} \to \mathrm{GL}_n(\mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1})))_\mathfrak{m}/J)$ satisfying the required condition at finite places $v \notin S \cup S^c$ is contained in Theorem 2.3.7. After conjugation, we can assume that $\rho_\mathfrak{m}$ mod $\mathfrak{m}$ equals $\overline{\rho}_\mathfrak{m}$. To prove the proposition, we need to show that for each $v \in S$, $\rho_\mathfrak{m}|_{G_{F_v}}$ is a lifting of $\overline{\rho}|_{G_{F_v}}$ of the appropriate type, and that for each $v \in S^c - S$, $\rho_\mathfrak{m}|_{G_{F_v}}$ is unramified and the characteristic polynomial of $\rho_\mathfrak{m}(\mathrm{Frob}_v)$ has the correct form. Theorem 4.5.1 shows that the Fontaine–Laffaille condition is satisfied for each $v|p$. We apply Theorem 3.1.1 with the set S of places there equal to $S \cup S^c$ and the set R equal to $S - S_p$. This shows that the appropriate condition on the characteristic polynomials of elements $\rho_\mathfrak{m}(\sigma)$ ($\sigma \in I_{F_v}$) is satisfied for each $v \in R$, and that $\rho_\mathfrak{m}|_{G_{F_v}}$ is unramified with the characteristic polynomial $\rho_\mathfrak{m}(\mathrm{Frob}_v)$ of the correct form for $v \in S^c - S$. □

Recall (as in (6.3.5)) that it makes sense to talk about the support of $H^*(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m}$ over $R_{\mathcal{S}_1}$, even though $H^*(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m}$ is not literally an $R_{\mathcal{S}_1}$-module. We can now state our first key technical result, which we will prove below.

THEOREM 6.5.4. *Under assumptions (1)–(17) above, $H^*(X_K, \mathcal{V}_\lambda(1))_\mathfrak{m}$ has full support over $R_{\mathcal{S}_1}$.*

COROLLARY 6.5.5. *Under assumptions (1)–(17) above, suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) *We have $\overline{\rho} \cong \overline{r_\iota(\pi)}$.*
(2) *For each place $v|p$ of F, $\rho|_{G_{F_v}}$ is crystalline. For each embedding $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$, we have*
$$\mathrm{HT}_\tau(\rho) = \{\lambda_{\iota\tau,1} + n - 1, \ldots, \lambda_{\iota\tau,n}\}.$$
(3) *For each finite place $v \notin S$ of F, $\rho|_{G_{F_v}}$ is unramified.*
(4) *For each place $v \in R$, $\rho|_{G_{F_v}}$ is unipotently ramified.*

Then ρ is automorphic: there exists a cuspidal, regular algebraic automorphic representation Π of weight λ such that $\rho \cong r_\iota(\Pi)$. Moreover, if v is a finite place of F such that $v|p$ or $v \notin S$, then Π_v is unramified.

Proof. After possibly enlarging the coefficient field E, and replacing ρ by a $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$-conjugate, we can assume that it takes values in $\mathrm{GL}_n(\mathcal{O})$, and that $\rho \bmod \varpi = \overline{\rho}_{\mathfrak{m}}$. Then ρ is a lifting of type $\mathcal{S}_1$, so it determines a homomorphism $f : R_{\mathcal{S}_1} \to E$. Theorem 6.5.4 implies that $\ker f$ is in the support of $H^*(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}[1/p]$; Theorem 2.4.10 then implies that there exists a cuspidal, regular algebraic automorphic representation Π of weight λ such that $\rho \cong r_\iota(\Pi)$ and $(\Pi^\infty)^K \neq 0$. This is the desired result. (Recall that $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ if $v|p$ or $v \notin S$.) $\square$

Before proceeding to the proof of Theorem 6.5.4, we need to introduce auxiliary level subgroups. These will be associated to a choice of Taylor–Wiles datum $(Q, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v \in Q})$ for $\mathcal{S}_1$; see Section 6.2.28. We assume that for each $v \in Q$, there exists an imaginary quadratic subfield of F in which the residue characteristic l_v of v splits. This Taylor–Wiles datum is automatically a Taylor–Wiles datum for all the global deformation problems $\mathcal{S}_\chi$, and so the auxiliary deformation problems $\mathcal{S}_{\chi,Q}$ are defined, and the deformation ring $R_{\mathcal{S}_{\chi,Q}}$ has a natural structure of $\mathcal{O}[\Delta_Q]$-algebra, where $\Delta_Q = \prod_{v \in Q} \Delta_v = \prod_{v \in Q} k(v)^\times(p)^n$. The constructions we are about to give necessarily involve a lot of notation. Accordingly, we invite the reader to review the notation related to Hecke algebras in Section 2.2.1 before continuing.

We define two auxiliary level subgroups $K_1(Q) \subset K_0(Q) \subset K$. They are good subgroups of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$, determined by the following conditions:

- If $v \notin S \cup Q$, then $K_1(Q)_v = K_0(Q)_v = K_v$.
- If $v \in Q$, then $K_0(Q)_v = \mathrm{Iw}_v$ and $K_1(Q)_v$ is the maximal pro-prime-to-p subgroup of Iw_v.

Then there are a natural isomorphism $K_0(Q)/K_1(Q) \cong \Delta_Q$, and surjective morphisms of $\mathbf{T}^{S \cup Q}$-algebras

$$\begin{aligned} {}^{K_0(Q)/K_1(Q)}\mathbf{T}^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1})) &\to \mathbf{T}^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1})) \\ &\to \mathbf{T}^{S\cup Q}(K, \mathcal{V}_\lambda(\chi^{-1})). \end{aligned} \tag{6.5.6}$$

The first of these arises by taking $K_0(Q)$-invariants (cf. Section 2.2.1, and note $\mathcal{O}[\Delta_Q]$ acts trivially on invariants), and the second is given by the formula $t \mapsto [K : K_0(Q)]^{-1} \pi_{Q,*} \circ t \circ \pi_Q^*$, where $\pi_Q : X_{K_0(Q)} \to X_K$ is the canonical projection; note that $[K : K_0(Q)] \equiv (n!)^{|Q|} \bmod p$ is a unit in $\mathcal{O}$ because of our assumption that $p > n$. We define

$$\mathbf{T}_Q^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1})) \subset \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1})))$$

as in Section 3.1; it is the commutative $\mathbf{T}^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1}))$-subalgebra generated by the operators $U_{v,i}$ ($v \in Q$, $i = 1, \dots, n$), or equivalently the image

of the algebra $\mathbf{T}_Q^{S\cup Q}$ defined in Section 3.1. Similarly, we define

$$\mathbf{T}_Q^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1})) \subset \mathrm{End}_{\mathbf{D}(\mathcal{O}[\Delta_Q])}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1})));$$

it is an $\mathcal{O}[\Delta_Q]$-algebra, which coincides with the image of the algebra $\mathbf{T}_Q^{S\cup Q}$. The first map in (6.5.6) extends to a surjective homomorphism

$$\mathbf{T}_Q^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1})) \to \mathbf{T}_Q^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1})) \tag{6.5.7}$$

that takes $U_{v,i}$ to $U_{v,i}$ for each $v \in Q$ and for each $i = 1, \dots, n$.

We define $\mathfrak{m}^Q \subset \mathbf{T}^{S\cup Q}(K, \mathcal{V}_\lambda(\chi^{-1}))$ to be the pullback of $\mathfrak{m}$ under the inclusion

$$\mathbf{T}^{S\cup Q}(K, \mathcal{V}_\lambda(\chi^{-1})) \subset \mathbf{T}^S(K, \mathcal{V}_\lambda(\chi^{-1})).$$

We define

$$\mathfrak{m}_0^Q \subset \mathbf{T}^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1}))$$

to be the pullback of $\mathfrak{m}^Q$ and define

$$\mathfrak{m}_1^Q \subset_{K_0(Q)/K_1(Q)} \mathbf{T}^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1}))$$

to be the pullback of $\mathfrak{m}_0^Q$, these pullbacks being taken under the maps in (6.5.6). We define $\mathfrak{n}_0^Q \subset \mathbf{T}_Q^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1}))$ to be the ideal generated by $\mathfrak{m}_0^Q$ and the elements $U_{v,i} - q_v^{i(1-i)/2}\alpha_{v,1}\cdots\alpha_{v,i}$ for each $v \in Q$ and $i = 1, \dots, n$. We define $\mathfrak{n}_1^Q \subset \mathbf{T}_Q^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1}))$ to be the pre-image of $\mathfrak{n}_0^Q$ under the map (6.5.7).

Lemma 6.5.8. *Each ideal* $\mathfrak{m}^Q$, $\mathfrak{m}_0^Q$, $\mathfrak{m}_1^Q$, $\mathfrak{n}_0^Q$, *and* $\mathfrak{n}_1^Q$ *is a* (*proper*) *maximal ideal.*

Proof. This is clear for the ideals $\mathfrak{m}^Q$, $\mathfrak{m}_0^Q$, and $\mathfrak{m}_1^Q$. Since $\mathfrak{n}_1^Q$ is the pre-image of $\mathfrak{n}_0^Q$ under a surjective algebra homomorphism, we just need to check that $\mathfrak{n}_0^Q$ is a proper ideal. Equivalently, we must check that

$$H^*(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1})/\varpi)[\mathfrak{m}_0^Q]$$

contains a non-zero vector on which each operator $U_{v,i}$ ($v \in Q$, $i = 1, \dots, n$) acts by the scalar $\alpha_{v,1}\cdots\alpha_{v,i}$. This will follow from [KT17, Lem. 5.3] (or rather its proof) if we can show that $H^*(X_K, \mathcal{V}_\lambda(\chi^{-1}))[\mathfrak{m}^Q]$ is annihilated by a power of $\mathfrak{m}$. This follows from the existence of $\overline{\rho}_\mathfrak{m}$ and its local-global compatibility at the places $v \in Q$. □

We can therefore form the localized complexes

$$R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_\mathfrak{m}, R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}^Q},$$
$$R\Gamma(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}_0^Q}, R\Gamma(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_0^Q},$$

$$R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}_1^Q}, R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}.$$

The first four lie in $\mathbf{D}(\mathcal{O})$, and the last two lie in $\mathbf{D}(\mathcal{O}[\Delta_Q])$.

LEMMA 6.5.9. *The natural morphisms*

$$R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}^Q} \to R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}},$$
$$R\Gamma(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_0^Q} \to R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}^Q},$$

and

$$R\Gamma(\Delta_Q, R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}) \to R\Gamma(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_0^Q}$$

in $\mathbf{D}(\mathcal{O})$ *are isomorphisms.*

Proof. We must show that these morphisms in the derived category give isomorphisms at the level of cohomology. For the first morphism, it is enough to show that $\mathfrak{m}$ is the unique maximal ideal of $\mathbf{T}^{S\cup Q}(K_0(Q), \mathcal{V}_\lambda(\chi^{-1}))$ lying above $\mathfrak{m}^Q$, and we have seen this already in the proof of Lemma 6.5.8. It is clear from the definitions for the third morphism. For the second, it is enough to check that we have an isomorphism after applying the functor $-\otimes^{\mathbf{L}}_{\mathcal{O}} k : \mathbf{D}(\mathcal{O}) \to \mathbf{D}(k)$. We are therefore reduced to showing that the map of k-vector spaces

$$\mathrm{tr}_{K/K_0(Q)} : H^*(X_{K_0(Q)}, \mathcal{V}_\lambda(\chi^{-1})/\varpi)_{\mathfrak{n}_0^Q} \to H^*(X_K, \mathcal{V}_\lambda(\chi^{-1})/\varpi)_{\mathfrak{m}^Q}$$

is an isomorphism. This is the content of [KT17, Lem. 5.4]. □

We see that there is a surjective homomorphism

$$\begin{aligned} &{}_{K_0(Q)/K_1(Q)}\mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}) \\ &\qquad \to \mathbf{T}^{S\cup Q}(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}^Q}) = \mathbf{T}^{S\cup Q}(K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}^Q}. \end{aligned} \tag{6.5.10}$$

The first ring ${}_{K_0(Q)/K_1(Q)}\mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q})$ is a local $\mathcal{O}[\Delta_Q]$-algebra, its unique maximal ideal being identified with the pre-image of $\mathfrak{m}^Q$ under the surjective homomorphism (6.5.10); indeed, this follows from the fact that it acts nearly faithfully on $H^*(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}$. (We recall ([Tay08, Def. 2.1]) that a finitely generated module over a Noetherian local ring is said to be nearly faithful if its annihilator is a nilpotent ideal.) We can now state a result asserting the existence of Galois representations valued with coefficients in this Hecke algebra.

PROPOSITION 6.5.11. *There exist an integer* $\delta \geq 1$, *depending only on* n *and* $[F:\mathbf{Q}]$, *an ideal* $J \subset {}_{K_0(Q)/K_1(Q)}\mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q})$ *such that* $J^\delta = 0$, *and a continuous surjective* $\mathcal{O}[\Delta_Q]$*-algebra homomorphism*

$$f_{\mathcal{S}_{\chi,Q}} : R_{\mathcal{S}_{\chi,Q}} \to_{K_0(Q)/K_1(Q)} \mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q})/J$$

such that for each finite place $v \notin S \cup Q$ of F, the characteristic polynomial of $f_{\mathcal{S}_{\chi,Q}} \circ \rho_{\mathcal{S}_{\chi,Q}}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in

$$_{K_0(Q)/K_1(Q)}\mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q})/J.$$

Proof. To save notation, let

$$\mathbf{T} =_{K_0(Q)/K_1(Q)} \mathbf{T}^{S\cup Q}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q})$$

and $\mathbf{T}' = \mathbf{T}_Q^{S\cup Q}(K_0(Q)/K_1(Q), \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}$. Then $\mathbf{T} \subset \mathbf{T}'$, and the inclusion $\mathbf{T} \to \mathbf{T}'$ is a local homomorphism of finite $\mathcal{O}[\Delta_Q]$-algebras. By Theorem 2.3.7, there are a nilpotent ideal $J' \subset \mathbf{T}'$ and a Galois representation $\rho_{\mathfrak{n}_1^Q} : G_{F,S\cup Q} \to \mathrm{GL}_n(\mathbf{T}'/J')$ satisfying local-global compatibility at unramified places. After conjugation, we can assume that $\rho_{\mathfrak{n}_1^Q} \bmod \mathfrak{n}_1^Q$ equals $\overline{\rho}_{\mathfrak{m}}$. We first need to show that $\rho_{\mathfrak{n}_1^Q}$ is a lifting of $\overline{\rho}_{\mathfrak{m}}$ of type $\mathcal{S}_{\chi,Q}$. The necessary conditions at places of S can be checked just as in the proof of Proposition 6.5.3. There is no condition at places of Q, so we obtain a morphism $f_{\mathcal{S}_{\chi,Q}} : R_{\mathcal{S}_{\chi,Q}} \to \mathbf{T}'/J'$ (which in fact factors through the image of $\mathbf{T}$ in $\mathbf{T}'/J'$).

It remains to check that $f_{\mathcal{S}_{\chi,Q}}$ is a homomorphism of $\mathcal{O}[\Delta_Q]$-algebras. Equivalently, we must check that it is a homomorphism of $\mathcal{O}[\Delta_v]$-algebras for each place $v \in Q$. To this end, let us fix a place $v \in Q$. For each $i = 1, \dots, n$, we define a character $\psi_{v,i} : W_{F_v} \to (\mathbf{T}')^\times$ by the formula $\psi_{v,i}(\mathrm{Art}_{F_v}(\alpha)) = t_{v,i}(\alpha)$ (notation as in Section 2.2.5). Theorem 3.1.1 shows that (after possibly enlarging J') for each $\sigma \in W_{F_v}$, we have the identity

$$\det(X - \rho_{\mathfrak{n}_1^Q}(\sigma)) = \prod_{i=1}^n (X - \psi_{v,i}(\sigma)).$$

Observe that the characters $\psi_{v,i} \bmod \mathfrak{n}_1^Q$ are pairwise distinct (because they take Frobenius to $\alpha_{v,i}$, and these elements of k are pairwise distinct, by definition of a Taylor–Wiles datum). We can therefore apply [BC09, Prop. 1.5.1] to conclude that $\rho_{\mathfrak{n}_1^Q}|_{W_{F_v}}$ is isomorphic to $\oplus_{i=1}^n \psi_{v,i}$, which shows that $f_{\mathcal{S}_{\chi,Q}}$ is indeed a homomorphism of $\mathcal{O}[\Delta_v]$-algebras; cf. Section 6.2.18 for the definition of the $\mathcal{O}[\Delta_v]$-algebra structure on $R_{\mathcal{S}_{\chi,Q}}$. The proof is complete on taking J to be the kernel of the map $\mathbf{T} \to \mathbf{T}'/J'$. □

We are now ready to begin the proof of Theorem 6.5.4.

Proof of Theorem 6.5.4. Let

$$q = h^1(F_S/F, \mathrm{ad}\,\overline{\rho}_{\mathfrak{m}}(1)) \quad \text{and} \quad g = qn - n^2[F^+ : \mathbf{Q}],$$

and set $\Delta_\infty = \mathbf{Z}_p^{nq}$. Let $\mathcal{T}$ be a power series ring over $\mathcal{O}$ in $n^2|S| - 1$ many variables, and let $S_\infty = \mathcal{T}[\![\Delta_\infty]\!]$. Viewing S_∞ as an augmented $\mathcal{O}$-algebra, we let $\mathfrak{a}_\infty$ denote the augmentation ideal.

Enlarging E if necessary, we can assume that E contains a primitive pth root of unity. Then since $p > n$, for each $v \in R$ we can choose a tuple of pairwise distinct characters $\chi_v = (\chi_{v,1}, \dots, \chi_{v,n})$, with $\chi_{v,i} \colon \mathcal{O}_{F_v}^\times \to \mathcal{O}^\times$ trivial modulo ϖ. We write χ for the tuple $(\chi_v)_{v \in R}$ as well as for the induced character $\chi = \prod_{v \in R} \chi_v \colon \prod_{v \in R} I_v \to \mathcal{O}^\times$. For each $N \geq 1$, we fix a choice of Taylor–Wiles datum $(Q_N, (\alpha_{v,1}, \dots, \alpha_{v,n})_{v \in Q_N})$ as in Proposition 6.2.33. (This is possible by our assumption that $\overline{r_\iota(\pi)}(G_{F(\zeta_p)})$ is enormous; we choose any imaginary quadratic subfield of F in the application of Proposition 6.2.33.) For $N = 0$, we set $Q_0 = \emptyset$. For each $N \geq 1$, we let $\Delta_N = \Delta_{Q_N}$ and fix a surjection $\Delta_\infty \to \Delta_N$. The kernel of this surjection is contained in $(p^N \mathbf{Z}_p)^{nq}$, since each $v \in Q$ satisfies $q_v \equiv 1 \bmod p^N$. We let Δ_0 be the trivial group, viewed as a quotient of Δ_∞.

For each $N \geq 0$, the auxiliary deformation problems $\mathcal{S}_{1,Q_N}$ and $\mathcal{S}_{\chi,Q_N}$ are defined, and we set $R_N = R_{\mathcal{S}_{1,Q_N}}$ and $R'_N = R_{\mathcal{S}_{\chi,Q_N}}$. Note that $R_0 = R_{\mathcal{S}_1}$ and $R'_0 = R_{\mathcal{S}_\chi}$. Let $R^{\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_1}$ and $R'^{\,\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$ denote the corresponding local deformation rings as in Section 6.2.22. For any $N \geq 1$, we have $R^{S,\mathrm{loc}}_{\mathcal{S}_{1,Q_N}} = R^{\mathrm{loc}}$ and $R^{S,\mathrm{loc}}_{\mathcal{S}_{\chi,Q_N}} = R'^{\,\mathrm{loc}}$. There are canonical isomorphisms $R^{\mathrm{loc}}/\varpi \cong R'^{\,\mathrm{loc}}/\varpi$ and $R_N/\varpi \cong R'_N/\varpi$ for all $N \geq 0$. For each $N \geq 1$, R_N and R'_N are canonically $\mathcal{O}[\Delta_N]$-algebras and there are canonical isomorphisms $R_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong R_0$ and $R'_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong R'_0$, which are compatible with the isomorphisms modulo ϖ. By Lemma 6.2.4, we have an R^{loc}-algebra structure on $R_N \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$ and an $R'^{\,\mathrm{loc}}$-algebra structure on $R'_N \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$. The canonical isomorphism $R^{\mathrm{loc}}/\varpi \cong R'^{\,\mathrm{loc}}/\varpi$ is compatible with these algebra structures and with the canonical isomorphisms $R_N/\varpi \cong R'_N/\varpi$. We let R_∞ and R'_∞ be formal power series rings in g variables over R^{loc} and $R'^{\,\mathrm{loc}}$, respectively. Using Proposition 6.2.25 when $N = 0$ (noting that $H^0(F_S/F, \operatorname{ad} \overline{\rho}_{\mathfrak{m}}(1)) = 0$, because $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ is irreducible and $\zeta_p \notin F$), and Proposition 6.2.32 when $N \geq 1$, there are local $\mathcal{O}$-algebra surjections $R_\infty \to R_N$ and $R'_\infty \to R'_N$ for any $N \geq 0$. We can (and do) assume that these are compatible with our fixed identifications modulo ϖ, and with the isomorphisms $R_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong R_0$ and $R'_N \otimes_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong R'_0$.

