RATIONALITY OF TWISTS OF THE SIEGEL MODULAR VARIETY OF GENUS 2 AND LEVEL 3

FRANK CALEGARI AND SHIVA CHIDAMBARAM

Abstract. Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a continuous Galois representation with cyclotomic similitude character. Equivalently, consider $\overline{\rho}$ to be the Galois representation associated to the 3-torsion of a principally polarized abelian surface $A/\mathbf{Q}$. We prove that the moduli space $\mathcal{A}_2(\overline{\rho})$ of principally polarized abelian surfaces $B/\mathbf{Q}$ admitting a symplectic isomorphism $B[3] \simeq \overline{\rho}$ of Galois representations is never rational over $\mathbf{Q}$ when $\overline{\rho}$ is surjective, even though it is both rational over $\mathbf{C}$ and unirational over $\mathbf{Q}$ via a map of degree 6.

1. Introduction

Let p be a prime and suppose that $A/\mathbf{Q}$ is an abelian variety of dimension g with a polarization of degree prime to p. Associated to the action of the absolute Galois group $G_{\mathbf{Q}}$ on $A[p]$ there exists a Galois representation

$$\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_{2g}(\mathbf{F}_p)$$

such that the corresponding similitude character is the mod-p cyclotomic character ε. One can ask, conversely, whether any such representation comes from an abelian variety in infinitely many ways. When $g = 1$, this question is well-studied, and has a positive answer exactly for $p = 2$, 3, and 5. Indeed, the corresponding twists $X(\overline{\rho})$ of the modular curve $X(p)$ are rational over $\mathbf{Q}$ for $p = 2$, 3, and 5, and have higher genus for larger p.

In [BCGP18], this question arose for abelian surfaces ($g = 2$) when $p = 3$. (The case $p = 2$, which is also discussed in that paper, is understood by analyzing the branch points of the hyperelliptic involution.) Let $\mathcal{A}_2(3)$ denote the Siegel modular variety of genus 2 and level 3. It is the moduli space of principally polarized abelian surfaces together with a symplectic isomorphism $A[3] \simeq (\mathbf{Z}/3\mathbf{Z})^2 \oplus (\mu_3)^2$. Given a $\overline{\rho}$ as above, one can form the corresponding moduli space $\mathcal{A}_2(\overline{\rho})$ where now one insists that there is a symplectic isomorphism $A[3] \simeq V$, where V is the representation space of $\overline{\rho}$ with its symplectic structure. The variety $\mathcal{A}_2(3)$ is well-known to be birational to the Burkhardt quartic, which is rational over $\mathbf{Q}$ ([BN18]). It is clear that $\mathcal{A}_2(\overline{\rho})$ is isomorphic to $\mathcal{A}_2(3)$ over $\mathbf{C}$ (and even over the fixed field of the kernel of $\overline{\rho}$), and hence $\mathcal{A}_2(\overline{\rho})$ is *geometrically* rational. If $\mathcal{A}_2(\overline{\rho})$ was in fact *rational* (by which we always mean rational over the base field), then indeed the answer to the question above would be positive, just as for elliptic curves when $p \leq 5$. In [BCGP18, Prop 10.2.3], a weaker result was established: The variety $\mathcal{A}_2(\overline{\rho})$ is unirational over $\mathbf{Q}$ via a map of degree at most 6. As a consequence, any such $\overline{\rho}$ *does* arise from (infinitely many) abelian surfaces. We refer the reader to [CCR20]

Both authors were supported in part by NSF Grants DMS-1701703. The first author was supported in part by DMS-2001097, and the second author was supported in part by the Simons Foundation (grant 550033).

which produces explicit polynomials describing the universal family over a rational cover of $\mathcal{A}_2(\overline{\rho})$ of degree 6. However, the question as to whether $\mathcal{A}_2(\overline{\rho})$ was actually rational was left open. We address this question here.