Let $\mathcal{C}_0 = R\operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}, \mathcal{O})[-d]$, and let $T_0 = \mathbf{T}^S(K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}$. Then $H^i(\mathcal{C}_0)[1/p] \cong \operatorname{Hom}_E(H^{d-i}(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}[1/p], E)$ as T_0-modules. Also, we let $\mathcal{C}'_0 = R\operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}}, \mathcal{O})[-d]$ and $T'_0 = \mathbf{T}^S(K, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{m}}$. For any $N \geq 1$, we let

$$\mathcal{C}_N = R\operatorname{Hom}_{\mathcal{O}[\Delta_N]}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(1))_{\mathfrak{n}_1^Q}, \mathcal{O}[\Delta_N])[-d]$$

and

$$T_N =_{K_0(Q)/K_1(Q)} \mathbf{T}^{S \cup Q_N}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(1))_{\mathfrak{n}_1^Q}).$$

Similarly, we let

$$\mathcal{C}'_N = R\operatorname{Hom}_{\mathcal{O}[\Delta_N]}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}, \mathcal{O}[\Delta_N])[-d]$$

and

$$T'_N =_{K_0(Q)/K_1(Q)} \mathbf{T}^{S\cup Q_N}(R\Gamma_{K_0(Q)/K_1(Q)}(X_{K_1(Q)}, \mathcal{V}_\lambda(\chi^{-1}))_{\mathfrak{n}_1^Q}).$$

For any $N \geq 0$, there are canonical isomorphisms $\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} k[\Delta_N] \cong \mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} k[\Delta_N]$ in $\mathbf{D}(k[\Delta_N])$. Using this isomorphism to identify $\mathrm{End}_{\mathbf{D}(\mathcal{O})}(\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}} k) = \mathrm{End}_{\mathbf{D}(\mathcal{O})}(\mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}} k)$, the images of T_N and T'_N in this endomorphism algebra are the same, and we denote it by $\overline{T}_N$. By Lemma 6.5.9, there are canonical isomorphisms $\mathcal{C}_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong \mathcal{C}_0$ and $\mathcal{C}'_N \otimes^{\mathbf{L}}_{\mathcal{O}[\Delta_N]} \mathcal{O} \cong \mathcal{C}'_0$ in $\mathbf{D}(\mathcal{O})$, and these isomorphisms are compatible with our fixed isomorphisms modulo ϖ. By Proposition 6.5.11 we have nilpotent ideals I_N of T_N and I'_N of T'_N for each $N \geq 0$, both of nilpotence degree $\leq \delta$, and local $\mathcal{O}[\Delta_N]$-algebra surjections $R_N \to T_N/I_N$ and $R'_N \to T'_N/I'_N$. The surjections are compatible with the canonical isomorphisms modulo ϖ. Moreover, using the isomorphism $R_N/\varpi \cong R'_N/\varpi$ and letting $\overline{I}_N$ and $\overline{I}'_N$ denote the images of I_N and I'_N, respectively, in $\overline{T}_N$, the induced surjections $R_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ and $R'_N/\varpi \to \overline{T}_N/(\overline{I}_N + \overline{I}'_N)$ agree. The maps $T_N \otimes_{\mathcal{O}[\Delta]} \mathcal{O} \to T_0$ and $T'_N \otimes_{\mathcal{O}[\Delta]} \mathcal{O} \to T'_0$ induce surjections onto T_0/I_0 and T'_0/I'_0 respectively. (Surjectivity follows from Chebotarev density and the existence of the Galois representations with coefficients in T_0/I_0 and T_0/I'_0.)

The objects introduced above satisfy the setup described in Section 6.4.1. We can then apply the results of Section 6.4.2 and obtain the following:

- Bounded complexes $\mathcal{C}_\infty$ and $\mathcal{C}'_\infty$ of free S_∞-modules, subrings
$$T_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty) \text{ and } T'_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}'_\infty),$$
and ideals I_∞ and I'_∞ satisfying $I_\infty^\delta = 0$ and $I'^\delta_\infty = 0$. We also have S_∞-algebra structures on R_∞ and R'_∞ and S_∞-algebra surjections $R_\infty \to T_\infty/I_\infty$ and $R'_\infty \to T'_\infty/I'_\infty$; see Proposition 6.4.12 and Remark 6.4.14.
- Surjections of local $\mathcal{O}$-algebras $R_\infty/\mathfrak{a}_\infty \to R_0$ and $R'_\infty/\mathfrak{a}_\infty \to R'_0$. We have isomorphisms $\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong \mathcal{C}_0$ and $\mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong \mathcal{C}'_0$ in $\mathbf{D}(\mathcal{O})$, inducing maps $T_\infty \to T_0$ and $T'_\infty \to T'_0$ that become surjective when composed with the projections $T_0 \to T_0/I_0$ and $T'_0 \to T'_0/I'_0$, respectively. We let $I_{\infty,0}$ and $I'_{\infty,0}$ denote the images of I_∞ and I'_∞, respectively, under these surjective maps. Then the induced maps $R_\infty/\mathfrak{a}_\infty \to (T_0/I_0)/I_{\infty,0}$ and $R'_\infty/\mathfrak{a}_\infty \to (T'_0/I'_0)/I'_{\infty,0}$ factor through $R_\infty/\mathfrak{a}_\infty \to R_0$ and $R'_\infty/\mathfrak{a}_\infty \to R'_0$, respectively; see Lemma 6.4.15 and Proposition 6.4.16.
- An isomorphism
$$\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi \cong \mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi$$
in $\mathbf{D}(S_\infty/\varpi)$. Under this identification, T_∞ and T_∞ have the same image $\overline{T}_\infty$ in
$$\mathrm{End}_{\mathbf{D}(S_\infty/\varpi)}(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi) = \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi).$$

Let $\overline{I}_\infty$ and $\overline{I}'_\infty$ denote the images of I_∞ and I'_∞, respectively, in $\overline{T}_\infty$. Then the actions of $R_\infty/\varpi \cong R'_\infty/\varpi$ on

$$H^*(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} \varpi)/(\overline{I}_\infty + \overline{I}'_\infty) \cong H^*(\mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty)$$

are identified via $\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi \cong \mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi$; see Proposition 6.4.17.

Recall that R_∞ and R'_∞ are power series rings over R^{loc} and R'^{loc}, respectively, in $g = qn - n[F^+ : \mathbf{Q}]$ many variables. By Lemma 6.2.26, we have

- Each generic point of $\operatorname{Spec} R_\infty/\varpi$ is the specialization of a unique generic point of $\operatorname{Spec} R_\infty$, and every generic point of $\operatorname{Spec} R_\infty$ has characteristic zero. Also, $\operatorname{Spec} R'_\infty$ is irreducible and has characteristic zero generic point.
- R_∞ is equidimensional, and R_∞ and R'_∞ have the common dimension

$$1 + g + n^2|S| + \tfrac{n(n-1)}{2}[F : \mathbf{Q}].$$

Since F is CM, the quantity l_0 for the locally symmetric space X_K is $l_0 = n[F^+ : \mathbf{Q}] - 1$. Then since $\dim S_\infty = n^2|S| + qn$ and $g = qn - n[F^+ : \mathbf{Q}]$, we have

$$\dim R_\infty = \dim R'_\infty = \dim S_\infty - l_0.$$

Finally, the isomorphism $\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong \mathcal{C}_0$ implies that $(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty)[1/p]$ has cohomology isomorphic to $\operatorname{Hom}_E(H^{d-*}(X_K, \mathcal{V}_\lambda(1))_{\mathfrak{m}}[1/p], E)$. So Theorem 2.4.10 implies that $H^*(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty)[1/p] \neq 0$ and that the cohomology is concentrated in degrees $[q_0, q_0 + l_0]$. We have now satisfied all the assumptions of Section 6.3.5, so we can apply Proposition 6.3.8 to conclude that $H^*(\mathcal{C}_\infty)$ has full support over R_∞, hence that $H^*(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty) = H^*(\mathcal{C}_0)$ has full support over $R_\infty/(\mathfrak{a}_\infty)$, hence that $H^*(\mathcal{C}_0)$ has full support over $R_{\mathcal{S}_1}$. This concludes the proof. □

6.5.12. *End of the proof* (*Fontaine–Laffaille case*). We now deduce Theorem 6.1.1 from Corollary 6.5.5. The proof will be an exercise in applying soluble base change. We first state the results that we need. Note that while up to now E has denoted the coefficient field of our Galois representations, having carried out our patching argument we no longer need this notation; we find it convenient to use E to denote a number field in the rest of the proof.

PROPOSITION 6.5.13. *Fix an integer $n \geq 2$, a prime p, and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Let F be an imaginary CM or totally real number field, and let E/F be finite Galois extension such that* $\mathrm{Gal}(E/F)$ *is soluble and E is also imaginary CM or totally real. Then*

(1) *Let π be a cuspidal, regular algebraic automorphic representation of* $\mathrm{GL}_n(\mathbf{A}_F)$ *of weight $\lambda = (\lambda_\tau)_{\tau \in \mathrm{Hom}(F,\mathbf{C})}$. Suppose that $r_\iota(\pi)|_{G_E}$ is irreducible. Then there exists a cuspidal, regular algebraic automorphic representation π_E of* $\mathrm{GL}_n(\mathbf{A}_E)$ *of weight $\lambda_{E,\tau} = \lambda_{\tau|_E}$ such that $r_\iota(\pi_E) \cong r_\iota(\pi)|_{G_E}$. If w is a*

finite place of E lying above the place v of F, then we have $\mathrm{rec}_{E_w}(\pi_E) = \mathrm{rec}_{F_v}(\pi)|_{W_{E_w}}$.

(2) *Let $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous representation such that $\rho|_{G_E}$ is irreducible. Suppose that there exists a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_E)$ of weight λ such that $\rho|_{G_E} \cong r_\iota(\pi)$. Define $\lambda_F = (\lambda_{F,\tau})_{\tau \in \mathrm{Hom}(F,\mathbf{C})}$ by the formula $\lambda_{F,\tau} = \lambda_{\tau'}$, where $\tau' : E \hookrightarrow \mathbf{C}$ is any extension of τ from F to E. Then λ_F is independent of any choices, and there exists a cuspidal, regular algebraic automorphic representation π_F of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ_F such that $\rho \cong r_\iota(\pi_F)$. If w is a finite place of E lying above the place v of F, then we have* $\mathrm{rec}_{E_w}(\pi) = \mathrm{rec}_{F_v}(\pi_F)$.

Proof. In either case we can reduce, by induction, to the case that E/F is cyclic of prime order. Let $\sigma \in \mathrm{Gal}(E/F)$ be a generator of the Galois group, and let $\eta : \mathrm{Gal}(E/F) \to \mathbf{C}^\times$ be a non-trivial character. We first treat the first part of the proposition. We claim that $\pi \otimes (\eta \circ \mathrm{Art}_F^{-1}) \not\cong \pi$. Otherwise, there would be an isomorphism

$$r_\iota(\pi) \otimes \iota^{-1}\eta \cong r_\iota(\pi),$$

implying that $r_\iota(\pi)|_{G_E}$ is reducible. We can therefore apply [AC89, Ch. 3, Th. 4.2] and [AC89, Ch. 3, Th. 5.1] to conclude the existence of a cuspidal, regular algebraic automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_E)$ of weight λ_E such that for almost all finite places w of E such that $\pi_{w|_F}$ is unramified, Π_w is a lift of $\pi_{w|_F}$. The Chebotarev density theorem then implies that we must have $r_\iota(\Pi) \cong r_\iota(\pi)|_{G_E}$, so we can take $\pi_E = \Pi$.

We now treat the second part of the proposition. The isomorphism $\rho|_{G_E} \cong r_\iota(\pi)$, together with strong multiplicity one for GL_n, implies that we have $\pi^\sigma \cong \pi$. By [AC89, Ch 3, Th. 4.2] and [AC89, Ch. 3, Th. 5.1], there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$, which is regular algebraic of weight λ_F, such that for almost all finite places w of E such that $\Pi_{w|_F}$ is unramified, π_w is a lift of $\Pi_{w|_F}$. The Chebotarev density theorem then implies that we must have $r_\iota(\Pi)|_{G_E} \cong r_\iota(\pi) \cong \rho|_{G_E}$. Using the irreducibility of $\rho|_{G_E}$, we conclude that there is a twist $\Pi \otimes (\eta \circ \mathrm{Art}_F^{-1})^i$ such that $r_\iota(\Pi \otimes (\eta \circ \mathrm{Art}_F^{-1})^i) \cong \rho$. We are done on taking $\pi_F = \Pi \otimes (\eta \circ \mathrm{Art}_F^{-1})^i$. □

Proof of Theorem 6.1.1. For the convenience of the reader, we recall the hypotheses of Theorem 6.1.1. Let F be an imaginary CM or totally real field, and let $c \in \mathrm{Aut}(F)$ be complex conjugation. We are given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:

(1) ρ is unramified almost everywhere.
(2) For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is crystalline. The prime p is unramified in F.

(3) $\overline{\rho}$ is absolutely irreducible and decomposed generic (Definition 4.3.1). The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is enormous (Definition 6.2.29).
(4) There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n^2$.
(5) There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:
   (a) π is regular algebraic of weight λ, this weight satisfying

$$\lambda_{\tau,1} + \lambda_{\tau c,1} - \lambda_{\tau,n} - \lambda_{\tau c,n} < p - 2n.$$

   (b) There exists an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{\rho} \cong \overline{r_\iota(\pi)}$ and the Hodge–Tate weights of ρ satisfy the formula for each $\tau : F \hookrightarrow \overline{\mathbf{Q}}_p$:

$$\mathrm{HT}_\tau(\rho) = \{\lambda_{\iota\tau,1} + n - 1, \lambda_{\iota\tau,2} + n - 2, \ldots, \lambda_{\iota\tau,n}\}.$$

   (c) If $v|p$ is a place of F, then π_v is unramified.

The case where F is a totally real field can be reduced to the case where F is totally imaginary by base change. We therefore assume now that F is imaginary and write F^+ for its maximal totally real subfield. Let $K/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$. Choose finite sets V_0, V_1, V_2 of finite places of F having the following properties:

- For each $v \in V_0$, v splits in $F(\zeta_p)$. For each proper subfield $K/K'/F(\zeta_p)$, there exists $v \in V_0$ such that v splits in $F(\zeta_p)$ but does not split in K'.
- For each proper subfield $K/K'/F$, there exists $v \in V_1$ that does not split in K'.
- There exists a rational prime $p_0 \neq p$ that is decomposed generic for $\overline{\rho}$, and V_2 is equal to the set of p_0-adic places of F.
- For each $v \in V_0 \cup V_1 \cup V_2$, $v \nmid 2$, $v \nmid p$, and ρ and π are both unramified at v.

If E/F is any finite Galois extension that is $V_0 \cup V_1 \cup V_2$-split, then $\overline{\rho}|_{G_E}$ has the following properties:

- $\overline{\rho}(G_E) = \overline{\rho}(G_F)$ and $\overline{\rho}(G_{E(\zeta_p)}) = \overline{\rho}(G_{F(\zeta_p)})$. In particular, $\overline{\rho}|_{G_{E(\zeta_p)}}$ has enormous image and there exists $\sigma \in G_E - G_{E(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar.
- $\overline{\rho}|_{G_E}$ is decomposed generic. Indeed, the rational prime p_0 splits in E.

Let E_0/F be a soluble CM extension satisfying the following conditions:

- Each place of $V_0 \cup V_1 \cup V_2$ splits in E_0, and the rational prime p is unramified in E_0.
- For each finite place w of E_0, $\pi_{E_0,w}^{\mathrm{Iw}_w} \neq 0$.
- For each finite prime-to-p place w of E_0, either both $\pi_{E_0,w}$ and $\rho|_{G_{E_{0,w}}}$ are unramified or $\rho|_{G_{E_{0,w}}}$ is unipotently ramified, $q_w \equiv 1 \bmod p$, and $\overline{\rho}|_{G_{E_{0,w}}}$ is trivial.

- For each place $\overline{w}|p$ of E_0^+, $\overline{w}$ splits in E_0 and there exists a place $\overline{w}' \neq \overline{w}$ of E_0^+ such that $\overline{w}'|p$ and

$$\sum_{\overline{w}'' \neq \overline{w}, \overline{w}'} [E^+_{0,\overline{w}''} : \mathbf{Q}_p] > \frac{1}{2}[E_0^+ : \mathbf{Q}].$$

We can find imaginary quadratic fields E_a, E_b, E_c satisfying the following conditions:

- Each rational prime lying below a place of $V_0 \cup V_1 \cup V_2$ splits in $E_a \cdot E_b \cdot E_c$. The prime p is unramified in $E_a \cdot E_b \cdot E_c$.
- The primes $2, p$ split in E_a.
- If $l \notin \{2, p\}$ is a rational prime lying below a place of E_0 at which $\pi_{E_0,w}$ or $\rho|_{E_{0,w}}$ is ramified, or that is ramified in $E_0 \cdot E_a \cdot E_c$, then l splits in E_b.
- If $l \notin \{2, p\}$ is a rational prime that is ramified in E_b, then l splits in E_c.

For example, we can choose any E_a satisfying the given condition. Then we can choose $E_b = \mathbf{Q}(\sqrt{-p_b})$, where p_b is a prime satisfying $p_b \equiv 1 \bmod 4$ and $p_b \equiv -1 \bmod l$ for any prime $l \notin \{2, p\}$ either lying below a place w of E_0 at which $\pi_{E_0,w}$ or $\rho|_{E_{0,w}}$ is ramified, or ramified in $E_0 \cdot E_a$, and $E_c = \mathbf{Q}(\sqrt{-p_c})$, where $p_c \equiv 1 \bmod 4p_b$ is any prime not equal to p. (Use quadratic reciprocity to show that p_c splits in E_b.)

We let $E = E_0 \cdot E_a \cdot E_b \cdot E_c$. Then E/F is a soluble CM extension in which each place of $V_0 \cup V_1 \cup V_2$ splits, and the following conditions hold by construction:

- The prime p is unramified in E.
- Let R denote the set of prime-to-p places w of E such that $\pi_{E,w}$ or $\rho|_{G_{E_w}}$ is ramified. Let S_p denote the set of p-adic places of E. Let $S' = S_p \cup R$. Then if l is a prime lying below an element of S', or which is ramified in E, then E contains an imaginary quadratic field in which l splits.
- If $w \in R$ then $\overline{\rho}|_{G_{E_w}}$ is trivial and $q_w \equiv 1 \bmod p$.
- The image of $\overline{\rho}|_{G_{E(\zeta_p)}}$ is enormous. The representation $\overline{\rho}|_{G_E}$ is decomposed generic.
- There exists $\sigma \in G_E - G_{E(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar.
- For each place $\overline{w}|p$ of E^+, there exists a place $\overline{w}' \neq \overline{w}$ of E^+ such that $\overline{w}'|p$ and

$$\sum_{\overline{w}'' \neq \overline{w}, \overline{w}'} [E^+_{\overline{w}''} : \mathbf{Q}_p] > \frac{1}{2}[E^+ : \mathbf{Q}].$$

By the Chebotarev density theorem, we can find infinitely many places v_0 of E of degree 1 over $\mathbf{Q}$ such that $\overline{\rho}(\mathrm{Frob}_{v_0})$ is scalar and $q_{v_0} \not\equiv 1 \bmod p$, $v_0 \notin S' \cup R^c$ and the residue characteristic of v_0 is odd. Then $H^2(E_{v_0}, \mathrm{ad}\,\overline{\rho}) = H^0(E_{v_0}, \mathrm{ad}\,\overline{\rho}(1))^\vee = 0$. We choose v_0, v_0' with distinct residue characteristics satisfying these conditions, and we set $S = S' \cup \{v_0, v_0'\}$. Note that if l_0, l_0'

denotes the residue characteristic of v_0, v_0', then l_0, l_0' splits in any imaginary quadratic subfield of E.

We see that the hypotheses (1)–(17) of Section 6.5.1 are now satisfied for E, π_E, and the set S. We can therefore apply Corollary 6.5.5 to $\rho|_{G_E}$ and Proposition 6.5.13 to conclude that ρ is associated to a cuspidal, regular algebraic automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ. Taking into account the final sentence of Corollary 6.5.5, we see that $\Pi_{E,w}$ is unramified if $w \notin S$.

To finish the proof, we must show that Π_v is unramified if v is a finite place of F such that $v \nmid p$ and both ρ and π are unramified at v. Using our freedom to vary the choice of places v_0, v_0', we see that if $v \nmid p$ is a place of F such that both ρ and π are unramified at v, then $\Pi_{E,w}$ is unramified for any place $w|v$ of E. This implies that $\mathrm{rec}_{F_v}(\Pi_v)$ is a finitely ramified representation of the Weil group W_{F_v}. Using the main theorem of [Var14] and the fact that ρ is unramified at v, we see that $\mathrm{rec}_{F_v}(\Pi_v)$ is unramified, hence that Π_v itself is unramified. This concludes the proof. □

6.6. *The proof of Theorem* 6.1.2. We proceed to the proof of the second main theorem of this chapter (Theorem 6.1.2). As in the case of the first theorem, we begin by establishing the result under additional conditions (Section 6.6.1), then reduce the general case to this one by using soluble base change (Section 6.6.10).

6.6.1. *Application of the patching argument* (*ordinary case*). We take F to be an imaginary CM number field, and fix the following data:

(1) an integer $n \geq 2$ and a prime $p > n$;
(2) a finite set S of finite places of F, including the places above p;
(3) a (possibly empty) subset $R \subset S$ of places prime to p;
(4) a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, regular algebraic of some weight μ;
(5) a choice of isomorphism $\iota : \overline{\mathbf{Q}}_p \cong \mathbf{C}$.

We assume that the following conditions are satisfied:

(6) If l is a prime lying below an element of S, or that is ramified in F, then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.
(7) The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible.
(8) If $v \in S_p$, then $\pi_v^{\mathrm{Iw}_v(1,1)} \neq 0$ and π is ι-ordinary at v (in the sense of [Ger19, Def. 5.3]).
(9) If $v \in R$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$.
(10) If $v \in S - (R \cup S_p)$, then π_v is unramified, $v \notin R^c$, and $H^2(F_v, \mathrm{ad}\,\overline{r_\iota(\pi)}) = 0$.

(11) $S-(R\cup S_p)$ contains at least two places with distinct residue characteristics.
(12) If $v\notin S$ is a finite place of F, then π_v is unramified.
(13) If $v\in R$, then $q_v\equiv 1 \bmod p$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.
(14) The representation $\overline{r_\iota(\pi)}$ is decomposed generic and the image of $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ is enormous.
(15) If $v\in S_p$, then $[F_v:\mathbf{Q}_p]>\frac{n(n+1)}{2}+1$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.

THEOREM 6.6.2. *With assumptions (1)–(15) as above, suppose given a continuous representation $\rho: G_F\to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ and a weight $\lambda\in(\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ satisfying the following conditions*:

(1) *We have $\overline{\rho}\cong\overline{r_\iota(\pi)}$.*
(2) *For each place $v|p$, there is an isomorphism*

$$\rho|_{G_{F_v}}\sim\begin{pmatrix}\psi_{v,1} & * & * & *\\ 0 & \psi_{v,2} & * & *\\ \vdots & \ddots & \ddots & *\\ 0 & \cdots & 0 & \psi_{v,n}\end{pmatrix},$$

where for each $i=1,\dots,n$ the character $\psi_{v,i}: G_{F_v}\to\overline{\mathbf{Q}}_p^\times$ agrees with the character

$$\sigma\in I_{F_v}\mapsto\prod_{\tau\in\mathrm{Hom}(F_v,\overline{\mathbf{Q}}_p)}\tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$

on the inertia group I_{F_v}.
(3) *For each place $v|p$ of F, for each $i=1,\dots,n$, and for each p-power root of unity $x\in\mathcal{O}_{F_v}$, we have*

$$\prod_{\tau\in\mathrm{Hom}(F_v,\overline{\mathbf{Q}}_p)}\tau(x)^{\lambda_{\tau,n+1-i}-\mu_{\iota\tau,n+1-i}}=1.$$

(4) *For each finite place $v\notin S$ of F, $\rho|_{G_{F_v}}$ is unramified.*
(5) *For each place $v\in R$, $\rho|_{G_{F_v}}$ is unipotently ramified.*

Then ρ is ordinarily automorphic of weight $\iota\lambda$: there exists an ι-ordinary cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\iota\lambda$ such that $\rho\cong r_\iota(\Pi)$. Moreover, if v is a finite place of F and $v\notin S$, then Π_v is unramified.

Note that we do not prove an analogue of Theorem 6.5.4 here, but rather only an analogue of Corollary 6.5.5. This is due to our poor understanding of the irreducible components of the local lifting rings of type $\mathcal{D}_v^{\mathrm{det,ord}}$. Before giving the proof of Theorem 6.6.2, we need to introduce some deformation rings, Hecke algebras, and complexes on which they act. These complexes will represent the ordinary part of completed *homology* with $\mathcal{O}$-coefficients. We will use the notation for ordinary parts established in Section 5.1.

We define an open compact subgroup $K=\prod_v K_v$ of $\mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ as follows:

- If $v \notin S$, then $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.
- If $v \in S_p$, then $K_v = \mathrm{Iw}_v(1,1)$.
- If $v \in R$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (R \cup S_p)$, then $K_v = \mathrm{Iw}_{v,1}$ is the pro-v Iwahori subgroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$.

Then (by Lemma 6.5.2) K is neat, and so is a good subgroup of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$. By Theorem 2.4.10, we can find a coefficient field $E \subset \overline{\mathbf{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K,\mu)^{\mathrm{ord}}$ of residue field k such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{r_\iota(\pi)}$. If $v \in S_p$, we let $\Lambda_{1,v} = \mathcal{O}[\![\mathcal{O}_{F_v}^\times(p)^n]\!]$. We define $\Lambda_1 = \widehat{\otimes}_{v \in S_p} \Lambda_{1,v}$, the completed tensor product being over $\mathcal{O}$. The n-tuple of characters

$$\chi_{\mu,v,i} : \mathcal{O}_{F_v}^\times(p) \to \mathcal{O}^\times,\ x \mapsto \prod_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v,E)} \tau(x)^{-(\mu_{\iota\tau,n-i+1}+i-1)}\ (i = 1,\dots,n)$$

determines a homomorphism $p_{\mu,v} : \Lambda_{1,v} \to \mathcal{O}$. We define $\wp_{\mu,v} = \ker p_{\mu,v}$ and write $\wp_{0,v}$ for the unique minimal prime of $\Lambda_{1,v}$ that is contained in $\wp_{\mu,v}$. We set $\Lambda_v = \Lambda_{1,v}/\wp_{0,v}$ and $\Lambda = \widehat{\otimes}_{v \in S_p} \Lambda_v$. We write $p_\mu : \Lambda \to \mathcal{O}$ for the homomorphism induced by the $p_{\mu,v}$ and the universal property of the completed tensor product, and we set $\wp_\mu = \ker p_\mu$. We use similar notation for p_λ; note that condition (3) in the statement of the theorem implies that $\wp_{0,v}$ is also the unique minimal prime contained in $\wp_{\lambda,v}$ for each $v \in S_p$.

We define a global deformation problem for each character $\chi : K_R \to \mathcal{O}^\times$ that is trivial modulo ϖ by the formula

$$\mathcal{S}_\chi = (\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{O}\}_{v \in S - S_p} \cup \{\Lambda_v\}_{v \in S_p}, \{\mathcal{D}_v^{\det,\mathrm{ord}}\}_{v \in S_p} \cup \{\mathcal{D}_v^\chi\}_{v \in R} \cup \{\mathcal{D}_v^\square\}_{v \in S - (R \cup S_p)}).$$

We fix representatives $\rho_{\mathcal{S}_\chi}$ of the universal deformations that are identified modulo ϖ (via the identifications $R_{\mathcal{S}_\chi}/\varpi \cong R_{\mathcal{S}_1}/\varpi$). We define an $\mathcal{O}[K_S]$-module $\mathcal{V}_\mu(\chi^{-1}) = \mathcal{V}_\mu \otimes_{\mathcal{O}} \mathcal{O}(\chi^{-1})$, where K_S acts on $\mathcal{V}_\mu$ by projection to K_p and on $\mathcal{O}(\chi^{-1})$ by projection to K_R. After possibly enlarging E, we can assume that ρ takes values in $\mathcal{O}$ and that $\rho \bmod \varpi = \overline{\rho}_{\mathfrak{m}}$; then ρ is a lifting of $\overline{\rho}_{\mathfrak{m}}$ of type $\mathcal{S}_1$.

If $c \geq 1$ is an integer, then we define

$$\Lambda_{1,c} = \mathcal{O}[\prod_{v \in S_p} \ker(T_n(\mathcal{O}_{F_v}/\varpi_v^c) \to T_n(\mathcal{O}_{F_v}/\varpi_v))];$$

it is naturally a quotient of Λ_1. For any $c \geq 1$, the complex $R\Gamma(X_{K(c,c)}, \mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}$ is defined as an object of $\mathbf{D}(\Lambda_{1,c})$. We define

$$A_1(\mu,\chi,c) = R\,\mathrm{Hom}_{\Lambda_{1,c}}(R\Gamma(X_{K(c,c)}, \mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}, \Lambda_{1,c})[-d].$$

It is a perfect complex in $\mathbf{D}(\Lambda_{1,c})$ (because $R\Gamma(X_{K(c,c)}, \mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}$ is). The Hecke algebra $\mathbf{T}^{S,\mathrm{ord}}$ acts on this complex by transpose. Moreover, Corollary 5.2.16 shows that for any $c' \geq c \geq 1$, there is a $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism

$$A_1(\mu, \chi, c') \otimes^{\mathbf{L}}_{\Lambda_{1,c'}} \Lambda_{1,c} \cong A_1(\mu, \chi, c) \tag{6.6.3}$$

in $\mathbf{D}(\Lambda_{1,c})$. By construction, there are canonical $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphisms

$$A_1(\mu, \chi, c) \otimes^{\mathbf{L}}_{\Lambda_{1,c}} \Lambda_{1,c}/\varpi \cong A_1(\mu, 1, c) \otimes^{\mathbf{L}}_{\Lambda_{1,c}} \Lambda_{1,c}/\varpi \tag{6.6.4}$$

in $\mathbf{D}(\Lambda_{1,c}/\varpi)$. By [KT17, Lem. 2.13], we can find a perfect complex $A_1(\mu, \chi) \in \mathbf{D}(\Lambda_1)$ that comes equipped an action by $\mathbf{T}^{S,\mathrm{ord}}$ and with $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphisms

$$A_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_{1,c} \cong A_1(\mu, \chi, c)$$

in $\mathbf{D}(\Lambda_{1,c})$ (for each $c \geq 1$) and

$$A_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_1/\varpi \cong A_1(\mu, 1) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_1/\varpi$$

in $\mathbf{D}(\Lambda_1/\varpi)$. These isomorphisms are compatible with the isomorphisms (6.6.3) for $c' \geq c$ and with the isomorphisms (6.6.4) for varying characters χ, trivial modulo ϖ. Finally, we define $A(\mu, \chi) = A_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda \in \mathbf{D}(\Lambda)$.

Let $\nu \in X^*((\mathrm{Res}_{F/\mathbf{Q}} T)_E) = (\mathbf{Z}^n)^{\mathrm{Hom}(F,E)}$ be defined by

$$\nu_\tau = (0, 1, \ldots, n-1)$$

for all $\tau \in \mathrm{Hom}(F, E)$. We define $B_1(\mu, \chi) = A_1(\mu, \chi) \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G \mu)^{-1}$, where $\mathcal{O}(\nu + w_0^G \mu)^{-1}$ is the $\mathcal{O}[T_n(F_p)]$-module described in Section 5.2.1. (In particular, the action of $T_n(\mathcal{O}_{F,p})$ extends uniquely to an action of the completed group algebra $\mathcal{O}[\![T_n(\mathcal{O}_{F,p})]\!]$.) Thus $B_1(\mu, \chi)$ is a perfect complex in $\mathbf{D}(\Lambda_1)$, on which the algebra $\mathbf{T}^{S,\mathrm{ord}}$ acts. We define $B(\mu, \chi) = B_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda$.

Lemma 6.6.5. *The complex $B_1(\mu, \chi)$ is independent of μ. More precisely, for any $\mu' \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$, there is a $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism $B_1(\mu, \chi) \cong B_1(\mu', \chi)$ in $\mathbf{D}(\Lambda_1)$.*

Proof. This follows from Proposition 5.2.17 and [KT17, Lem. 2.13]. □

Corollary 6.6.6. *Let $\mu' \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,E)}$. Then there is a $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism in $\mathbf{D}(\mathcal{O})$:*

$$B_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \mathcal{O}(\nu + w_0^G \mu')^{-1} \cong A_1(\mu', \chi, 1) \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G \mu')^{-1}.$$

Proof. By the lemma, it suffices to treat the case $\mu' = \mu$. In this case the left-hand side may be identified with

$$A_1(\mu, \chi) \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G \mu)^{-1} \otimes^{\mathbf{L}}_{\Lambda_1} \mathcal{O}(\nu + w_0^G \mu)^{-1} \cong A_1(\mu, \chi) \otimes^{\mathbf{L}}_{\Lambda_1} \mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G \mu)^{-1}.$$

Essentially by definition, this complex admits a $\mathbf{T}^{S,\mathrm{ord}}$-equivariant isomorphism to $A_1(\mu,\chi,1)\otimes_{\mathcal{O}}\mathcal{O}(\nu+w_0^G\mu)^{-1}$ in $\mathbf{D}(\mathcal{O})$. This completes the proof. □

Let $\mathbf{T}^{S,\Lambda_1}=\mathbf{T}^S\otimes_{\mathcal{O}}\Lambda_1\subset\mathbf{T}^{S,\mathrm{ord}}$.

PROPOSITION 6.6.7. *There exist an integer $\delta\geq 1$, depending only on n and $[F:\mathbf{Q}]$, an ideal $J\subset\mathbf{T}^{S,\Lambda_1}(A(\mu,\chi)_{\mathfrak{m}}\otimes_{\mathcal{O}}\mathcal{O}(\nu+w_0^G\mu)^{-1})$ such that $J^\delta=0$, and a continuous surjective homomorphism of Λ-algebras*

$$f_{\mathcal{S}_\chi}:R_{\mathcal{S}_\chi}\to\mathbf{T}^{S,\Lambda_1}(A(\mu,\chi)_{\mathfrak{m}}\otimes_{\mathcal{O}}\mathcal{O}(\nu+w_0^G\mu)^{-1})/J$$

such that for each finite place $v\notin S$ of F, the characteristic polynomial of $f_{\mathcal{S}_\chi}\circ\rho_{\mathcal{S}_\chi}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $\mathbf{T}^{S,\Lambda_1}(A(\mu,\chi)_{\mathfrak{m}}\otimes_{\mathcal{O}}\mathcal{O}(\nu+w_0^G\mu)^{-1})/J$.

Proof. We will construct a compatible family of homomorphisms

$$f_{\mathcal{S}_\chi,c}:R_{\mathcal{S}_\chi}\to\mathbf{T}^{S,\Lambda_1}(A(\mu,\chi,c)_{\mathfrak{m}}\otimes_{\mathcal{O}}\mathcal{O}(\nu+w_0^G\mu)^{-1})/J_c,$$

one for each $c\geq 1$. The desired homomorphism $f_{\mathcal{S}_\chi}$ is then obtained by passage to the limit, in a similar way to the proof of Theorem 4.5.1. It even suffices to construct a family of homomorphisms

$$R_{\mathcal{S}_\chi}\to\mathbf{T}^{S,\Lambda_1}(R\Gamma(X_{K(c,c)},\mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}_{\mathfrak{m}})/J_c;$$

in fact, the Hecke algebras are the same (the isomorphism being given by transpose and twist by $\mathcal{O}(\nu+w_0^G\mu)$). Finally, it even suffices to construct a family of homomorphisms

$$R_{\mathcal{S}_\chi}\to\mathbf{T}^{S,\mathrm{ord}}(R\Gamma(X_{K(c,c)},\mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}})_{\mathfrak{m}}/J_c;$$

an application of Carayol's lemma (cf. [CHT08, Lem. 2.1.10]) then implies that the image of $R_{\mathcal{S}_\chi}$ is in fact contained in a nilpotent quotient of the subalgebra

$$\mathbf{T}^{S,\Lambda_1}(R\Gamma(X_{K(c,c)},\mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}_{\mathfrak{m}})\subset\mathbf{T}^{S,\mathrm{ord}}(R\Gamma(X_{K(c,c)},\mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}})_{\mathfrak{m}}.$$

This family of homomorphisms can be constructed exactly as in the proof of Proposition 6.5.3, with the appeal to Theorem 4.5.1 being replaced instead with an appeal to Theorem 5.5.1; here we are using the characterization of the deformation functor $\mathcal{D}_v^{\mathrm{det,ord}}$ given in Section 6.2.6. □

We now need to describe the auxiliary objects associated to a choice of Taylor–Wiles datum $(Q,(\alpha_{v,1},\dots,\alpha_{v,n})_{v\in Q})$ for $\mathcal{S}_1$ (see Section 6.2.28), where each place of Q is assumed to have residue characteristic split in some imaginary quadratic subfield of F. Once again, this datum is automatically a Taylor–Wiles datum for all the global deformation problems $\mathcal{S}_\chi$, and so the auxiliary deformation problems $\mathcal{S}_{\chi,Q}$ are defined, and the deformation ring $R_{\mathcal{S}_{\chi,Q}}$ has a natural structure of $\mathcal{O}[\Delta_Q]$-algebra, where $\Delta_Q=\prod_{v\in Q}\Delta_v=\prod_{v\in Q}k(v)^\times(p)^n$.

If $c\geq 1$ is an integer, then we define two auxiliary level subgroups

$$K(c,c)_1(Q)\subset K(c,c)_0(Q)\subset K(c,c).$$

They are good subgroups of $\mathrm{GL}_n(\mathbf{A}_F^\infty)$, determined by the following conditions:

- If $v \notin S \cup Q$, then $K(c,c)_1(Q)_v = K(c,c)_0(Q)_v = K(c,c)_v$.
- If $v \in Q$, then $K(c,c)_0(Q)_v = \mathrm{Iw}_v$ and $K(c,c)_1(Q)_v$ is the maximal pro-prime-to-p subgroup of Iw_v.

Then there is a natural isomorphism $K(c,c)_0(Q)/K(c,c)_1(Q) \cong \Delta_Q$. We define $A_1(\mu,\chi,Q,c)$ to be

$$R\,\mathrm{Hom}_{\Lambda_{1,c}[\Delta_Q]}(R\Gamma_{K(c,c)_0(Q)/K(c,c)_1(Q)}(X_{K(c,c)_1(Q)}, \mathcal{V}_\mu(\chi^{-1}))^{\mathrm{ord}}, \Lambda_{1,c}[\Delta_Q])[-d],$$

an object of $\mathbf{D}(\Lambda_{1,c}[\Delta_Q])$. The algebra $\mathbf{T}_Q^{S\cup Q,\mathrm{ord}} = \mathbf{T}^{S\cup Q,\mathrm{ord}} \otimes_{\mathbf{T}^{S\cup Q}} \mathbf{T}_Q^{S\cup Q}$ acts on $A_1(\mu,\chi,Q,c)$ by transpose. As in the case where Q is empty, we can pass to the limit with respect to c to obtain a complex $A_1(\mu,\chi,Q) \in \mathbf{D}(\Lambda_1[\Delta_Q])$ that comes equipped with an action of $\mathbf{T}_Q^{S\cup Q,\mathrm{ord}}$ and with $\mathbf{T}_Q^{S\cup Q,\mathrm{ord}}$-equivariant isomorphisms

$$A_1(\mu,\chi,Q) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_{1,c} \cong A_1(\mu,\chi,Q,c)$$

in $\mathbf{D}(\Lambda_{1,c})$ (for each $c \geq 1$) and

$$A_1(\mu,\chi,Q) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_1/\varpi \cong A_1(\mu,1,Q) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda_1/\varpi$$

in $\mathbf{D}(\Lambda_1/\varpi)$, all compatible with the similar data at level c. We define $\mathfrak{m}^Q$ to be the contraction of $\mathfrak{m}$ to $\mathbf{T}^{S\cup Q,\mathrm{ord}}$, and $\mathfrak{n}^Q$ to be the ideal of $\mathbf{T}_Q^{S\cup Q,\mathrm{ord}}$ generated by $\mathfrak{m}^Q$ and the elements $U_{v,i} - \alpha_{v,1}\cdots\alpha_{v,i}$ $(v \in Q, i = 1,\ldots,n)$.

Lemma 6.6.8. *The ideal $\mathfrak{n}^Q$ occurs in the support of $H^*(A_1(\mu,\chi,Q))$. There are $\mathbf{T}^{S\cup Q,\mathrm{ord}}$-equivariant isomorphisms*

$$A_1(\mu,\chi,Q)_{\mathfrak{n}^Q} \otimes^{\mathbf{L}}_{\Lambda_1[\Delta_Q]} \Lambda_1 \cong A_1(\mu,\chi)_{\mathfrak{m}^Q} \cong A_1(\mu,\chi)_{\mathfrak{m}}.$$

Proof. These properties can be established in the same way as in the finite level (Fontaine–Laffaille) case. See Section 6.5.1. We omit the details. □

We define $A(\mu,\chi,Q) = A_1(\mu,\chi,Q) \otimes^{\mathbf{L}}_{\Lambda_1} \Lambda$ and ${}_{\Delta_Q}\mathbf{T}^{S\cup Q,\Lambda_1} = \mathbf{T}^{S\cup Q,\Lambda_1} \otimes_{\mathcal{O}} \mathcal{O}[\Delta_Q]$. Note that this acts on $A(\Lambda,\chi,Q)_{\mathfrak{n}^Q}$ via our identifications

$$K(c,c)_0(Q)/K(c,c)_1(Q) \cong \Delta_Q$$

for each c and passing to the limit. Thus ${}_{\Delta_Q}\mathbf{T}^{S\cup Q,\Lambda_1}(A(\Lambda,\chi,Q)_{\mathfrak{n}^Q})$ is a local $\Lambda[\Delta_Q]$-algebra.

Proposition 6.6.9. *There exist an integer $\delta \geq 1$, depending only on n and $[F:\mathbf{Q}]$, an ideal $J \subset_{\Delta_Q} \mathbf{T}^{S\cup Q,\Lambda_1}(A(\Lambda,\chi,Q)_{\mathfrak{n}^Q} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1})$ such that $J^\delta = 0$, and a continuous surjective homomorphism of $\Lambda[\Delta_Q]$-algebras*

$$f_{\mathcal{S}_{\chi,Q}} : R_{\mathcal{S}_{\chi,Q}} \to_{\Delta_Q} \mathbf{T}^{S\cup Q,\Lambda_1}(A(\mu,\chi,Q)_{\mathfrak{n}^Q} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1})/J$$

such that for each finite place $v \notin S$ of F, the characteristic polynomial of $f_{\mathcal{S}_\chi} \circ \rho_{\mathcal{S}_\chi}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in ${}_{\Delta_Q}\mathbf{T}^{S\cup Q,\Lambda_1}(A(\mu,\chi,Q)_{\mathfrak{n}^Q} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1})/J$.