Theorem 1. *Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a representation with cyclotomic similitude character. Suppose that the order of $\mathrm{im}(\overline{\rho})$ is greater than 96. Then $\mathcal{A}_2(\overline{\rho})$ is not rational over $\mathbf{Q}$.*

More refined results can be extracted directly from the table in §3. Since $\overline{\rho}$ has cyclotomic similitude character, the restriction of $\overline{\rho}$ to G_E, where $E = \mathbf{Q}(\sqrt{-3})$, has image contained in $\mathrm{Sp}_4(\mathbf{F}_3)$. If we let H denote the projection of $\mathrm{im}(\overline{\rho}|_{G_E})$ to the simple group $\mathrm{PSp}_4(\mathbf{F}_3)$, then we prove that $\mathcal{A}_2(\overline{\rho})$ is not rational over $\mathbf{Q}$ for all but 26 of the 116 conjugacy classes of subgroups of $\mathrm{PSp}_4(\mathbf{F}_3)$. With the exception of three cases (including when H is trivial) where the methods of [BN18] may be applied (see §2.3), we do not know what happens in the remaining 23 cases, nor do we even know whether the rationality of $\mathcal{A}_2(\overline{\rho})$ depends only on $\mathrm{im}(\overline{\rho})$ or not. One easy remark is that, for a quadratic character χ, there is an isomorphism $\mathcal{A}_2(\overline{\rho}) \simeq \mathcal{A}_2(\overline{\rho} \otimes \chi)$, and so the rationality of $\mathcal{A}_2(\overline{\rho})$ depends only on the image of $\overline{\rho}|_{G_E}$ in $\mathrm{PSp}_4(\mathbf{F}_3)$.

The case of a surjective representation $\overline{\rho}$ is of special interest, since this is what happens generically for the three-torsion Galois representations of abelian surfaces.

Theorem 2. *Suppose that $\overline{\rho}$ is surjective. Then $\mathcal{A}_2(\overline{\rho})$ is not rational over $\mathbf{Q}$, and the minimal degree of any rational cover is 6.*

In light of the result [BCGP18, Prop 10.2.3] mentioned above, the constant 6 is best possible in this case.

The key ingredient in our results is the explicit description of the cohomology of the compactified Siegel modular variety $\mathcal{A}_2^*(3)$ given in [HW01]. We use it to study the Galois module $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(\overline{\rho}))$. The Galois action over $E = \mathbf{Q}(\sqrt{-3})$ factors through the projectivization of $\overline{\rho}$ turning it into a H-module. We then calculate group cohomology of this module for various subgroups $P \subset H$, and employ a necessary criterion for rationality (see Theorem 3) to deduce our results.

1.1. **Acknowledgments.** We thank Jason Starr and Yuri Tschinkel for discussions about rationality versus geometric rationality for smooth varieties over number fields, Steven Weintraub for a suggestion on how to explicitly extract a description of $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$ as a $G = \mathrm{PSp}_4(\mathbf{F}_3)$-module from Theorem 4.9 of [HW01], and Mark Watkins with help using `Magma` [BCP97]. We thank the anonymous referees for useful comments and corrections, and we also thank Nils Bruin for explaining to us many of the ideas in section 2.3.

2. Strategy

The main idea behind the proof is to follow a strategy employed by Manin for cubic surfaces. Recall [Man86, §A.1] that a continuous G_K-module with the discrete topology is called a *permutation module* if it admits a finite free $\mathbf{Z}$-basis on which G_K acts (via a finite quotient) via permutations, and that two G_K-modules M and N are *similar* if $M \oplus P \simeq N \oplus Q$ for some permutation modules P and Q. In particular, we employ the following theorem.

Theorem 3. [Man86, §A.1 Theorem 2] *Let Z be a smooth projective algebraic variety over a number field K. Suppose that Z is rational over K. Then* $\mathrm{Pic}_{\overline{K}} Z$

as a G_K-module is stably permutation. In other words, it is similar to the zero module.

The Shimura variety $\mathcal{A}_2(3)$ admits a smooth toroidal projective compactification $\mathcal{A}_2^*(3)$, the (canonical) toroidal compactification constructed by Igusa [Igu67]. The automorphism group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$ is the group $G = \mathrm{PSp}_4(\mathbf{F}_3)$, the simple group of order 25920, which acts over the field $E = \mathbf{Q}(\sqrt{-3})$. It will be convenient from this point onwards to always work over the field E. (Certainly rationality over $\mathbf{Q}$ implies rationality over E, so non-rationality over E implies non-rationality over $\mathbf{Q}$.) This action on $\mathcal{A}_2(3)$ arises explicitly from the action of G on the 3-torsion $A[3] = (\mathbf{Z}/3\mathbf{Z})^2 \oplus (\mu_3)^2 \simeq (\mathbf{Z}/3\mathbf{Z})^4$ over E. We will apply Theorem 3 to the corresponding twist $\mathcal{A}_2^*(\overline{\rho})$. We then make crucial use of very explicit description of the cohomology of this compactification given by Hoffman and Weintraub [HW01]. We recall some facts from that paper here now.