Proof. The existence of a Λ-algebra homomorphism

$$R_{\mathcal{S}_{\chi,Q}} \to_{\Delta_Q} \mathbf{T}^{S\cup Q,\Lambda_1}(A(\mu,\chi,Q)_{\mathfrak{n}^Q} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1})/J$$

satisfying the given condition at finite places $v \notin S \cup Q$ of F is proved just as in the proof of Proposition 6.6.7 above. The key point is to show that this is a homomorphism of $\Lambda[\Delta_Q]$-algebras. This can be proved in the same way as in the proof of Proposition 6.5.11, by considering the enlarged algebra $\mathbf{T}_Q^{S\cup Q,\mathrm{ord}}(A(\mu,\chi,Q) \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1})_{\mathfrak{n}^Q}$. □

We are now ready to begin the proof of Theorem 6.6.2.

Proof of Theorem 6.6.2. We recall that we have constructed a homomorphism $f : R_{\mathcal{S}_1} \to \mathcal{O}$, classifying the representation ρ that we wish to show is automorphic. We will show that $\ker f$ is in the support of

$$H^*(B(\mu,1)_{\mathfrak{m}} \otimes^{\mathbf{L}}_{\Lambda} \mathcal{O}(\nu + w_0^G\lambda)^{-1}).$$

By Corollary 6.6.6, this will show that $\ker f$ is in the support of

$$H^*(A(\lambda,1,1)_{\mathfrak{m}} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\lambda)^{-1})[1/p],$$

in turn a quotient of

$$\mathrm{Hom}_E(H^{d-*}(X_{K(1,1)},\mathcal{V}_\lambda)_{\mathfrak{m}}, \mathcal{O}(\nu + w_0^G\lambda)^{-1}[1/p]).$$

The ι-ordinary automorphy of ρ will then follow from Theorem 2.4.10.

Our proof now closely follows the proof of Theorem 6.5.4. Let

$$q = h^1(F_S/F, \mathrm{ad}\,\overline{\rho}_{\mathfrak{m}}(1)) \quad \text{and} \quad g = qn - n^2[F^+ : \mathbf{Q}],$$

and set $\Delta_\infty = \mathbf{Z}_p^{nq}$. Let $\mathcal{T}$ be a power series ring over Λ in $n^2|S| - 1$ many variables, and let $S_\infty = \mathcal{T}[\![\Delta_\infty]\!]$. Viewing S_∞ as an augmented Λ-algebra, we let $\mathfrak{a}_\infty$ denote the augmentation ideal.

As in the proof of Theorem 6.5.4, we choose a character

$$\chi = \prod_{v\in R} \chi_v \colon \prod_{v\in R} \mathrm{Iw}_v \to \mathcal{O}^\times$$

such that for each $v \in R$, the n characters $\chi_{v,i} \colon k(v)^\times \to \mathcal{O}^\times$ are trivial modulo ϖ and pairwise distinct.

Let $R^{\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_1^{\mathrm{ord}}}$ and $R'^{\mathrm{loc}} = R^{S,\mathrm{loc}}_{\mathcal{S}_\chi^{\mathrm{ord}}}$ denote the corresponding local deformation rings as in Section 6.2.22. We let R_∞ and R'_∞ be formal power series rings in g variables over R^{loc} and R'^{loc}, respectively.

We can then apply the results of Section 6.4.2 to complexes $A(\mu,\chi,Q)_{\mathfrak{n}^Q} \otimes_{\mathcal{O}} \mathcal{O}(\nu + w_0^G\mu)^{-1}$ (for choices of Taylor–Wiles data $(Q,(\alpha_{v,1},\ldots,\alpha_{v,n})_{v\in Q})$, proved to exist using Proposition 6.2.33) and obtain the following:

- Bounded complexes $\mathcal{C}_\infty$ and $\mathcal{C}'_\infty$ of free S_∞-modules, subrings
$$T_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}_\infty) \text{ and } T'_\infty \subset \mathrm{End}_{\mathbf{D}(S_\infty)}(\mathcal{C}'_\infty),$$
and ideals I_∞ and I'_∞ satisfying $I_\infty^\delta = 0$ and $I_\infty'^\delta = 0$. We also have S_∞-algebra structures on R_∞ and R'_∞ and S_∞-algebra surjections $R_\infty \to T_\infty/I_\infty$ and $R'_\infty \to T'_\infty/I'_\infty$.
- Surjections of local Λ-algebras $R_\infty/\mathfrak{a}_\infty \to R_{\mathcal{S}_1^{\mathrm{ord}}}$ and $R'_\infty/\mathfrak{a}_\infty \to R_{\mathcal{S}_\chi^{\mathrm{ord}}}$.
- Isomorphisms $\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong A(\mu,1)_\mathfrak{m} \otimes_\mathcal{O} \mathcal{O}(\nu + w_0^G\mu)^{-1} = B(\mu,1)_\mathfrak{m}$ and $\mathcal{C}'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong A(\mu,\chi)_\mathfrak{m} \otimes_\mathcal{O} \mathcal{O}(\nu + w_0^G\mu)^{-1} = B(\mu,\chi)_\mathfrak{m}$ in $\mathbf{D}(\Lambda)$.

This gives the necessary input for Section 6.3.5. Recall that R_∞ and R'_∞ are power series rings over R^{loc} and $R'^{\,\mathrm{loc}}$, respectively, in $g = qn - n[F^+ : \mathbf{Q}]$ many variables. It follows from parts (1) and (2) of Lemma 6.2.27 that we have satisfied assumptions (1) and (2) of Section 6.3.5. To verify assumption (3), if we let $\mathfrak{p}$ denote the inverse image in S_∞ of $\mathrm{Ann}_\Lambda(\mathcal{O}(\nu + w_0^G\mu)^{-1})$, then (Corollary 6.6.6) the complex
$$(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{p})[1/p] \cong (B(\mu,1)_\mathfrak{m} \otimes^{\mathbf{L}}_\Lambda \mathcal{O}(\nu + w_0^G\mu)^{-1})[1/p]$$
has cohomology isomorphic to a quotient of $\mathrm{Hom}_E(H^{d-*}(X_{K(1,1)}, \mathcal{V}_\mu)_\mathfrak{m}[1/p], E)$. Since π contributes to this quotient, Theorem 2.4.10 implies that $H^*(\mathcal{C}_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{p})[1/p] \neq 0$ and that the cohomology is concentrated in degrees $[q_0, q_0 + l_0]$.

We have now satisfied all the assumptions of Section 6.3.5, and we apply Corollary 6.3.9 with $x \in \mathrm{Spec}(R_\infty)$ the inverse image of $\ker f$, so $y \in \mathrm{Spec}(S_\infty)$ is the inverse image of $\mathrm{Ann}_\Lambda(\mathcal{O}(\nu + w_0^G\lambda)^{-1})$. For each $v \in S_p$, the inertial characters on the diagonal of $\rho|_{G_{F_v}}$ are distinct, so x lies on a maximal dimension irreducible component of $\mathrm{Spec}(R_\infty)$ by part (3) of Lemma 6.2.27, and this corollary does apply. We deduce that the support of
$$H^*(B(\mu,1)_\mathfrak{m} \otimes^{\mathbf{L}}_\Lambda \mathcal{O}(\nu + w_0^G\lambda)^{-1})\ [1/p]$$
contains $\ker f$. This completes the proof. □

6.6.10. *End of the proof* (*ordinary case*). We can now deduce Theorem 6.1.2, our main automorphy lifting result in the ordinary case, from Theorem 6.6.2. The proof is a minor variation of the proof of our main automorphy lifting result in the Fontaine–Laffaille case; see Section 6.5.12.

Proof of Theorem 6.1.2. For the convenience of the reader, we recall the hypotheses of Theorem 6.1.2. Let F be an imaginary CM or totally real field, and let $c \in \mathrm{Aut}(F)$ be complex conjugation. We are given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:

(1) ρ is unramified almost everywhere.
(2) For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is potentially semi-stable, ordinary with regular Hodge–Tate weights. In other words, there exists

a weight $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ such that for each place $v|p$, there is an isomorphism

$$\rho|_{G_{F_v}} \sim \begin{pmatrix} \psi_{v,1} & * & * & * \\ 0 & \psi_{v,2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \psi_{v,n} \end{pmatrix},$$

where for each $i = 1, \ldots, n$ the character $\psi_{v,i} : G_{F_v} \to \overline{\mathbf{Q}}_l^\times$ agrees with the character

$$\sigma \in I_{F_v} \mapsto \prod_{\tau \in \mathrm{Hom}(F_v,\overline{\mathbf{Q}}_p)} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$

on an open subgroup of the inertia group I_{F_v}.

(3) $\overline{\rho}$ is absolutely irreducible and generic. The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is enormous. There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar. We have $p > n$.

(4) There exist a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that π is ι-ordinary and $\overline{r_\iota(\pi)} \cong \rho$.

The case where F is a totally real field can be reduced to the case where F is totally imaginary by base change. We therefore assume now that F is imaginary, and we write F^+ for its maximal totally real subfield. Let $K/F(\zeta_p)$ be the extension cut out by $\overline{\rho}|_{G_{F(\zeta_p)}}$. Choose finite sets V_0, V_1, V_2 of finite places of F having the following properties:

- For each $v \in V_0$, v splits in $F(\zeta_p)$. For each proper subfield $K/K'/F(\zeta_p)$, there exists $v \in V_0$ such that v splits in $F(\zeta_p)$ but does not split in K'.
- For each proper subfield $K/K'/F$, there exists $v \in V_1$ that does not split in K'.
- There exists a rational prime $p_0 \neq p$ that is decomposed generic for $\overline{\rho}$, and V_2 is equal to the set of p_0-adic places of F.
- For each $v \in V_0 \cup V_1 \cup V_2$, $v \nmid 2$, $v \nmid p$, and ρ and π are both unramified at v.

If E/F is any finite Galois extension that is $V_0 \cup V_1 \cup V_2$-split, then $\overline{\rho}|_{G_E}$ has the following properties:

- $\overline{\rho}(G_E) = \overline{\rho}(G_F)$ and $\overline{\rho}(G_{E(\zeta_p)}) = \overline{\rho}(G_{F(\zeta_p)})$. In particular, $\overline{\rho}|_{G_{E(\zeta_p)}}$ has enormous image and there exists $\sigma \in G_E - G_{E(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar.
- $\overline{\rho}|_{G_E}$ is decomposed generic. Indeed, the rational prime p_0 splits in E.

Let E_0/F be a soluble CM extension satisfying the following conditions:

- Each place of $V_0 \cup V_1 \cup V_2$ splits in E_0.
- For each finite place w of E_0, $\pi_{E_0,w}^{\mathrm{Iw}_w} \neq 0$.

- For each finite prime-to-p place w of E_0, either both $\pi_{E_0,w}$ and $\rho|_{G_{E_{0,w}}}$ are unramified or $\rho|_{G_{E_{0,w}}}$ is unipotently ramified, $q_w \equiv 1 \bmod p$, and $\overline{\rho}|_{G_{E_{0,w}}}$ is trivial.
- For each place $w|p$ of E_0, $\overline{\rho}|_{G_{E_{0,w}}}$ is trivial and $[E_{0,w} : \mathbf{Q}_p] > n(n+1)/2 + 1$.
- For each place $v|p$ of F, for each $w|v$ of E_0, and for each $i = 1, \ldots, n$, the character $\psi_{v,i} : G_{F_v} \to \overline{\mathbf{Q}}_p^\times$ agrees with the character
$$\sigma \in I_{F_v} \mapsto \prod_{\tau \in \mathrm{Hom}(F_v, \overline{\mathbf{Q}}_p)} \tau(\mathrm{Art}_{F_v}^{-1}(\sigma))^{-(\lambda_{\tau,n-i+1}+i-1)}$$
on the whole of the inertia subgroup $I_{E_{0,w}} \subset I_{F_v}$.
- Let μ denote the weight of π_{E_0}. Then for each place $w|p$ of E_0, and for each p-power root of unity $x \in E_{0,w}$, we have
$$\psi_{v,i}(\mathrm{Art}_{E_{0,w}}(x)) \prod_{\tau \in \mathrm{Hom}(E_{0,w}, \overline{\mathbf{Q}}_p)} \tau(x)^{\mu_{\iota\tau,n-i+1}+i-1} = 1.$$

We can find imaginary quadratic fields E_a, E_b, E_c satisfying the following conditions:

- Each rational prime lying below a place of $V_0 \cup V_1 \cup V_2$ splits in $E_a \cdot E_b \cdot E_c$.
- The primes $2, p$ split in E_a.
- If $l \notin \{2, p\}$ is a rational prime lying below a place of E_0 at which $\pi_{E_0,w}$ or $\rho|_{E_{0,w}}$ is ramified, or that is ramified in $E_0 \cdot E_a \cdot E_c$, then l splits in E_b.
- If $l \notin \{2, p\}$ is a rational prime that is ramified in E_b, then l splits in E_c.

For example, we can choose any E_a satisfying the given conditions. Then we can choose $E_b = \mathbf{Q}(\sqrt{-p_b})$, where p_b is a prime satisfying $p_b \equiv 1 \bmod 4$ and $p_b \equiv -1 \bmod l$ for any prime $l \notin \{2, p\}$ either lying below a place w of E_0 at which $\pi_{E_0,w}$ or $\rho|_{E_{0,w}}$ is ramified, or ramified in $E_0 \cdot E_a$, and $E_c = \mathbf{Q}(\sqrt{-p_c})$, where $p_c \equiv 1 \bmod 4p_b$ is a prime. (Use quadratic reciprocity to show that p_c splits in E_b.)

We let $E = E_0 \cdot E_a \cdot E_b \cdot E_c$. Then E/F is a soluble CM extension in which each place of $V_0 \cup V_1 \cup V_2$ splits, and the following conditions hold by construction:

- Let R denote the set of prime-to-p places w of E such that $\pi_{E,w}$ or $\rho|_{G_{E_w}}$ is ramified. Let S_p denote the set of p-adic places of E. Let $S' = S_p \cup R$. Then if l is a prime lying below an element of S', or which is ramified in E, then E contains an imaginary quadratic field in which l splits.
- If $w \in R$, then $\overline{\rho}|_{G_{E_w}}$ is trivial and $q_w \equiv 1 \bmod p$.
- The image of $\overline{\rho}|_{G_{E(\zeta_p)}}$ is enormous. The representation $\overline{\rho}|_{G_E}$ is decomposed generic.
- There exists $\sigma \in G_E - G_{E(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar.
- For each place $w|p$ of E, $\overline{\rho}|_{G_{E_w}}$ is trivial and $[E_w : \mathbf{Q}_p] > n(n+1)/2 + 1$.

- Let π_E denote the base change of π to E, which exists, by Proposition 6.5.13. Then π_E is ι-ordinary, by [Ger19, Lem. 5.7].

By the Chebotarev density theorem, we can find infinitely many places v_0 of E of degree 1 over $\mathbf{Q}$ such that $\overline{\rho}(\mathrm{Frob}_{v_0})$ is scalar and $q_{v_0} \not\equiv 1 \bmod p$, $v_0 \notin S' \cup R^c$ and the residue characteristic of v_0 is odd. Then $H^2(E_{v_0}, \mathrm{ad}\,\overline{\rho}) = H^0(E_{v_0}, \mathrm{ad}\,\overline{\rho}(1))^\vee = 0$. We choose v_0, v_0' with distinct residue characteristics satisfying these conditions, and we set $S = S' \cup \{v_0, v_0'\}$. Note that if l_0, l_0' denotes the residue characteristic of v_0, v_0', then l_0, l_0' splits in any imaginary quadratic subfield of E.

We see that the hypotheses (1)–(15) of Section 6.6.1 are now satisfied for E, π_E, and the set S. We can therefore apply Theorem 6.6.2 to $\rho|_{G_E}$ to conclude the existence of a cuspidal, regular algebraic automorphic representation Π_E of $\mathrm{GL}_n(\mathbf{A}_E)$ such that Π_E is ι-ordinary of weight λ_E and $r_\iota(\Pi_E) \cong \rho|_{G_E}$. By Proposition 6.5.13 and [Ger19, Lem. 5.7], we can descend Π_E to obtain a cuspidal, regular algebraic automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that Π is ι-ordinary of weight λ and $r_\iota(\Pi) \cong \rho$. Taking into account the final sentence of the statement of Theorem 6.6.2, we see that $\Pi_{E,w}$ is unramified if $w \notin S$.

To finish the proof, we must show that Π_v is unramified if v is a finite place of F such that $v \nmid p$ and both ρ and π are unramified at v. Using our freedom to vary the choice of places v_0 v_0', we see that if $v \nmid p$ is a place of F such that both ρ and π are unramified at v, then $\Pi_{E,w}$ is unramified for any place $w|v$ of E. This implies that $\mathrm{rec}_{F_v}(\Pi_v)$ is a finitely ramified representation of the Weil group W_{F_v}. Using the main theorem of [Var14] and the fact that ρ is unramified at v, we see that $\mathrm{rec}_{F_v}(\Pi_v)$ is unramified, hence that Π_v itself is unramified. This concludes the proof. □

7. Applications

7.1. *Compatible systems.* Suppose that F is a number field. We will use a slight weakening of the definition of a weakly compatible system from [BLGGT14]: By a *rank n very weakly compatible system $\mathcal{R}$ of l-adic representations of G_F defined over M* we shall mean a 5-tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\}),$$

where

(1) M is a number field.
(2) S is a finite set of primes of F.
(3) For each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$.
(4) For $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of n integers.
(5) For each prime λ of M (with residue characteristic l say),

$$r_\lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{M}_\lambda)$$

is a continuous, semi-simple representation such that

(a) if $v \notin S$ and $v \nmid l$ is a prime of F, then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$;

(b) for l outside a set of primes of Dirichlet density 0, the representation $r_\lambda|_{G_{F_v}}$ is crystalline for all $v|l$, and for any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over M, we have $\mathrm{HT}_\tau(r_\lambda) = H_\tau$;

(c) for all λ, we have $\mathrm{HT}_\tau(\det r_\lambda) = \sum_{h \in H_\tau} h$.

If we further drop hypothesis (5b), then we say that $\mathcal{R}$ is an *extremely weakly compatible system*. The only dependence of an extremely weakly compatible system on H_τ is via the condition on the determinant via hypothesis (5c). The difference between very weakly compatible systems and the (merely) weakly compatible systems in [BLGGT14] is that if $v|l$, then we only insist that $r_\lambda|_{G_{F_v}}$ is de Rham for l in a set of Dirichlet density 1. The notion of an extremely weakly compatible system is what used to be known as a compatible system, but we use this language so as to emphasize that the condition of being a very weakly compatible system is more stringent than being an extremely weakly compatible system. (Here we implicitly use the following fact: *any* compatible family of one-dimensional representations is always de Rham [Hen82].) Of course, we expect that any extremely weakly compatible system should give rise to a weakly compatible system for an appropriate choice of H_τ. We have adopted the present definition so that, as a consequence of Theorem 4.5.1, we can deduce that the Galois representations constructed in [HLTT16] for $n = 2$ form a very weakly compatible system; see Lemma 7.1.10.

We will often write l for the residue characteristic of a prime λ of M without comment. We shall write $\overline{r}_\lambda$ for the semi-simplified reduction of r_λ. The representation $\overline{r}_\lambda$ is a priori defined over the algebraic closure of $\mathcal{O}_M/\lambda$. However, because its trace lies in $\mathcal{O}_M/\lambda$ and because the Brauer groups of all finite fields are trivial, it is actually a representation

$$\overline{r}_\lambda : G_F \longrightarrow \mathrm{GL}_n(\mathcal{O}_M/\lambda).$$

We recall some further definitions from Section 5.1 of [BLGGT14] that apply *mutatis mutandis* to both very weakly and extremely weakly compatible families:

A very (or extremely) weakly compatible system $\mathcal{R}$ is *regular* if, for each τ, the set H_τ has n distinct elements.

A very (or extremely) weakly compatible system $\mathcal{R}$ is *irreducible* if there is a set $\mathcal{L}$ of rational primes of Dirichlet density 1 such that for $\lambda|l \in \mathcal{L}$, the representation r_λ is irreducible. We say that it is *strongly irreducible* if for all finite extensions F'/F the compatible system $\mathcal{R}|_{G_{F'}}$ is irreducible.

LEMMA 7.1.1. *If $\mathcal{R}$ is an extremely weakly compatible system of rank* 2, *then either r_λ is irreducible for all λ or there exist weakly compatible systems $\mathcal{X}_1$ and $\mathcal{X}_2$ of rank* 1 *with $r_\lambda \cong \chi_{1,\lambda} \oplus \chi_{2,\lambda}$ for all λ.*

Proof. Suppose that for one prime λ_0 the representation r_{λ_0} is a sum of characters $r_{\lambda_0} = \chi_1 \oplus \chi_2$. By the main result of [Hen82], we see that r_{λ_0} is de Rham. Hence each χ_i is also de Rham and so there are weakly compatible systems $\mathcal{X}_1$ and $\mathcal{X}_2$ of rank 1 with $\chi_{i,\lambda_0} = \chi_i$ for $i = 1, 2$. Then for all λ, we have $r_\lambda \cong \chi_{1,\lambda} \oplus \chi_{2,\lambda}$. □

In view of Lemma 7.1.1, we say that an extremely weakly compatible system of rank 2 is *reducible* if it is not irreducible, in which case every representation r_λ is reducible. Say that a very (or extremely) weakly compatible system of rank 2 is *Artin up to twist* if there exists an irreducible Artin representation $\rho : G_F \to \mathrm{GL}_2(\overline{M})$ with traces in M (possibly after increasing M) and a weakly compatible system of one-dimensional representations χ_λ such that $r_\lambda \simeq \rho \otimes \chi_\lambda$.

LEMMA 7.1.2. *If $\mathcal{R}$ is an extremely weakly compatible system of rank* 2 *and $\mathcal{R}$ is irreducible, then either*

(1) *$\mathcal{R}$ is strongly irreducible; or*
(2) *$\mathcal{R}$ is Artin up to twist; or*
(3) *there is a quadratic extension F'/F and a weakly compatible system $\mathcal{X}$ of characters of $G_{F'}$ such that*

$$\mathcal{R} \cong \mathrm{Ind}_{G_{F'}}^{G_F} \mathcal{X},$$

in which case we say that $\mathcal{R}$ is induced.

Proof. Suppose that $\mathcal{R}$ is not strongly irreducible, so that there exists a finite extension E/F such that $\mathcal{R}|_{G_E}$ is reducible. We may suppose that E/F is Galois. Choose a prime λ of M of residue characteristic greater than 2. Write $r_\lambda|_{G_E} = \chi_1 \oplus \chi_2$.

Suppose that $\chi_1 = \chi_2 = \chi$. As in the proof of Lemma 7.1.1, we deduce that χ is de Rham. On the other hand, let ϕ denote the determinant of r_λ, and let $\langle\phi\rangle$ be the character such that $\phi/\langle\phi\rangle$ is the Teichmüller lift of the reduction $\overline{\phi}$ of ϕ. Since $\overline{\phi}$ is a finite order character, we may assume (increasing E if necessary) that this character is trivial after restriction to G_E. By construction, $\overline{\langle\phi\rangle} = 1$ and thus (because λ is assumed to have odd residue characteristic) $\langle\phi\rangle$ admits a square root character ψ as a representation of G_F. But then $\psi^2|_{G_E}$ and χ^2 coincide as representations of G_E, since they are both equal to the determinant of $r_\lambda|_{G_E}$. In particular, their ratio is a character of order dividing 2. Increasing E by a finite extension if necessary, we may assume that $\psi|_{G_E} = \chi$. Hence $\psi|_{G_E}$ is de Rham, and thus ψ is de Rham and extends to a compatible

system of characters of G_F. After twisting $\mathcal{R}$ by this compatible system, we may assume that $r_\lambda|_{G_E}$ is trivial. In particular, r_λ factors through $\mathrm{Gal}(E/F)$, and is thus coming from an Artin representation $\rho_\lambda : G_F \to \mathrm{GL}_2(\overline{M}_\lambda)$, which automatically extends to a (strongly) compatible system coming from an Artin representation $\rho : \mathrm{Gal}(E/F) \to \mathrm{GL}_2(\overline{M})$ with traces in some finite extension of M (specifically, the extension of M coming from the coefficient field of the compatible family ψ). Hence $\mathcal{R}$ is Artin up to twist in this case.

Now assume that $\chi_1 \neq \chi_2$. The group $\mathrm{Gal}(E/F)$ permutes the two characters χ_i and, because r_λ is irreducible, this action is transitive. Let F' denote the stabilizer of χ_1. Then χ_1 extends to a character of $G_{F'}$ and $r_\lambda = \mathrm{Ind}_{G_{F'}}^{G_F} \chi_1$. As in the proof of Lemma 7.1.1, there is a weakly compatible system of characters $\mathcal{X}$ of $G_{F'}$ with $\chi_\lambda = \chi_1$. Then $\mathcal{R} \cong \mathrm{Ind}_{G_{F'}}^{G_F} \mathcal{X}$, as desired. □

Lemma 7.1.3. *If $\mathcal{R}$ is an extremely weakly compatible system of rank 2 and $\mathcal{R}$ is irreducible, then for all l in a set of Dirichlet density 1 and all $\lambda|l$, the residual representation $\overline{r}_\lambda$ is absolutely irreducible.*

If, moreover, $\mathcal{R}$ is neither induced nor Artin up to twist and $\widetilde{F}$ denotes the normal closure of $F/\mathbf{Q}$, then one may additionally assume that the image $\overline{r}_\lambda(G_{\widetilde{F}})$ contains $\mathrm{SL}_2(\mathbf{F}_l)$.

Proof. This is immediate if $\mathcal{R}$ is Artin up to twist. If $\mathcal{R} \cong \mathrm{Ind}_{G_{F'}}^{G_F} \mathcal{X}$ then choose a prime $v \notin S$ of F that splits in F' and such that $Q_v(X)$ has distinct roots. (If no such prime v existed, then we would have $\mathcal{X} = {}^\sigma\mathcal{X}$, where $1 \neq \sigma \in \mathrm{Gal}(F'/F)$, contradicting the irreducibility of $\mathcal{R}$.) Then for any λ not dividing the residue characteristic of v and modulo that $Q_v(X)$ still has distinct roots, we see that $\overline{r}_\lambda$ is irreducible.

Hence we may assume that $\mathcal{R}$ is strongly irreducible. In particular, since the only connected Zariski closed subgroups of GL_2 that act irreducibly contain SL_2, it follows that the Zariski closure of the image of r_λ contains $\mathrm{SL}_2(\overline{M}_\lambda)$ for all λ. We first prove, replacing M by a finite extension if necessary, that the Galois representations r_λ can all be made to land inside $\mathrm{GL}_2(M_\lambda)$.

The image of r_λ contains an element with distinct eigenvalues. Hence, by the Cebotarev density theorem, there exists an auxiliary prime $v \notin S$ such that $r_\lambda(\mathrm{Frob}_v)$ has distinct eigenvalues. These eigenvalues are defined over a (at most) quadratic extension of M. By enlarging M if necessary, we deduce that the images of r_λ for all $\lambda \nmid N(v)$ contain an element with distinct eigenvalues in M_λ, which allows one to conjugate the representation r_λ to land in M_λ. By choosing a second auxiliary prime of different residue characteristic and enlarging M once again, we may ensure the image of r_λ lies in $\mathrm{GL}_2(M_\lambda)$ for all λ.

Let

$$s_l = \bigoplus_{\lambda|l} r_\lambda : G_F \longrightarrow \mathrm{GL}_{2[M:\mathbf{Q}]}(\mathbf{Q}_l),$$

so that $\mathcal{S} = \{s_l\}$ form an extremely weakly compatible system with coefficients $\mathbf{Q}$. Let $\mathbf{G}_l$ denote the Zariski closure of the image of s_l. It is contained in $(\mathrm{Res}_{\mathbf{Q}}^M \mathrm{GL}_2) \times_{\mathbf{Q}} \mathbf{Q}_l$. The pushforward of $\mathbf{G}_l$ to $\mathrm{GL}_2/\overline{\mathbf{Q}}_l$ via any embedding of $M \hookrightarrow \overline{\mathbf{Q}}_l$ will contain SL_2. We will write $\mathbf{G}_l^\circ$ for the connected component of the identity of $\mathbf{G}_l$, $\mathbf{G}_l^{\mathrm{ad}}$ for the quotient of $\mathbf{G}_l$ by its center and $\mathbf{G}_l^{\mathrm{sc}}$ for the (simply connected) universal cover of $\mathbf{G}_l^{\mathrm{ad}}$. Then $\mathbf{G}_l^0$ is unramified for all $l \in \mathcal{L}$ a set of rational primes of Dirichlet density 1 (see [LP92, Prop. 8.9]). Also over $\overline{\mathbf{Q}}_l$, we see that $\mathbf{G}_l^{\mathrm{ad}}$ is contained in $\mathrm{PGL}_2^{[M:\mathbf{Q}]}$ and surjects onto each factor.

The following facts are either well known or easy to check in the order indicated:

(1) The only morphisms $\mathrm{PGL}_2 \to \mathrm{PGL}_2$ over $\overline{\mathbf{Q}}_l$ are the trivial map and conjugation by an element of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_l)$.
(2) The only morphisms $\mathrm{PGL}_2^r \to \mathrm{PGL}_2$ over $\overline{\mathbf{Q}}_l$ are the trivial map and projection onto one factor composed with conjugation by an element of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_l)$.
(3) If I and J are finite sets, then up to conjugation by an element of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_l)^J$, the only morphisms $\mathrm{PGL}_2^I \to \mathrm{PGL}_2^J$ over $\overline{\mathbf{Q}}_l$ are induced by a pair (J_0, ϕ) where $J_0 \subset J$ and $\phi : J_0 \to I$.
(4) If I is a finite set, then the automorphism group of PGL_2^I is $\mathrm{PGL}_2^I \rtimes S_I$, where S_I is the group of permutations of I.
(5) If J is a finite set and G is a connected algebraic subgroup of PGL_2^J over $\overline{\mathbf{Q}}_l$ that surjects onto PGL_2 via each projection, then $G \cong \mathrm{PGL}_2^I$ and the inclusion $\mathrm{PGL}_2^I \hookrightarrow \mathrm{PGL}_2^J$ corresponds, up to conjugation by an element of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_l)^I$ to a map $\phi : J \twoheadrightarrow I$. (Use induction on $\#J$ and Goursat's lemma.)
(6) If $M/\mathbf{Q}_l$ is a finite extension, then $(\mathrm{Res}_{\mathbf{Q}_l}^M \mathrm{PGL}_2) \times_{\mathbf{Q}_l} \overline{\mathbf{Q}}_l \cong \mathrm{PGL}_2^{\mathrm{Hom}_{\mathbf{Q}_l}(M, \overline{\mathbf{Q}}_l)}$ and the action of $G_{\mathbf{Q}_l}$ is via the map $G_{\mathbf{Q}_l} \to S_{\mathrm{Hom}_{\mathbf{Q}_l}(M, \overline{\mathbf{Q}}_l)}$ where $G_{\mathbf{Q}_l}$ acts by left translation.
(7) Forms of PGL_2^r are classified by the middle term of the (split) exact sequence of pointed sets

$$H^1(\mathbf{Q}_l, \mathrm{PGL}_2^r/\overline{\mathbf{Q}}_l) \to H^1(\mathbf{Q}_l, \mathrm{Aut}(\mathrm{PGL}_2^r/\overline{\mathbf{Q}}_l)) \to H^1(\mathbf{Q}_l, S_r).$$

In order to split over an unramified extension, the image in $H^1(\mathbf{Q}_l, S_r) = \mathrm{Hom}(G_{\mathbf{Q}_l}, S_r)$ must be unramified and hence land in $H^1(\mathbf{F}_l, S_r)$. Every class in $H^1(\mathbf{F}_l, S_r)$ comes from the image of a group of the form $G = \prod_i \mathrm{Res}_{\mathbf{Q}_l}^{N_i} \mathrm{PGL}_2$, where $N_i/\mathbf{Q}_l$ are unramified extensions. On the other hand, the fibres of $[G] \in H^1(\mathbf{Q}_l, S_r)$ are inner forms of G, and there is a unique quasi-split form amongst all inner forms. Since G is quasi-split, the only forms of PGL_2^r that are unramified (that is, quasi-split and split over an unramified extension) are thus given by $\prod_i \mathrm{Res}_{\mathbf{Q}_l}^{N_i} \mathrm{PGL}_2$ for unramified N_i.

(8) Suppose that for $j \in J$, a finite set, $M_j/\mathbf{Q}_l$ is a finite extension, and that $G \subset \prod_{j\in J} \operatorname{Res}_{\mathbf{Q}_l}^{M_j} \mathrm{PGL}_2$ is an unramified connected algebraic subgroup over $\mathbf{Q}_l$ such that, after base change to $\overline{\mathbf{Q}}_l$, the projection of G onto each factor of $\prod_{j\in J}(\operatorname{Res}_{\mathbf{Q}_l}^{M_j} \mathrm{PGL}_2) \times_{\mathbf{Q}_l} \overline{\mathbf{Q}}_l \cong \mathrm{PGL}_2^{\coprod_j \operatorname{Hom}_{\mathbf{Q}_l}(M_j,\overline{\mathbf{Q}}_l)}$ is surjective. Then there are unramified extensions $N_i/\mathbf{Q}_l$ for i in some finite set I such that $G \cong \prod_{i\in I} \operatorname{Res}_{\mathbf{Q}_l}^{N_i} \mathrm{PGL}_2$. Moreover for each $j \in J$ and each $\tau : M_j \hookrightarrow \overline{\mathbf{Q}}_l$, the projection of the base change of G to $\overline{\mathbf{Q}}_l$ to the (j,τ) factor of $\prod_{j\in J}(\operatorname{Res}_{\mathbf{Q}_l}^{M_j} \mathrm{PGL}_2) \times_{\mathbf{Q}_l} \overline{\mathbf{Q}}_l \cong \mathrm{PGL}_2^{\coprod_j \operatorname{Hom}_{\mathbf{Q}_l}(M_j,\overline{\mathbf{Q}}_l)}$ is conjugate by an element of $\mathrm{PGL}_2(\overline{\mathbf{Q}}_l)$ to projection onto one of the factors of $\prod_{i\in I}(\operatorname{Res}_{\mathbf{Q}_l}^{N_i} \mathrm{PGL}_2) \times_{\mathbf{Q}_l} \overline{\mathbf{Q}}_l \cong \mathrm{PGL}_2^{\coprod_i \operatorname{Hom}_{\mathbf{Q}_l}(N_i,\overline{\mathbf{Q}}_l)}$.

Thus for $l \in \mathcal{L}$, there are finite unramified extensions $N_{l,i}/\mathbf{Q}_l$ for i in some finite index set I_l such that $\mathbf{G}_l^{\mathrm{ad}} \cong \prod_{i\in I_l} \operatorname{Res}_{\mathbf{Q}_l}^{N_{l,i}} \mathrm{PGL}_2$. Moreover for any prime λ of M, there are an index $i \in I_l$ and an embedding $\tau : N_{l,i} \hookrightarrow \overline{M}_\lambda$ such that the projection of $\mathbf{G}_l^{\mathrm{ad}} \times_{\mathbf{Q}_l} \overline{M}_\lambda$ to $\mathrm{PGL}_2/\overline{M}_\lambda$ is conjugate by an element of $\mathrm{PGL}_2(\overline{M}_\lambda)$ to to the projection onto the (i,τ) factor of $\mathbf{G}_l^{\mathrm{ad}} \times_{\mathbf{Q}_l} \overline{M}_\lambda \cong \mathrm{PGL}_2^{\coprod_{i\in I_l} \operatorname{Hom}_{\mathbf{Q}_l}(N_{l,i},\overline{M}_\lambda)}$.

Let Γ_l denote the image of s_l, let $\Gamma_l^\circ = \Gamma_l \cap \mathbf{G}^\circ$, and let Γ^{ad} denote the image of Γ_l^0 in $\mathbf{G}_l^{\mathrm{ad}}$. By [Lar95, Th. 3.17], after replacing $\mathcal{L}$ by a smaller set of Dirichlet density 1, we may suppose that for $l \in \mathcal{L}$, the group Γ_l^{ad} contains a conjugate of $\prod_{i\in I_l} \mathrm{SL}_2(\mathcal{O}_{N_i})/\{\pm 1_2\}$. Thus, for $l \in \mathcal{L}$ and $\lambda | l$, we may suppose that the image of $r_\lambda(G_F)$ in $\mathrm{PGL}_2(\overline{M}_\lambda)$ contains $\mathrm{SL}_2(\mathbf{Z}_l)/\{\pm 1_2\}$ and the image $\overline{r}_\lambda(G_F)$ in $\mathrm{PGL}_2(\mathcal{O}_M/\lambda)$ contains $\mathrm{SL}_2(\mathbf{F}_l)/\{\pm 1_2\}$.

Now we may suppose that $l \in \mathcal{L}$ implies that $l > 3$ so that $\mathrm{SL}_2(\mathbf{F}_l)$ is perfect. Suppose $\lambda | l \in \mathcal{L}$. For every $g \in \mathrm{SL}_2(\mathbf{F}_l)$, the image of $\overline{r}_\lambda$ contains an element $z(g)g$ where $z(g) \in (\mathcal{O}_M/\lambda)^\times$ and is well defined modulo $Z = (\mathcal{O}_M/\lambda)^\times \cap \operatorname{im} \overline{r}$. Then z defines a homomorphism $\mathrm{SL}_2(\mathbf{F}_l) \to (\mathcal{O}_M/\lambda)^\times/Z$ that must be identically 1. Thus $\mathrm{SL}_2(\mathbf{F}_l)$ is contained in the image of $\overline{r}_\lambda$.

Finally if we remove finitely many primes from $\mathcal{L}$, we may suppose that $\mathrm{PSL}_2(\mathbf{F}_l)$ is not a subquotient of $\mathrm{Gal}(\widetilde{F}/F)$ from which the last assertion follows. □

We now prove some further preliminary lemmas concerning enormous and decomposed generic representations.

Lemma 7.1.4. *If $n \geq 2$ and $l > 2n+1$ and H is a finite subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_l)$ containing $\mathrm{SL}_2(\mathbf{F}_l)$, then $\operatorname{Symm}^{n-1} H \subset \mathrm{GL}_2(\overline{\mathbf{F}}_l)$ is enormous.*

Proof. The image of H in $\mathrm{PGL}_2(\overline{\mathbf{F}}_l)$ must be conjugate to $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some finite extension $k/\mathbf{F}_l$; see, for instance, [DDT97, Th. 2.47(b)]. Thus

$$\overline{\mathbf{F}}_l^\times \operatorname{Symm}^{n-1} \mathrm{GL}_2(k) \supset H \supset \operatorname{Symm}^{n-1} \mathrm{SL}_2(k),$$

and the lemma follows from [GN22, Lem. 3.2.5]. □

LEMMA 7.1.5. *Suppose that L is a number field, that k is a finite field of characteristic l and that $\overline{r} : G_L \to GL_n(k)$ is a continuous representation. Let M denote the normal closure over $\mathbf{Q}$ of $\overline{F}^{\ker \operatorname{ad} \overline{r}}$. If M does not contain a primitive l^{th} root of unity, then $\overline{r}$ is decomposed generic.*

Proof. If a rational prime p splits completely in M, but not in $M(\zeta_l)$, then p is decomposed generic for $\overline{r}$. □

LEMMA 7.1.6. *Suppose that $F/\mathbf{Q}$ is a finite extension with normal closure $\widetilde{F}/\mathbf{Q}$ and that $m \in \mathbf{Z}_{>0}$. Suppose also that $l > 2m+3$ is a rational prime and that $\overline{r} : G_F \longrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_l)$ is a continuous representation such that $\overline{r}(G_{\widetilde{F}}) \supset \mathrm{SL}_2(\mathbf{F}_l)$. Finally suppose that F'/F is a finite extension that is linearly disjoint from $\overline{F}^{\ker \overline{r}}$ over F.*

(1) *If l is unramified in $F'/\mathbf{Q}$, then $\zeta_l \notin \overline{F}^{\ker \operatorname{ad} \operatorname{Symm}^m \overline{r}} F'$.*
(2) *$(\operatorname{Symm}^m \overline{r})(G_{F'(\zeta_l)})$ is enormous.*
(3) *Let $\widetilde{F'}$ denote the normal closure of F' over $\mathbf{Q}$. Suppose that $\operatorname{ad} \overline{r}(G_{\widetilde{F'}}) \supset \mathrm{PSL}_2(\mathbf{F}_l)$. Then $\operatorname{Symm}^m \overline{r}|_{G_{F'}}$ is decomposed generic.*
(4) *Suppose that $F'/\mathbf{Q}$ is unramified at l and that no quotient of $\operatorname{im} \operatorname{ad} \overline{r}$ is unramified at all primes above l. Then $\operatorname{Symm}^m \overline{r}|_{G_{F'}}$ is decomposed generic.*
(5) *If $l > [\widetilde{F} : F]$, then $\operatorname{Symm}^m \overline{r}$ is decomposed generic.*

Proof. The image $\overline{r}_\lambda(G_F)$ in $\mathrm{PGL}_2(\overline{\mathbf{F}}_l)$ must be conjugate to $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some finite extension $k/\mathbf{F}_l$. (See, for instance, [DDT97, Th. 2.47(b)].)

For assertion (1), it suffices to treat the case $m = 1$, in which case the assertion follows because $\mathrm{Gal}(F'(\zeta_l)/F') \cong (\mathbf{Z}/l\mathbf{Z})^\times$, while $(\operatorname{ad} \overline{r})(G_{F'}) = (\operatorname{ad} \overline{r})(G_F)$ does not surject onto $(\mathbf{Z}/l\mathbf{Z})^\times$.

For assertion (2), note that $\overline{r}(G_{F'}) = \overline{r}(G_F) \supset \mathrm{SL}_2(\mathbf{F}_l)$ and so, because $\mathrm{SL}_2(\mathbf{F}_l)$ is perfect, we have that $\overline{r}(G_{F'(\zeta_l)}) \supset \mathrm{SL}_2(\mathbf{F}_l)$. The assertion now follows from Lemma 7.1.4.