2.1. **Picard group.** The Picard group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$ is a free $\mathbf{Z}$-module of rank 61. It is generated by two natural sets of classes. The first is a 40-dimensional space explained by the 40 connected components of the boundary. The second is a 45-dimensional space explained by divisors coming from Humbert surfaces. These are also in one to one correspondence with the 45 nodes on the Burkhardt quartic. Together, these generate the Picard group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$, which is free of rank 61. Indeed, the Betti cohomology of $\mathcal{A}_2^*(3)$ over $\mathbf{Z}$ is free of degrees $1, 0, 61, 0, 61, 0, 1$ for $i = 0, \ldots 6$ by [HW01, Theorem 1.1]. Furthermore, all of these classes are trivial under the action of G_E.

Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a continuous Galois representation with cyclotomic similitude character. The assumption on the similitude character implies that the restriction of $\overline{\rho}$ to E is valued in $\mathrm{Sp}_4(\mathbf{F}_3)$. Let

$$\varrho : G_E \to G = \mathrm{PSp}_4(\mathbf{F}_3)$$

denote the projectivization of the representation $\overline{\rho}$ restricted to E. The group G acts over E on $\mathcal{A}_2^*(3)$ via automorphisms, and $\mathcal{A}_2^*(\overline{\rho})$ is the twist of $\mathcal{A}_2^*(3)$ by ϱ. The group $\mathrm{Pic}_{\overline{\mathbf{Q}}}\mathcal{A}_2^*(\overline{\rho})$ as a G_E-module is obtained by considering $\mathrm{Pic}_{\overline{\mathbf{Q}}}\mathcal{A}_2^*(3)$ as a G-module and then obtaining the Galois action via the map $\varrho : G_E \to G$. Thus it remains to closely examine $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3))$ as a G-module over $\mathbf{Z}$. In fact, we can quickly prove a weaker version of Theorem 2 by studying this G-module over $\mathbf{Q}$. The group G admits a unique conjugacy class G_{45} of subgroups of index 45, but two conjugacy classes of index 40; let G_{40} denote the (conjugacy class of) subgroups which fix a point in the tautological action of $G \subset \mathrm{PGL}_4(\mathbf{F}_3)$ on $\mathbf{P}^3(\mathbf{F}_3)$. The following is an easy consequence of the calculations of [HW01] (and is also confirmed by our `Magma` code).

Lemma 1. *As $\mathbf{Q}[G]$-modules, there is an equality of virtual representations*

$$H^2(\mathcal{A}_2^*(3), \mathbf{Q}) \simeq \mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3)) \otimes \mathbf{Q} = \mathbf{Q}[G/G_{40}] + \mathbf{Q}[G/G_{45}] - [\chi_{24}],$$

where $\chi_{24} \otimes_{\mathbf{Q}} \mathbf{C}$ is the unique absolutely irreducible 24-dimensional representation of G.

Now, assuming that ϱ is *surjective*, we can prove that $\mathcal{A}_2^*(\overline{\rho})$ is not rational simply by proving that χ_{24} is not virtually equal to a sum of permutation representations. If $R_{\mathbf{Q}}(G)$ denotes the representation ring of G, this is equivalent to proving that $\chi_{24} \in R_{\mathbf{Q}}(G)$ does not lie in the Burnside subring generated by permutation representations. But one may compute (using `Magma` or otherwise) that the Burnside cokernel of G has order 2 and is generated by χ_{24}. This proves a weaker version of Theorem 2 showing that any rational cover of $\mathcal{A}_2(\overline{\rho})$ should have degree at least 2, although it is softer in that it only needs the $\mathbf{Q}[G]$-representation rather than the $\mathbf{Z}[G]$-module. This argument also applies if one only assumes that the image of ϱ is $H \subset G$, as long as the restriction of χ_{24} to H is still non-trivial in the Burnside cokernel, which it is for precisely 8 of the 116 conjugacy classes of subgroups of G.