For assertion (3), it suffices to prove that $\operatorname{Symm}^m \overline{r}|_{G_{F'}}$ is decomposed generic after replacing F' with some finite extension. We first replace F' by $\widetilde{F'}(\zeta_l)$, which we can do as $\mathrm{PSL}_2(\mathbf{F}_l)$ is perfect. Then, as above, (up to conjugacy) the image of $(\operatorname{ad} \overline{r})(G_{F'})$ is $\mathrm{PSL}_2(k)$ or $\mathrm{PGL}_2(k)$ for some finite extension $k/\mathbf{F}_l$. Perhaps making a further extension, we may assume that $\operatorname{ad} \overline{r}(G_{F'}) = \mathrm{PSL}_2(k)$ for some finite extension $k/\mathbf{F}_l$, while maintaining the fact that $F'/\mathbf{Q}$ is Galois. Let H/F' denote the finite Galois extension with Galois group $\mathrm{PSL}_2(k)$ cut out by this projective representation, and let H' denote its normal closure over $\mathbf{Q}$. Using the simplicity of $\mathrm{PSL}_2(k)$, we deduce, from Goursat's Lemma, that $\mathrm{Gal}(H'/F') = \mathrm{PSL}_2(k)^n$ for some n. Moreover the conjugation action of any $\sigma \in \mathrm{Gal}(H'/\mathbf{Q})$ on $\mathrm{Gal}(H'/F') \cong \mathrm{PSL}_2(k)^n$ is via an element of $\mathrm{Aut}(\mathrm{PSL}_2(k)^n) \cong (\mathrm{PGL}_2(k) \rtimes \mathrm{Gal}(k/\mathbf{F}_l))^n \rtimes S_n$. (To see this note

two things. Firstly $\mathrm{PSL}_2(k)$ has automorphism group $\mathrm{PGL}_2(k) \rtimes \mathrm{Gal}(k/\mathbf{F}_l)$; see, for instance, [Die51]. Secondly the only normal subgroups of $\mathrm{PSL}_2(k)^n$ are $\mathrm{PSL}_2(k)^I$ for $I \subset \{1, \dots, n\}$, as can be seen by induction on n, and so any automorphism of $\mathrm{PSL}_2(k)^n$ permutes the n factors of this product.)

There exists an element $A \in \mathrm{PSL}_2(\mathbf{F}_l) \subset \mathrm{PSL}_2(k)$ such that a preimage in $\mathrm{SL}_2(\mathbf{F}_l)$ has two distinct $\mathbf{F}_l$-rational eigenvalues with ratio α satisfying $\alpha^{\pm i} \neq 1$ for $1 \leq i \leq m$. By the Cebotarev density theorem, there exists a rational prime p such that $\langle \mathrm{Frob}_p \rangle$ in $\mathrm{Gal}(H'/\mathbf{Q})$ is (the conjugacy class of) the element $(A, \dots, A)$ in $\mathrm{PSL}_2(k)^n = \mathrm{Gal}(H'/F')$. The image of this element is trivial in the quotient $\mathrm{Gal}(F'/\mathbf{Q})$, and thus, in addition, we see that p splits completely in F' and (hence) that $p \equiv 1 \mod l$. By construction, the ratio of any two roots of the characteristic polynomial of Frobenius of *any* prime above p in $\mathrm{Symm}^m \overline{r}$ is given by $\alpha^{\pm i}$ for $i = 1, \dots, m$. In particular, these ratios are not equal to $p \equiv 1 \mod l$. Hence $\mathrm{Symm}^m \overline{r}|_{G_{F'}}$ is decomposed generic.

Assertion (4) follows from assertion (3), because $\widetilde{F'}$ is unramified above l so that $(\widetilde{F'} \cap \overline{F}^{\ker \mathrm{ad}\, \overline{r}})/F$ is unramified above l and hence $\widetilde{F'} \cap \overline{F}^{\ker \mathrm{ad}\, \overline{r}} = F$ and $\widetilde{F'}$ is linearly disjoint from $\overline{F}^{\ker \mathrm{ad}\, \overline{r}}$ over F.

For assertion (5), note that $[\widetilde{F} \cap \overline{F}^{\ker \mathrm{ad}\, \overline{r}} : \overline{F}^{\ker \mathrm{ad}\, \overline{r}}] < l$, so that we have $(\mathrm{ad}\, \overline{r})(G_{\widetilde{F} \cap \overline{F}^{\ker \mathrm{ad}\, \overline{r}}}) \supset \mathrm{PSL}_2(\mathbf{F}_l)$. However, being Galois extensions of F, the fields $\widetilde{F}$ and $\overline{F}^{\ker \mathrm{ad}\, \overline{r}}$ are linearly disjoint over $\widetilde{F} \cap \overline{F}^{\ker \mathrm{ad}\, \overline{r}}$, so that again the result follows from assertion (3). □

Lemma 7.1.7. *Suppose that $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_l)$ is decomposed generic and absolutely irreducible. Let $E/\mathbf{Q}$ be a Galois extension that is linearly disjoint from the Galois closure of $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over $\mathbf{Q}$. Then $\overline{r}|_{G_{FE}}$ is decomposed generic and absolutely irreducible.*

Proof. The irreducibility claim is clear. Write H for the Galois closure of $\overline{F}^{\ker \overline{r}}(\zeta_l)$ over $\mathbf{Q}$. As in the proof of Lemma 4.3.2, there exists a conjugacy class of elements $\sigma \in \mathrm{Gal}(H/\mathbf{Q})$ such that any rational prime unramified in H whose Frobenius element corresponds to σ is decomposed generic for $\overline{r}$. By assumption, $\mathrm{Gal}(HE/\mathbf{Q}) = \mathrm{Gal}(H/\mathbf{Q}) \times \mathrm{Gal}(E/\mathbf{Q})$, and now any rational prime whose conjugacy class in $\mathrm{Gal}(HE/\mathbf{Q})$ is of the form $(\sigma, 1) \in \mathrm{Gal}(H/\mathbf{Q}) \times \mathrm{Gal}(E/\mathbf{Q})$ will be decomposed generic for $\overline{r}|_{G_{FE}}$. □

Lemma 7.1.8.

(1) *Suppose that $K/\mathbf{Q}_l$ is an unramified extension and that $r : G_K \to \mathrm{GL}_2(\mathbf{Z}_l)$ is a crystalline representation with Hodge–Tate numbers $\{0, 1\}$ for each embedding $K \hookrightarrow \overline{\mathbf{Q}}_l$. Either $\overline{r}|_{I_l}^{\mathrm{ss}} \cong 1 \oplus \overline{\epsilon}_l^{-1}$ or $\overline{r}|_{I_l} \cong \omega_{l,2}^{-1} \oplus \omega_{l,2}^{-l}$.*

(2) *Suppose that K is a number field in which l is unramified and that $r : G_K \to \mathrm{GL}_2(\mathbf{Z}_l)$ is a crystalline representation with Hodge–Tate numbers*

$\{0,1\}$ for each embedding $K \hookrightarrow \overline{\mathbf{Q}}_l$. If the image of $\overline{r}$ contains $\mathrm{SL}_2(\mathbf{F}_l)$, then the only subextension of $\overline{K}^{\ker \overline{r}}/K$ unramified at all primes above l is K itself.

Proof. The first part is presumably well known, but for lack of a reference we give a proof. (Note the slight subtlety that the result would be false if we replaced the coefficients $\mathbf{Z}_l$ with the ring of integers in an arbitrary extension of $\mathbf{Q}_l$, which is one obstacle to finding a suitable reference.) Recall that $r^\vee$ arises from the Tate module of a height 2 l-divisible group G over $\mathcal{O}_K$ (see [Bre00] and [Kis06].) Moreover $G \neq G^0 \neq (0)$, as otherwise we would have Hodge–Tate numbers $\{1,1\}$ or $\{0,0\}$. Thus there is a finite flat group scheme H over the ring of integers of the completion of the maximal unramified extension of K of order l^2 killed by l giving rise to $\overline{r}^\vee$. Moreover $H \neq H^0 \neq (0)$. By [Ray74, Prop. 3.2.1, Th. 3.4.3], either $\overline{r}|_{I_l}^{\mathrm{ss}} \cong 1 \oplus \overline{\epsilon}_l^{-1}$ or $\overline{r}|_{I_l} \cong \omega_{l,2}^{-1} \oplus \omega_{l,2}^{-l}$ or $\overline{r}|_{I_l}^{\mathrm{ss}} \cong 1 \oplus 1$ or $\overline{r}|_{I_l}^{\mathrm{ss}} \cong \overline{\epsilon}_l^{-1} \oplus \overline{\epsilon}_l^{-1}$. If $l = 2$, then $1 = \overline{\epsilon}_l^{-1}$ and we are done, so suppose that $l > 2$. Then since $\det r$ is a crystalline character with all Hodge–Tate weights equal to 1, we have $\det r|_{I_l} = \epsilon_l^{-1}$, so the last two possibilities cannot occur, and the first part follows.

Consider now the second part. It follows from the first part that the image under $\det \overline{r}$ of any inertia group above l is $\mathbf{F}_l^\times$, and so $\operatorname{im} \overline{r} = \mathrm{GL}_2(\mathbf{F}_l)$. Let Δ denote the subgroup of $\mathrm{GL}_2(\mathbf{F}_l)$ generated by the images of all inertia groups above l. It is a normal subgroup of $\mathrm{GL}_2(\mathbf{F}_l)$ that surjects under the determinant map onto $\mathbf{F}_l^\times$. But any normal subgroup of $\mathrm{GL}_2(\mathbf{F}_l)$ either contains $\mathrm{SL}_2(\mathbf{F}_l)$ or is central, and so $\Delta = \mathrm{GL}_2(\mathbf{F}_l)$ and the second part follows. □

A very (or extremely) weakly compatible system $\mathcal{R}$ is defined to be *pure* of weight w if

- for each $v \notin S$, each root α of $Q_v(X)$ in $\overline{M}$ and each $\imath : \overline{M} \hookrightarrow \mathbf{C}$, we have
$$|\imath\alpha|^2 = q_v^w;$$
- and for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}(\overline{M}/\mathbf{Q})$, we have
$$H_{c\tau} = \{w - h : \ h \in H_\tau\}.$$

If $\mathcal{R}$ is rank one, then it is automatically pure; see [Ser98]. The same is true if $\mathcal{R}$ is induced from an extremely weakly compatible system of characters over a finite extension of F, or if $\mathcal{R}$ is Artin up to twist.

If $\mathcal{R}$ is pure of weight w and if $\imath : M \hookrightarrow \mathbf{C}$, then the partial L-function $L^S(\imath\mathcal{R}, s)$ is defined as an analytic function in $\Re s > 1 + w/2$. If $\mathcal{R}$ is pure and regular and if v is an infinite place of F, then the Euler factor $L_v(\imath\mathcal{R}, s)$ can be defined (see [BLGGT14, §5.1]).

The very (or extremely) weakly compatible system $\mathcal{R}$ is defined to be *automorphic* if there are a regular algebraic, cuspidal automorphic representation

π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an embedding $\imath : M \hookrightarrow \mathbf{C}$, such that if $v \notin S$, then π_v is unramified and $\mathrm{rec}(\pi_v |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath(Q_v(X))$. Note that if $\mathcal{R}$ is automorphic, then $L^S(\imath\mathcal{R}, s)$ defines an analytic function in $\Re s \gg 0$ that, for $n > 1$, has analytic continuation to the whole complex plane. It follows from [Clo90, Th. 3.13] that if $\mathcal{R}$ is automorphic, then for any embedding $\imath' : M \hookrightarrow \mathbf{C}$, there is a regular algebraic, cuspidal automorphic representation $\pi_{\imath'}$ of $\mathrm{GL}_n(\mathbf{A}_F)$ such that if $v \notin S$, then $\pi_{\imath',v}$ is unramified and $\mathrm{rec}(\pi_{\imath',v} |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\imath'(Q_v(X))$.

Suppose that F is a CM field and π is a regular algebraic cuspidal automorphic representation on $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $(a_{\tau,i})$. From the main theorems of [HLTT16] and [Var14] we may associate to π an extremely weakly compatible system

$$\mathcal{R}_\pi = (M_\pi, S_\pi, \{Q_{\pi,v}(X)\}, \{r_{\pi,\lambda}\}, \{H_{\pi,\tau}\}),$$

where

- $M_\pi \subset \mathbf{C}$ is the fixed field of $\{\sigma \in \mathrm{Aut}(\mathbf{C}) : {}^\sigma\pi^\infty \cong \pi^\infty\}$;
- S_π is the set of primes of F with π_v ramified;
- $Q_{\pi,v}(X)$ is the characteristic polynomial of $\mathrm{rec}(\pi_v |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$;
- $H_{\pi,\tau} = \{a_{\tau,1} + n - 1, \ldots, a_{\tau,n}\}$.

We now note that this can be upgraded to a very weakly compatible system under some hypotheses.

Lemma 7.1.9. *Let F be a CM field, and let π be a regular algebraic cuspidal automorphic representation on $\mathrm{GL}_n(\mathbf{A}_F)$ of weight $\xi = (a_{\tau,i})$. Suppose that the following hypothesis holds*:

(DGI) (*decomposed generic and absolutely irreducible*). *For a set of primes l of Dirichlet density one, the representations*

$$\overline{r}_{\pi,\lambda} : G_F \to \mathrm{GL}_n(\mathcal{O}_M/\lambda)$$

are decomposed generic and absolutely irreducible for all $\lambda \mid l$.

Then $\mathcal{R}_\pi$ is a very weakly compatible system.

Proof. The lemma follows from Theorem 4.5.1 (taking p there to be our l). Indeed, the assumption that $\mathfrak{m}$ is not Eisenstein is implied (for a set l of density one) by hypothesis (DGI). Conditions (3), (4), and (5) hold automatically for large enough l. Similarly, l will be unramified in F for large enough l. Conditions (1), (2), and (6), can always be satisfied after after making a solvable Galois base change F'/F (using [AC89]) that is disjoint over F from the Galois closure of $\overline{F}^{\ker \overline{r}}$ over $\mathbf{Q}$ and in which all primes dividing either S or l are unramified. (We are free to make a different such base change for each prime l.) In particular, one can take the compositum of F with a Galois extension $E/\mathbf{Q}$ that is the compositum of various imaginary quadratic fields in which all primes

dividing S or l split completely for (1), (2), and the compositum with a large totally real cyclic extension $E/\mathbf{Q}$ in which l splits completely for condition (6), where E may be easily be chosen to be linearly disjoint over $\mathbf{Q}$ from $\overline{F}^{\ker \overline{r}_{\pi,\lambda}}(\zeta_l)$. By Lemma 7.1.7, hypothesis (DGI) is preserved under such base extensions. Condition (8a) holds by the existence of π, and finally, condition (7) holds for l for a set of l of density one, by hypothesis (DGI). □

LEMMA 7.1.10. *Let F be a CM field, and let π be a regular algebraic cuspidal automorphic representation on $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $\xi = (a_{\tau,i})$. Then the extremely weakly compatible system $\mathcal{R}_\pi$ is irreducible. Moreover hypothesis* (DGI) *of Lemma 7.1.9 holds and $\mathcal{R}_\pi$ is a very weakly compatible system.*

Proof. If $\mathcal{R}_\pi$ were reducible, then by Lemma 7.1.1 and the automorphy of all weakly compatible systems of rank 1, we see that there would be grossencharacters χ_1 and χ_2 of $\mathbf{A}_F^\times/F^\times$ such that $\pi_v \cong \chi_{1,v} \boxplus \chi_{2,v}$ for all but finitely many v. By [JS81a], this would contradict the cuspidality of π. Thus $\mathcal{R}_\pi$ is irreducible.

By Lemma 7.1.9 it only remains to verify hypothesis (DGI). The absolute irreducibility condition follows from Lemma 7.1.3. For the decomposed generic condition, we treat the three possibilities of Lemma 7.1.2 separately.

Suppose first that $\mathcal{R}_\pi$ is strongly irreducible. By Lemma 7.1.3 and part (5) of Lemma 7.1.6, we deduce that hypothesis (DGI) holds and so $\mathcal{R}_\pi$ is very weakly compatible.

Suppose second that $\mathcal{R}_\pi \cong \mathrm{Ind}_{G_E}^{G_F} \mathcal{X}$ for some quadratic extension E/F and some very weakly compatible system of characters $\mathcal{X}$ of G_E. Let $\widetilde{F}$ (resp. $\widetilde{E}$) denote the normal closure of F (resp. E) over $\mathbf{Q}$, so that $\mathrm{Gal}(\widetilde{E}/\widetilde{F})$ is an elementary abelian 2-group. Let $1 \neq \tau \in \mathrm{Gal}(E/F)$. Then $\mathrm{Gal}(E/F)$ acts on $\mathrm{Gal}(\overline{E}^{\ker \chi_\lambda/\chi_\lambda^\tau}/E)$ via the non-trivial character $\mathrm{Gal}(E/F) \to \{\pm 1\}$. If $L^{(\lambda)}$ denotes the normal closure of $\overline{E}^{\ker \overline{\chi}_\lambda/\overline{\chi}_\lambda^\tau}$ over $\mathbf{Q}$, then $L^{(\lambda)}/\widetilde{E}$ is the compositum of abelian Galois extensions on which some subgroup of $\mathrm{Gal}(\widetilde{E}/\mathbf{Q})$ acts by a non-trivial character of order 2.

Suppose that a rational prime l is unramified in $\widetilde{E}$. Then $\overline{\epsilon}_l(\mathrm{Gal}(L^{(\lambda)}/\widetilde{E}))$ can have order at most 2 (as any subgroup of $\mathrm{Gal}(\widetilde{E}/\mathbf{Q})$ will act trivially on it). Thus, if $l > 3$, then $\zeta_l \notin L^{(\lambda)}$ for any λ. It follows from Lemma 7.1.5 that if λ lies above a rational prime $l > 3$ that is unramified in $\widetilde{E}$, then $\overline{r}_{\pi,\lambda}$ is decomposed generic.

Finally suppose that $\mathcal{R}_\pi$ is Artin up to twist; i.e., there exists an irreducible Artin representation $\rho : G_F \to GL_2(\overline{M}_\pi)$ such that for all λ, the representation $r_{\pi,\lambda}$ is the twist of ρ by some character. In particular, $\overline{F}^{\ker \mathrm{ad}\, \overline{r}_{\pi,\lambda}} \subset \overline{F}^{\ker \mathrm{ad}\, \rho}$. Let L denote the normal closure of $\overline{F}^{\ker \mathrm{ad}\, \rho}$ over $\mathbf{Q}$. If $l > 2$ is unramified in L, then $\zeta_l \notin L$, and by Lemma 7.1.5 we see that $\overline{r}_{\pi,\lambda}$ is decomposed generic for all $\lambda | l$. □

If $\imath_0$ is the canonical embedding $M_\pi \hookrightarrow \mathbf{C}$, then $L^S(\imath_0\mathcal{R}_\pi, s) = L^S(\pi, s)$. If moreover $\mathcal{R}_\pi$ is pure, and hypothesis (DGI) of Lemma 7.1.9 holds, then for each infinite place v of F we have $L_v(\imath_0\mathcal{R}_\pi, s) = L_v(\pi, s)$. (This follows from the definition of $L_v(\imath_0\mathcal{R}_\pi, s)$ in [BLGGT14, §5.1] together with the determination of the Hodge–Tate weights of $\mathcal{R}_\pi$ in Lemma 7.1.9 and, in the case that F is totally real, the main result of [CLH16].)

The following is our main theorem.

Theorem 7.1.11. *Suppose that F/F_0 is a finite Galois extension of CM fields. Suppose also that F_0^{avoid} is a finite Galois extension of F and that $\mathcal{L}_0$ is a finite set of rational primes. Suppose, moreover, that $\mathcal{I}$ is a finite set and that for $i \in \mathcal{I}$, we are given $m_i \in \mathbf{Z}_{>0}$ and a strongly irreducible rank 2 very weakly compatible system of l-adic representations of G_F*

$$\mathcal{R}_i = (M_i, S_i, \{Q_{i,v}(X)\}, \{r_{i,\lambda}\}, \{\{0,1\}\})$$

with S_i disjoint from $\mathcal{L}_0$.

Then there are

- *a finite set $\mathcal{L} \supset \mathcal{L}_0$ of rational primes;*
- *a finite CM Galois extension F^{suffices}/F unramified above $\mathcal{L}$, such that F^{suffices} is Galois over F_0; and*
- *a finite Galois extension F^{avoid}/F containing F_0^{avoid}, which is linearly disjoint from F^{suffices} over F*

with the following property: For any finite CM extension F'/F containing F^{suffices} that is unramified above $\mathcal{L}$ and linearly disjoint from F^{avoid} over F, the representations $\mathrm{Symm}^{m_i}\mathcal{R}_i|_{G_{F'}}$ are all automorphic, and each arises from an automorphic representation unramified above $\mathcal{L}_0$.

We have phrased this in a rather technical way in the hope that it will be helpful for applications. However let us record a simpler immediate consequence.

Corollary 7.1.12. *Suppose that F is a CM field and that the 5-tuple $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{\{0,1\}\})$ is a strongly irreducible rank 2 very weakly compatible system of l-adic representations of G_F. If m is a non-negative integer, then there exists a finite Galois CM extension F'/F such that the weakly compatible system $\mathrm{Symm}^m\mathcal{R}|_{G_{F'}}$ is automorphic.*

Before proving Theorem 7.1.11 in the next section, we record some consequences.

Corollary 7.1.13 (Potential modularity and purity for rank two compatible systems over CM fields of weight zero and their symmetric powers). *Suppose that F is a CM field and that the 5-tuple $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ is an irreducible rank 2 very weakly compatible system of l-adic representations of*

G_F such that $H_\tau = \{0,1\}$ for all τ. Suppose further that m is a non-negative integer. Then

(1) *$\mathcal{R}$ is pure of weight* 1;
(2) *the partial L-functions $L^S(\imath \operatorname{Symm}^m \mathcal{R}, s)$ have meromorphic continuation to the entire complex plane*;
(3) *for $v \in S$, there are Euler factors $L_v(\imath \operatorname{Symm}^m \mathcal{R}, s) = P_{m,\imath,v}(q_v^{-s})^{-1}$, where $P_{m,\imath,v}$ is a polynomial of degree at most $m+1$ and q_v is the order of the residue field of v, such that*

$$\Lambda(\imath \operatorname{Symm}^m \mathcal{R}, s) = L^S(\imath \operatorname{Symm}^m \mathcal{R}, s) \prod_{v|\infty} L_v(\imath \operatorname{Symm}^m \mathcal{R}, s) \prod_{v \in S} L_v(\imath \operatorname{Symm}^m \mathcal{R}, s)$$

satisfies a functional equation of the form

$$\Lambda(\imath \operatorname{Symm}^m \mathcal{R}, s) = AB^s \Lambda(\imath \operatorname{Symm}^m \mathcal{R}^\vee, 1-s).$$

Suppose further that $\mathcal{R}$ is strongly irreducible and that $m > 0$. Then $L^s(\imath \operatorname{Symm}^m \mathcal{R}, s)$ is holomorphic and non-vanishing for $\operatorname{Re}(s) \geq m/2+1$ and, in particular, has neither a pole nor a zero at $s = m/2+1$.

Proof. If $\mathcal{R}$ is not strongly irreducible, then by Lemma 7.1.2 there are a quadratic extension F'/F and a weakly compatible system $\mathcal{X}$ of characters of $G_{F'}$ such that $\mathcal{R} = \operatorname{Ind}_{G_{F'}}^{G_F} \mathcal{X}$. In this case $\mathcal{X}$ is pure, necessarily of weight 1, and automorphic. The corollary follows easily.

So suppose that $\mathcal{R}$ is strongly irreducible. Then for any positive integer m, we see from Theorem 7.1.11 that there are a finite Galois CM extension F_m/F and, for any embedding $\imath : \overline{M} \hookrightarrow \mathbf{C}$, a cuspidal automorphic representation $\pi_{\imath,m}$ of $\mathrm{GL}_{m+1}(\mathbf{A}_{F_m})$, such that for each $w|v \notin S$, the roots of the characteristic polynomial of $\operatorname{rec}(\pi_{\imath,m,w}|\det|_w^{-m/2})(\operatorname{Frob}_w)$ are the images under $\imath$ of the roots of $Q_{\operatorname{Symm}^m \mathcal{R}|_{G_{F_m}},w}(X)$.

For the first part of the corollary, we combine the "Deligne–Langlands method" with our theorem: because $\det \mathcal{R}$ is pure of weight 2, it suffices to show that for every $v \notin S$, for every root α of $Q_{\mathcal{R},v}(X)$, and every $\imath : \overline{M} \hookrightarrow \mathbf{C}$, we have

$$|\imath\alpha| \leq q_v^{1/2}.$$

It even suffices to show that for every $m > 0$, for every $v \notin S$, for every root β of $Q_{\operatorname{Symm}^m \mathcal{R},v}(X)$ and every $\imath : \overline{M} \hookrightarrow \mathbf{C}$, we have

$$|\imath\beta| \leq q_v^{(m+1)/2}.$$

(For then $|\imath\alpha| \leq q_v^{1/2+1/(2m)}$.) Equivalently, it suffices to show that for every $m > 0$, for every $w|v \notin S$, for every root γ of $Q_{\operatorname{Symm}^m \mathcal{R}|_{G_{F_m}},w}(X)$, and every $\imath : \overline{M} \hookrightarrow \mathbf{C}$, we have

$$|\imath\gamma| \leq q_w^{(m+1)/2}.$$

If $\chi_{\pi_{\imath,m}}$ denotes the central character of $\pi_{\imath,m}$, we see that $\det \operatorname{Symm}^m \mathcal{R}|_{G_{F_m}}$ is equivalent to $\mathcal{R}_{\chi_{\pi_{\imath,m}}|\det||^{-m/2}}$ and so

$$|\chi_{\pi_{\imath,m}}(x)| = 1$$

for all $x \in \mathbf{A}_{F_m}^\times$. Thus $\pi_{\imath,m}$ is unitary and, applying the bound of [JS81b, Cor. 2.5] (which applies since each local factor of $\pi_{\imath,m}$ is generic, by the final corollary of [Sha74]), we see that the image under $\imath$ of all the roots of the characteristic polynomial of $\operatorname{rec}(\pi_{\imath,m,w})(\operatorname{Frob}_w)$ have absolute value $\leq q_w^{1/2}$. Thus the absolute value of the image under $\imath$ of any root of $Q_{\operatorname{Symm}^m \mathcal{R}|_{G_{F_m}},w}(X)$ is $\leq q_w^{(m+1)/2}$. The first part of the corollary follows.

The rest of the corollary follows on using the usual Brauer's theorem argument (together with known non-vanishing properties of automorphic L-functions as in [JS77]) as in [HSBT10, Th. 4.2]. □

Corollary 7.1.14 (Sato–Tate for Elliptic curves over CM fields). *Suppose that F is a CM field and that E/F is a non-CM elliptic curve. Then the numbers*

$$(1 + \#k(v) - \#E(k(v)))/2\sqrt{\#k(v)}$$

are equidistributed in $[-1, 1]$ with respect to the measure $(2/\pi)\sqrt{1-t^2}\,dt$.

Proof. This follows from Corollary 7.1.13 and the corollary to [Ser98, Th. 2], as explained on page I-26 of [Ser98]. □

Corollary 7.1.15 (Ramanujan conjecture for weight 0 automorphic representations for GL(2) over CM fields). *Suppose that F is a CM field and that π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of weight $(0)_{\tau,i}$. Then, for all primes v of F, the representation π_v is tempered.*

Proof. The result is immediate for all primes v such that π_v is a twist of the Steinberg representation. At the remaining places, since π_v is not a twist of the Steinberg representation, it follows from the main theorems of [HLTT16] and [Var14], together with [TY07, Lem. 1.4 (3)], that it suffices to prove that if $v \nmid l$, then the restriction to G_{F_v} of any of the l-adic Galois representations associated to π is pure in the sense of [TY07, §1]. By [TY07, Lem. 1.4 (2)] and solvable base change, we can reduce to the case that π_v is unramified, in which case the result follows from Corollary 7.1.13(1), after noting by Lemma 7.1.10 that the compatible system $\mathcal{R}$ associated to π is very weakly compatible of the expected Hodge–Tate weights. □

Corollary 7.1.16. *Suppose that F is a CM field and that the 5-tuples $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{\{0,1\}\})$ and $\mathcal{R}' = (M', S', \{Q'_v(X)\}, \{r'_\lambda\}, \{\{0,1\}\})$ are a pair of strongly irreducible rank 2 very weakly compatible systems of l-adic representations of G_F. Suppose further that m and m' are non-negative integers, and that $\mathcal{R}$ and $\mathcal{R}'$ are not twists of each other. Then $L^S(\imath \operatorname{Symm}^m \mathcal{R} \otimes$*

$\mathrm{Symm}^{m'}\mathcal{R}', s)$ *is meromorphic for* $s \in \mathbf{C}$, *has no zeroes or poles for* $\mathrm{Re}(s) \geq 1 + m/2 + m'/2$, *and satisfies a functional equation relating* $L^S(\imath\,\mathrm{Symm}^m\mathcal{R} \otimes \mathrm{Symm}^{m'}\mathcal{R}', s)$ *and* $L^S(\imath\,\mathrm{Symm}^m\mathcal{R}^\vee \otimes \mathrm{Symm}^{m'}(\mathcal{R}')^\vee, 1 + m + m' - s)$.

Proof. This follows from Theorem 7.1.11 by the same argument as [Har09] (for example, Theorem 5.3 of ibid.). (As usual, this argument involves the known non-vanishing results of Rankin–Selberg convolutions as established in Theorem 5.2 of [Sha81]). □

7.2. *Proof of the main potential automorphy theorem.*

7.2.1. *Preliminaries.* Before turning to the proof of Theorem 7.1.11, we record some preliminaries.

If $L/\mathbf{Q}_l$ is a finite extension and $\overline{\chi}$ (resp. χ) is an unramified character of G_L valued in $\mathbf{F}_l^\times$ (resp. $\mathbf{Z}_l^\times$), we will write $H^1_f(G_L, \mathbf{F}_l(\epsilon_l\overline{\chi}))$ (resp. $H^1_f(G_L, \mathbf{Z}_l(\epsilon_l\chi)))$ for the kernel of the composite

$$H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l\overline{\chi})) \longrightarrow H^1(G_{L^{\mathrm{nr}}}, \mathbf{F}_l(\overline{\epsilon}_l)) \cong L^{\mathrm{nr},\times}/(L^{\mathrm{nr},\times})^l \longrightarrow \mathbf{Z}/l\mathbf{Z}$$

(resp.

$$H^1(G_L, \mathbf{Z}_l(\epsilon_l\chi)) \longrightarrow H^1(G_{L^{\mathrm{nr}}}, \mathbf{Z}_l(\epsilon_l)) \cong \lim_{\leftarrow r} L^{\mathrm{nr},\times}/(L^{\mathrm{nr},\times})^{l^r} \longrightarrow \mathbf{Z}_l),$$

where the latter maps are induced by the valuation map. Note that if $\overline{\chi}$ (resp. χ) is non-trivial, then

$$H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l\overline{\chi})) = H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l\overline{\chi}))$$

(resp.

$$H^1_f(G_L, \mathbf{Z}_l(\epsilon_l\chi)) = H^1(G_L, \mathbf{Z}_l(\epsilon_l\chi))).$$

Also note that

$$H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l))/H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l)) \cong \mathbf{F}_l.$$

Lemma 7.2.2. *The map*

$$H^1_f(G_L, \mathbf{Z}_l(\epsilon_l\chi)) \longrightarrow H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l\overline{\chi}))$$

is always surjective.

Proof. We will consider three cases. If the reduction of χ is non-trivial, we may suppress the f and the cokernel is simply $H^2(G_L, \mathbf{Z}_l(\epsilon_l\chi))[l]$. Because $H^0(G_L, \mathbf{Q}_l/\mathbf{Z}_l(\chi^{-1})) = (0)$, Tate duality shows that this cokernel is zero.

Suppose now that χ is non-trivial but that $\overline{\chi}$ is trivial. Using duality as above, we have an exact sequence

$$H^1(G_L, \mathbf{Z}_l(\epsilon_l\chi)) \longrightarrow H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l)) \longrightarrow \mathbf{F}_l \longrightarrow (0).$$

The image of $H^1(G_L, \mathbf{Z}_l(\epsilon_l\chi)) = H^1_f(G_L, \mathbf{Z}_l(\epsilon_l\chi))$ in $H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l))$ is contained in $H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l))$. As $H^1(G_L, \mathbf{F}_l(\overline{\epsilon}_l))/H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l)) \cong \mathbf{F}_l$, we conclude that this image equals $H^1_f(G_L, \mathbf{F}_l(\overline{\epsilon}_l))$, as desired.

Suppose finally that $\chi = 1$. In this case the assertion of the lemma is just the surjectivity of

$$\lim_{\leftarrow r} \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^{l^r} \twoheadrightarrow \mathcal{O}_L^\times/(\mathcal{O}_L^\times)^l. \qquad \square$$

We will need a slight strengthening of [BLGGT14, Th. 3.1.2], which we now state. We will use the notation and definitions from [BLGGT14]. The proof of this theorem given in [BLGGT14] immediately proves this variant also.

Proposition 7.2.3. *Suppose that*

- *F/F_0 is a finite, Galois extension of totally real fields;*
- *$\mathcal{I}$ is a finite set;*
- *for each $i \in \mathcal{I}$, n_i is a positive even integer, l_i is an odd rational prime, and $\imath_i : \overline{\mathbf{Q}}_{l_i} \xrightarrow{\sim} \mathbf{C}$;*
- *F^{avoid}/F is a finite Galois extension;*
- *$\mathcal{L}$ is a finite set of rational primes that are unramified in F and not equal to l_i for any $i \in \mathcal{I}$; and*
- *$\overline{r}_i : G_F \to \mathrm{GSp}_{n_i}(\overline{\mathbf{F}}_{l_i})$ is a mod l_i Galois representation with open kernel and multiplier $\overline{\epsilon}_{l_i}^{1-n_i}$, which is unramified above $\mathcal{L}$.*

Then we can find finite Galois extensions $F^{\mathrm{suffices}}/F_0$ and $F_1^{\mathrm{avoid}}/\mathbf{Q}$, such that

- *F^{suffices} contains F and is linearly disjoint from $F^{\mathrm{avoid}}F_1^{\mathrm{avoid}}$ over F;*
- *F_1^{avoid} and F^{avoid} are linearly disjoint over $\mathbf{Q}$; and*
- *F^{suffices} is totally real and unramified above $\mathcal{L}$*

and that has the following property: For each finite totally real extension $F_1/F^{\mathrm{suffices}}$ which is linearly disjoint from $F^{\mathrm{avoid}}F_1^{\mathrm{avoid}}$ over F and for each $i \in \mathcal{I}$, there is a regular algebraic, cuspidal, polarized automorphic representation (π_i, χ_i) of $\mathrm{GL}_{n_i}(\mathbf{A}_{F_1})$ such that

(1) $(\overline{r}_{l_i,\imath_i}(\pi_i), \overline{r}_{l_i,\imath_i}(\chi_i)\overline{\epsilon}_{l_i}^{1-n_i}) \cong (\overline{r}_i|_{G_{F_1}}, \overline{\epsilon}_{l_i}^{1-n_i})$;
(2) *π_i is $\imath_i$-ordinary of weight* 0.

(In the notation of the proof of [BLGGT14, Th. 3.1.2] one must choose N not divisible by any prime in $\mathcal{L}$; $M_i/\mathbf{Q}$ unramified at primes in $\mathcal{L}$ and primes dividing N; q unramified in $F^{\mathrm{avoid}}(\zeta_{4N})$ and not in $\mathcal{L}$; ϕ_i unramified above $\mathcal{L}$ and all rational primes that ramify in F^{avoid}; and $l' \notin \mathcal{L}$ and not ramified in F^{avoid}. We set $F_1^{\mathrm{avoid}} = \overline{\mathbf{Q}}^{\ker \prod_i \overline{r}'_i}(\zeta_{l'})$. It is linearly disjoint from F^{avoid} over $\mathbf{Q}$ because no rational prime ramifies in both these fields. We choose $F'/F(\zeta_N)^+$ to be linearly disjoint from $F^{\mathrm{avoid}}F_1^{\mathrm{avoid}}F(\zeta_N)^+$ over $F(\zeta_N)^+$ with $F'/F(\zeta_N)^+$ unramified above $\mathcal{L}$. The last choice is possible because $\overline{r}_i$ and $\overline{r}'_i$ are unramified above $\mathcal{L}$,

so that $\overline{r}_i$ becomes isomorphic to $V_{n_i}[\lambda_i]((N-1-n_i)/2)_0$ and $\overline{r}'_i$ becomes isomorphic to $V_{n_i}[\lambda']((N-1-n_i)/2)_0$ over some unramified extension of $F(\zeta_N)^+_v$ for any prime v above $\mathcal{L}$. We take F^{suffices} to be the field F'. The fields F^{suffices} and $F^{\text{avoid}}F_1^{\text{avoid}}$ are linearly disjoint over F because $F^{\text{avoid}}F_1^{\text{avoid}}$ and $F(\zeta_N)^+$ are linearly disjoint over F, because, in turn, all primes dividing N are unramified in $F^{\text{avoid}}F_1^{\text{avoid}}$. The point $P \in \widetilde{T}(F')$ also provides a point of $\widetilde{T}(F_1)$. Moreover $\overline{r}'_i(G_{F_1(\zeta_{l'})})$ is adequate because F_1 is linearly disjoint from $\overline{F}^{\ker \overline{r}_i}(\zeta_{l'})$ over F.)

Corollary 7.2.4. *Suppose that $\mathcal{M}$ is a finite set of positive integers, that $E/\mathbf{Q}$ is a non-CM elliptic curve, and that $\mathcal{L}$ is a finite set of rational primes at which E has good reduction. Suppose also that $F^{\text{avoid}}/\mathbf{Q}$ is a finite extension.*

Then we can find

- *a finite Galois extension $F_2^{\text{avoid}}/\mathbf{Q}$ linearly disjoint from F^{avoid} over $\mathbf{Q}$; and*
- *a finite totally real Galois extension $F^{\text{suffices}}/\mathbf{Q}$ unramified above $\mathcal{L}$ such that F^{suffices} is linearly disjoint from $F^{\text{avoid}}F_2^{\text{avoid}}$ over $\mathbf{Q}$;*

that have the following property:

For any finite totally real extension F'/F^{suffices}, which is linearly disjoint from F_2^{avoid} over $\mathbf{Q}$, and for any $m \in \mathcal{M}$, there is a regular algebraic, cuspidal, polarizable automorphic representation π of $\mathrm{GL}_{m+1}(\mathbf{A}_{F'})$ of weight $(0)_{\tau,i}$ such that for some, and hence every, rational prime l and any $\imath : \overline{\mathbf{Q}}_l \cong \mathbf{C}$, we have

$$\mathrm{Symm}^m r_{E,l}|^{\vee}_{G_{F'}} \cong r_{l,\imath}(\pi).$$

Moreover, π is unramified above any prime where E has good reduction.

Proof. We may, and will, suppose that $F^{\text{avoid}}/\mathbf{Q}$ is Galois. Choose a rational prime $l \geq \max_{m\in\mathcal{M}} 2(m+2)$ such that E has good ordinary reduction at l, $\overline{r}_{E,l}$ has image $\mathrm{GL}_2(\mathbf{F}_l)$, $l \notin \mathcal{L}$, and l is unramified in F^{avoid}. (By [Ser81, Th. 20], the condition that E is ordinary at l excludes a set of primes of Dirichlet density 0. By the main result of [Ser72], each of the other conditions excludes a finite number of primes.) Note that $\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$ contains ζ_l and, by part 2 of Lemma 7.1.8, is linearly disjoint from F^{avoid} over $\mathbf{Q}$.

Choose an imaginary quadratic field L that is unramified at all primes in $\mathcal{L}$, at all primes where E has bad reduction, and all primes that ramify in F^{avoid}, and in which l splits. Also choose a rational prime $q \notin \mathcal{L} \cup \{l\}$ that splits as $v_q v'_q$ in L, that is unramified in F^{avoid} and at which E has good reduction.

If $m \in \mathcal{M}$ is even, also choose a character

$$\psi_m : G_L \longrightarrow \overline{\mathbf{Q}}_l^{\times}$$

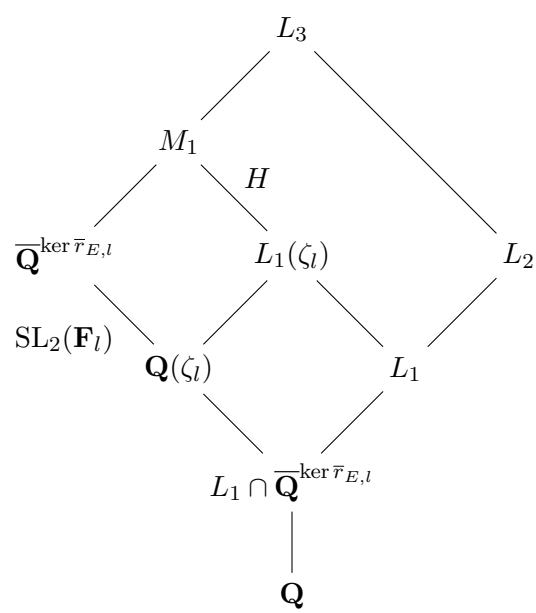
such that

- ψ_m is crystalline above l with Hodge–Tate numbers 0 at one place above l and $m+1$ at the other;

- $q|\#(\psi_m/\psi_m^c)(G_{F_{v_q}^{\mathrm{nr}}})$;
- ψ_m is unramified above $\mathcal{L}$ and all primes that ramify in F^{avoid} and all primes at which E has bad reduction;
- $\psi_m\psi_m^c = \epsilon_l^{-(m+1)}$.

([BLGGT14, Lem. A.2.5] tells us that this is possible.) The representation $\mathrm{Ind}_{G_L}^{G_{\mathbf{Q}}}\psi_m$ has determinant $\epsilon_l^{-(m+1)}$. (This is true on G_L by the construction of ψ_m and true on complex conjugation because m is even.)

Let L_2 denote the compositum of the $\overline{\mathbf{Q}}^{\ker \mathrm{Ind}_{G_{\mathbf{Q}}}^{G_L}\overline{\psi}_m}$ for $m \in \mathcal{M}$ even. Let L_1 denote the maximal sub-extension of L_2 ramified only at l, and let T denote the set of primes other than l that ramify in L_2. Then $L_2 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}} = L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$. Let $L_3 = L_2\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$. We will now show that F^{avoid} is linearly disjoint from L_3 over $\mathbf{Q}$; the argument is somewhat involved, and the reader may find it helpful to refer to the diagram of field extensions:



Let M_1 denote the maximal subfield of L_3 in which the primes of T are all unramified. Then $M_1 \supset \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$ (because $\overline{r}_{E,l}$ can only be ramified at l and places where E has bad reduction), and $M_1 \cap L_2 = L_1$ and $M_1 = L_1\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$. Thus

$$\begin{aligned}
&\mathrm{Gal}(M_1/(L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}})) \\
&\quad \cong \mathrm{Gal}(\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}/(L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}})) \times \mathrm{Gal}(L_1/(L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}})) \\
&\quad \cong \mathrm{Gal}(M_1/L_1) \times \mathrm{Gal}(M_1/\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}).
\end{aligned}$$

As $\mathrm{Gal}(L_1/\mathbf{Q})$ is soluble, we see that $L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}} \subset \mathbf{Q}(\zeta_l)$ and we see that $\mathrm{Gal}(M_1/\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}})$ is soluble and hence that $\mathrm{Gal}(M_1/(L_1 \cap \overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}))$ contains a unique copy of $\mathrm{SL}_2(\mathbf{F}_l)$ (because this latter group is perfect and, in particular, admits no solvable quotient), and this copy is therefore normal in $\mathrm{Gal}(M_1/\mathbf{Q})$. Its fixed field is $L_1(\zeta_l)$.

Let H be the subgroup of $\mathrm{Gal}(M_1/\mathbf{Q})$ generated by the inertia groups above l. The group H maps surjectively to $\mathrm{Gal}(\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}/\mathbf{Q})$ (because H is normal, and the only subfield of $\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}}$ unramified at l is $\mathbf{Q}$ itself, by Lemma 7.1.8) and so must contain the unique copy of $\mathrm{SL}_2(\mathbf{F}_l)$. Thus the maximal sub-extension M_0 of L_3 in which l and all elements of T are unramified is contained in $L_1(\zeta_l)$. This latter field is only ramified above l and so $M_0 = \mathbf{Q}$. Finally we deduce that F^{avoid} is linearly disjoint from L_3 over $\mathbf{Q}$ (using that all of the primes in $T \cup \{l\}$ are unramified in F^{avoid}).

If $m \in \mathcal{M}$ is odd, set

$$r_m = \mathrm{Symm}^m r_{E,l}^\vee : G_{\mathbf{Q}} \longrightarrow \mathrm{GSp}_{m+1}(\overline{\mathbf{Q}}_l).$$

It has multiplier ϵ_l^{-m}, is unramified above $\mathcal{L}$, and is crystalline and ordinary at l with Hodge–Tate numbers $\{0, 1, \ldots, m\}$. If $m \in \mathcal{M}$ is even, set

$$r_m = (\mathrm{Symm}^m r_{E,l}^\vee) \otimes \mathrm{Ind}_{G_F}^{G_{\mathbf{Q}}} \psi_m.$$

As the representation $(\mathrm{Symm}^m r_{E,l}^\vee)$ is orthogonal with multiplier ϵ_l^{-m}, we see that

$$r_m : G_{\mathbf{Q}} \longrightarrow \mathrm{GSp}_{2(m+1)}(\overline{\mathbf{Q}}_l)$$

with multiplier $\epsilon_l^{-(2m+1)}$. It is unramified above $\mathcal{L}$ and it is crystalline and ordinary at l with Hodge–Tate numbers $\{0, 1, \ldots, 2m+1\}$.

We apply Proposition 7.2.3 to $F = F_0 = \mathbf{Q}$, $\{r_m : m \in \mathcal{M}\}$, $\mathcal{L}$ and $F^{\mathrm{avoid}} L_3$, producing fields F_1^{avoid} and F^{suffices}. Set $F_2^{\mathrm{avoid}} = F_1^{\mathrm{avoid}} L_3$. Then F_2^{avoid} is linearly disjoint from F^{avoid} over $\mathbf{Q}$, and F^{suffices} is linearly disjoint from $F^{\mathrm{avoid}} F_2^{\mathrm{avoid}}$ over $\mathbf{Q}$.

Suppose that F'/F^{suffices} is a finite totally real extension linearly disjoint from F_2^{avoid} over $\mathbf{Q}$. Then $\mathrm{Symm}^m \mathrm{SL}_2(\mathbf{F}_l) \subset \mathrm{Symm}^m \overline{r}_{E,l}^\vee(G_{LF'(\zeta_l)})$, and so for $m \in \mathcal{M}$, the tautological representation of the subgroup of $\mathrm{Symm}^m \overline{r}_{E,l}^\vee(G_{LF'(\zeta_l)})$ generated by its elements of l-power order is absolutely irreducible. If $m \in \mathcal{M}$ is even, then $\overline{r}_m|_{G_{LF'(\zeta_l)}}$ is the direct sum of two absolutely irreducible constituents. The group $\mathrm{Gal}(L_3 F'/\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}} LF')$ acts by different characters on these two constituents, and $\mathrm{Gal}(\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}} LF'/\overline{\mathbf{Q}}^{\ker \overline{r}_{E,l}} F')$ interchanges these two characters. (Consider the action of inertia above q.) Thus $\overline{r}_m|_{G_{F'(\zeta_l)}}$ is absolutely irreducible. It follows from [BLGGT14, Prop. 2.1.2] that for $m \in \mathcal{M}$ odd or even, $\overline{r}_m(G_{F'(\zeta_l)})$ is adequate.

Combining Proposition 7.2.3 with [BLGGT14, Th. 2.4.1], we deduce that r_m is automorphic for $m \in \mathcal{M}$. It follows (using, in the case that m is even, [BLGGT14, Lem. 2.2.4] and the argument of [CHT08, Lem. 4.2.2]) that $\mathrm{Symm}^m r_{E,l}^\vee|_{G_{F'}}$ is automorphic. This finishes the proof of the corollary. □

7.2.5. *The main proof.* Finally we turn to the proof of Theorem 7.1.11.

Proof of Theorem 7.1.11. Choose a non-CM elliptic curve $E/\mathbf{Q}$ with good reduction above $\mathcal{L}_0$. Choose distinct rational primes l_1 and l_2 and a prime $\lambda_i|l_2$ of M_i for each $i \in \mathcal{I}$ such that

Assumption 7.2.6.

(1) l_2 splits completely in each M_i.

(2) The image of G_F on $E[l_1]$ contains $\mathrm{SL}_2(\mathbf{F}_{l_1})$, and $\overline{r}_{i,\lambda_i}(G_F)$ contains $\mathrm{SL}_2(\mathbf{F}_{l_2})$ for each $i \in \mathcal{I}$.

(3) l_1 and l_2 are unramified in F.

(4) E has good reduction above l_1 and l_2.

(5) l_1 and l_2 lie under no prime of any S_i.

(6) $l_1, l_2 > 2m_i + 3$ for all i.

This is possible because all the conditions are satisfied for a set of primes of Dirichlet density 1 (using Lemma 7.1.3), except for the first condition for l_2, which is satisfied for a set of primes of positive Dirichlet density.

Set $\mathcal{L} = \mathcal{L}_0 \cup \{l_1, l_2\}$. The weakly compatible system of characters

$$(M_i, S_i, l_i, \{q_v^{-1} Q_{i,v}(0)\}, \{\epsilon_l \det r_{i,\lambda}\}, \{\{0\}\})$$

has all Hodge–Tate numbers 0 and so there is a character $\psi_i : G_F \to M_i^\times$ with open kernel unramified outside S_i such that $\det r_{i,\lambda} = \psi_i \epsilon_l^{-1}$ for all λ (a prime of M_i with residue characteristic l). There is a sequence

$$\mathrm{Hom}(G_F, \overline{M}_i^\times) \xrightarrow{2} \mathrm{Hom}(G_F, \overline{M}_i^\times) \xrightarrow{\partial} H^2(G_F, \{\pm 1\}) = \mathrm{Br}_F[2] \hookrightarrow \oplus_v \mathrm{Br}_{F_v}[2],$$

which is exact at the second term. The element $\partial\psi_i$ is non-trivial only at places $v \in S_i$. We can find a soluble Galois totally real extension $F_1^+/\mathbf{Q}$, unramified above $\mathcal{L}$ and linearly disjoint over $\mathbf{Q}$ from the normal closure F_1^{avoid} of $F_0^{\mathrm{avoid}} \overline{F}^{\ker(\overline{r}_{E,l_1} \times \prod_i \overline{r}_{i,\lambda_i})}$ over $\mathbf{Q}$, such that for each $v \in \bigcup_{i \in \mathcal{I}} S_i$, the rational prime p_v below v has inertia degree in F_1^+ divisible by $2[F_v : \mathbf{Q}_{p_v}]$. (See, for instance, [CHT08, Lem. 4.1.2].) Then we see that for each $i \in \mathcal{I}$, $\partial\psi_i|_{G_{FF_1^+}}$ is trivial so that there is a continuous homomorphism

$$\phi_i : G_{FF_1^+} \longrightarrow \overline{M}_i^\times$$

such that $\phi_i^2 = \psi_i|_{G_{FF_1^+}}$. For $v|l \in \mathcal{L}$, we see that $\phi_i|^2_{I_{FF_1^+,v}} = 1$. By the Grunwald–Wang theorem (see Theorem 5 of Chapter X of [AT09]), we can find

a continuous character $\delta_i : G_{FF_1^+} \to \{\pm 1\}$ such that $\phi_i\delta_i$ is unramified at all places above $\mathcal{L}$. Replacing ϕ_i by $\phi_i\delta_i$ we may suppose that ϕ_i is unramified at all places above $\mathcal{L}$. Set

$$\mathcal{R}_i' = \mathcal{R}|_{G_{FF_1^+}} \otimes \phi_i^{-1}$$

with $S_i' = S_i$.

We apply Corollary 7.2.4 to $\{m_i : \ i \in \mathcal{I}\}$, E, $\mathcal{L}$ and $F_1^{\mathrm{avoid}}F_1^+$. We obtain a finite Galois extension $F_2^{\mathrm{avoid}}/\mathbf{Q}$ linearly disjoint from $F_1^{\mathrm{avoid}}F_1^+$ over $\mathbf{Q}$ and a finite totally real Galois extension $F^{+,\mathrm{suffices}}/\mathbf{Q}$, which is unramified above $\mathcal{L}$ and linearly disjoint from $F_1^{\mathrm{avoid}}F_2^{\mathrm{avoid}}F_1^+$ over $\mathbf{Q}$. Set $F^{\mathrm{avoid}} = F_1^{\mathrm{avoid}}F_2^{\mathrm{avoid}}$. It is Galois over $\mathbf{Q}$, and certainly contains $\overline{F}^{\ker(\overline{r}_{E,l_1} \times \prod_i \overline{r}_{i,\lambda_i})} \subset F_1^{\mathrm{avoid}}$ by definition. Moreover, it is linearly disjoint from $F_1^+F^{+,\mathrm{suffices}}$ over $\mathbf{Q}$. (See the diagram of field extensions later in this proof.)

Let $V_{\overline{r}'_{i,\lambda_i}}$ denote the vector space underlying $\overline{r}'_{i,\lambda_i}$, and give it a non-degenerate symplectic pairing, which $\overline{r}'_{i,\lambda_i}$ will then preserve up to multiplier $\overline{\epsilon}_{l_2}^{-1}$. Let Y_i/FF_1^+ denote the moduli space of elliptic curves D along with isomorphisms $\alpha_1 : E[l_1] \xrightarrow{\sim} D[l_1]$ and $\alpha_2 : V^\vee_{\overline{r}'_{i,\lambda_i}} \xrightarrow{\sim} D[l_2]$, which preserve symplectic pairings. Let $X_i/F^+F_1^+$ denote the restriction of scalars of Y_i.

If v is an infinite place of $F^+F_1^+$, then a point of $X_i((F^+F_1^+)_v)$ is the same as an $\overline{(FF_1^+)_v}$-point of Y_i and hence $X_i((F^+F_1^+)_v) \neq \emptyset$.

Suppose that w is a place of F_1^+F above $\mathcal{L}_0 \cup \{l_1\}$. Then we can find a positive integer f such that $\overline{r}'_{i,\lambda_i}(\mathrm{Frob}_w)^{-f} \sim \overline{r}_{E,l_2}(\mathrm{Frob}_w)^f$. Thus E gives rise to a point of Y_i over the unramified extension of degree f of $(F_1^+F)_w$. Hence X_i has a point over an unramified extension of $(F_1^+F^+)_v$ for every place v above $\mathcal{L}_0 \cup \{l_1\}$. Moreover this point corresponds to an elliptic curve with good reduction.

Now suppose that v is a place of $F_1^+F^+$ above l_2. We will show that X_i has a rational point over an unramified extension of $(F_1^+F^+)_v$ corresponding to an elliptic curve with good reduction. It suffices to show that Y_i has a point over an unramified extension of $(F_1^+F)_w$ for every prime w of F_1^+F over v and that this point corresponds to an elliptic curve with good reduction. Because $\mathbf{Q}_{l_2} \cong M_{i,\lambda_i}$, part 1 of Lemma 7.1.8 implies that the restriction $\overline{r}'_{i,\lambda_i}|_{G_{(F_1^+F)_w}}$ must have one of the following two forms:

(1) the induction from the unramified quadratic extension of $(F_1^+F)_w$ of $\omega_{l_2,2}^{-1}\delta$, where δ is the unramified quadratic character;

(2) $\begin{pmatrix} \overline{\chi} & * \\ 0 & \overline{\chi}^{-1}\overline{\epsilon}_{l_2}^{-1} \end{pmatrix}$, where $\overline{\chi}$ is unramified and where the extension class is peu ramifié in the sense that it lies in

$$H^1_f(G_{(F_1^+F)_w}, \mathbf{F}_{l_2}(\overline{\epsilon}_{l_2}\overline{\chi}^2)) \subset H^1(G_{(F_1^+F)_w}, \mathbf{F}_{l_2}(\overline{\epsilon}_{l_2}\overline{\chi}^2)).$$

(While the statement of Lemma 7.1.8 does not prescribe the direction of the extension in the second possibility, nor specify that it is peu ramifié, these follow easily from the connected-étale sequence for the finite flat group scheme H considered in the proof of Lemma 7.1.8.) In the first case, let $D/(F_1^+F)_w$ be an elliptic curve with good supersingular reduction. Choose a positive integer f such that $\overline{r}_{E,l_1}(\mathrm{Frob}_v)^{2f} = (-l_2)^f \bmod l_1$. Then D provides a point of Y_i over the unramified extension of $(F_1^+F)_w$ of degree $2f$.

In the second case, let $\overline{D}/k(w)$ be an ordinary elliptic curve, and let $\psi : G_{k(w)} \to \mathbf{Z}_{l_2}^\times$ denote the character by which $G_{k(w)}$ acts on the Tate module $T_{l_2}\overline{D}$. If L/F_w is a finite extension then, by Serre–Tate theory, liftings of $\overline{D}$ to $\mathcal{O}_L$ are parametrized by $H^1(G_L, \mathbf{Z}_{l_2}(\epsilon_{l_2}\psi^{-2})) = H^1_f(G_L, \mathbf{Z}_{l_2}(\epsilon_{l_2}\psi^{-2}))$, and we shall write D_e for the lifting corresponding to a class e. (Note that ψ^{-2} always has infinite order.) Then

$$\overline{r}_{D_e,l_2} \cong \begin{pmatrix} \overline{\epsilon}_{l_2}\overline{\psi}^{-1} & * \\ 0 & \overline{\psi} \end{pmatrix},$$

and the extension class is the image of e in $H^1_f(G_L, \mathbf{F}_{l_2}(\overline{\epsilon}_{l_2}\overline{\psi}^{-2}))$. Choose a positive integer f such that $\overline{\chi}^f = 1$ and $\psi^f \equiv 1 \bmod l_2$ and $\overline{r}_{E,l_1}(\mathrm{Frob}_{l_2})^f = 1$ and $\mathrm{Frob}_w^f = 1$ on $\overline{D}[l_1](\overline{k(w)})$. Let L/F_w denote the unramified extension of degree f, and let $e \in H^1_f(G_L, \mathbf{Z}_{l_2}(\epsilon_{l_2}\psi^{-2}))$ lift the negative of the class of $\overline{r}'_{i,\lambda_i}|_{G_L}$ in $H^1_f(G_L, k(w)(\overline{\epsilon}_{l_2}\overline{\chi}^2))$. (The existence of such an e follows from Lemma 7.2.2.) Then D_e/L has $D_e[l_1] \cong E[l_1]$ and $\overline{r}_{D_e,l_2} \cong (\overline{r}'_{i,\lambda_i})^\vee$.

It follows (for instance, by [BLGGT14, Prop. 3.1.1]) that there is a finite extension $F_2^+/F_1^+F^+$ such that

- F_2^+ is Galois over F_0^+;
- F_2^+ is totally real;
- all primes above $\mathcal{L}$ are unramified in F_2^+/F^+, and D_i has good reduction at all primes in $\mathcal{L}$;
- F_2^+ is linearly disjoint over $F_1^+F^+$ from $F^{\mathrm{avoid}}F^{+,\mathrm{suffices}}F_1^+$;
- $\prod_i X_i$ has an F_2^+-rational point, i.e., there exist elliptic curves D_i over F_2 such that $D_i[l_1] \cong E[l_1]|_{G_{F_2}}$ and $D_i[l_2] \cong \overline{r}'_{i,\lambda_i}|^\vee_{G_{F_2}}$.

Set $F^{\mathrm{suffices}} = F^{+,\mathrm{suffices}}F_2^+F$, a CM extension of F that is unramified above $\mathcal{L}$ and Galois over F_0. We now show that this is linearly disjoint from F^{avoid} over F; the reader may find it helpful to consult the diagram of field extensions below. As $F^{+,\mathrm{suffices}}$ is linearly disjoint from $F_1^+F^{\mathrm{avoid}}$ over $\mathbf{Q}$, we see that $F^+F_1^+F^{+,\mathrm{suffices}}$ is linearly disjoint from $F_1^+F^{\mathrm{avoid}}$ over $F^+F_1^+$, and so $F_2^+F^{+,\mathrm{suffices}}$ is linearly disjoint from $F_1^+F^{\mathrm{avoid}}$ over $F^+F_1^+$. Thus F^{suffices} is linearly disjoint from $F_1^+F^{\mathrm{avoid}}$ over FF_1^+. On the other hand, F_1^+ is linearly disjoint from F^{avoid} over $\mathbf{Q}$ and so F^{avoid} is linearly disjoint from FF_1^+ over F.

We conclude that F^{suffices} is linearly disjoint from F^{avoid} over F:

$$\begin{tikzcd}
 & F^{\text{suffices}} \arrow[dl, dash] \arrow[dr, dash] & & F_1^+ F^{\text{avoid}} \arrow[d, dash] \arrow[dr, dash] & \\
F^{+,\text{suffices}} F_2^+ \arrow[rr, dash] \arrow[dr, dash] & & F_2^+ \arrow[dr, dash] & F F_1^+ \arrow[d, dash] \arrow[dr, dash] & F^{\text{avoid}} \arrow[d, dash] \\
 & F^+ F_1^+ F^{+,\text{suffices}} \arrow[rr, dash] \arrow[d, dash] & & F^+ F_1^+ \arrow[d, dash] & F \arrow[dl, dash] \\
 & F^{+,\text{suffices}} \arrow[rr, dash] & & \mathbf{Q} &
\end{tikzcd}$$

Suppose that F'/F is a finite CM extension containing F^{suffices} and that is unramified above $\mathcal{L}$ and linearly disjoint from F^{avoid} over F.

By Corollary 7.2.4, there are regular algebraic, cuspidal, polarizable automorphic representations π_i of $\mathrm{GL}_{1+m_i}(\mathbf{A}_{F'})$ unramified above $\mathcal{L}$ and of weight $(0)_{\tau,z}$ such that for any $\imath : \overline{\mathbf{Q}}_{l_1} \cong \mathbf{C}$, we have

$$\mathrm{Symm}^{m_i} r_{E,l_1}|^{\vee}_{G_{F'}} \cong r_{l_1,\imath}(\pi_i).$$

Applying Theorem 6.1.1 (the conditions on the residual representations are satisfied by parts (1), (2) and (4) of Lemmas 7.1.6 and 7.1.8), we see that there are regular algebraic, cuspidal automorphic representations π_i' of $\mathrm{GL}_{1+m_i}(\mathbf{A}_{F'})$ unramified above $\mathcal{L}$ and of weight $(0)_\tau$ and $\imath : \overline{\mathbf{Q}}_{l_1} \cong \mathbf{C}$ such that

$$\mathrm{Symm}^{m_i} r_{D_i,l_1}|^{\vee}_{G_{F'}} \cong r_{l_1,\imath}(\pi_i'),$$

and so, for some $\imath : \overline{\mathbf{Q}}_{l_2} \cong \mathbf{C}$, we have

$$\mathrm{Symm}^{m_i} r_{D_i,l_2}|^{\vee}_{G_{F'}} \cong r_{l_2,\imath}(\pi_i').$$

Applying Theorem 6.1.1 again (the conditions on the residual representations again being satisfied by parts (1), (2) and (4) of Lemma 7.1.6 and Lemma 7.1.8) we see that there are regular algebraic, cuspidal automorphic representations π_i'' of $\mathrm{GL}_{1+m_i}(\mathbf{A}_{F'})$ unramified above $\mathcal{L}$ and of weight $(0)_{\tau,i}$, and $\imath : \overline{\mathbf{Q}}_{l_2} \cong \mathbf{C}$ such that

$$\mathrm{Symm}^{m_i} r'_{i,\lambda_i}|_{G_{F'}} \cong r_{l_2,\imath}(\pi_i'').$$

Untwisting completes the proof of Theorem 7.1.11. □

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