2.2. **Cohomological Obstructions.** From now on, we let H denote the image of $\varrho : G_E \to G = \mathrm{PSp}_4(\mathbf{F}_3)$. A second way to prove that a Galois module is not similar to the zero module is to use cohomology. If M is a permutation module of H, then the restriction of M to any subgroup P is also a permutation module, and thus a direct sum of P-modules of the form $\mathbf{Z}[P/Q]$ for subgroups Q of P. (Note that since a permutation module of a group G arises from a finite G-set, it always decomposes over $\mathbf{Z}$ into a direct sum of such irreducible permutation modules.) Then, Shapiro's Lemma implies that $H^1(P, M)$ is a direct sum of groups of the form

$$H^1(P, \mathbf{Z}[P/Q]) = H^1(Q, \mathbf{Z}) = 0,$$

where the second group vanishes because Q is finite. Moreover, the $\mathbf{Z}$-dual $M^\vee = \mathrm{Hom}(M, \mathbf{Z})$ of a permutation module is isomorphic to the same permutation module (a permutation matrix is its own inverse transpose). Thus one immediately has the following elementary criterion.

Lemma 2 (Cohomological Criterion for non-rationality)**.** *Let M denote the G-module $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3))$. Suppose $\mathcal{A}_2^*(\overline{\rho})$ is rational over $E = \mathbf{Q}(\sqrt{-3})$, and $\varrho|_{G_E}$ has image $H \subset G$. Then*

$$H^1(P, M^\vee) = H^1(P, M) = 0$$

for every subgroup $P \subset H$.

We note that this is not an "if and only if" criterion. In the language of [CTS77], the lemma is saying that M as a G_E-module is *flasque* and *coflasque* respectively. In general, this is weaker than being stably permutation (which itself is not enough to formally imply rationality).

In order to test this criterion in practice, we need an explicit description of M as a $\mathbf{Z}[G]$-module rather than a $\mathbf{Q}[G]$-module. In order to do this, we explain how an explicit description of M can be extracted from Theorem 4.9 of [HW01]. That theorem describes a set of elements which generate both $H_4(\mathcal{A}_2^*(3), \mathbf{Z})$ and $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$, and explicitly gives the intersection pairing between them. Moreover, the basis comes with a transparent action of the group G. Specifically, $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$ is given as a quotient of $\mathbf{Z}[G/G_{40}] \oplus \mathbf{Z}[G/G_{45}]$. Hence to compute $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$ as a G-module, it suffices to compute the quotient of $\mathbf{Z}[G/G_{40}] \oplus \mathbf{Z}[G/G_{45}]$ by the saturated subspace which pairs trivially with all elements of $H_4(\mathcal{A}_2^*(3), \mathbf{Z})$. Having carried out this computation, we obtain a free abelian group of rank 61 with an explicit action of G. We then do the following for every conjugacy class of subgroups $H \subset G$.

(1) Determine whether χ_{24} is non-trivial in the Burnside cokernel of H.
(2) Determine whether $H^1(P, M) \neq 0$ for any subgroup $P \subset H$.
(3) Determine whether $H^1(P, M^\vee) \neq 0$ for any subgroup $P \subset H$.

If any of these is non-trivial, this proves that $\mathcal{A}_2^*(\overline{\rho})$ is not rational. Moreover, the computation of these cohomology groups allows us to deduce our result about the minimal degree of any rational covering.

Lemma 3. *Let M denote the G-module $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3))$. Suppose $\varrho|_{G_E}$ has image $H \subset G$. Let n denote the least common multiple of the exponents of $H^1(P, M)$ and $H^1(P, M^\vee)$ as P varies over all subgroups of H. Suppose $f : X \to \mathcal{A}_2^*(\overline{\rho})$ is a rational cover of degree d defined over $\mathbf{Q}$. Then n divides d.*

Proof. The induced pullback map $f^* : \mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(\overline{\rho})) \to \mathrm{Pic}_{\overline{\mathbf{Q}}}(X)$ and pushforward map $f_* : \mathrm{Pic}_{\overline{\mathbf{Q}}}(X) \to \mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(\overline{\rho}))$ are Galois equivariant since f is defined over $\mathbf{Q}$. The composite map $g = f_* \circ f^*$ on $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(\overline{\rho}))$ is multiplication by d. The discussion in §2.1 shows that the G_E-module $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(\overline{\rho}))$ can be thought of as the H-module M.

By Theorem 3, we know that $\mathrm{Pic}_{\overline{\mathbf{Q}}}(X)$ is stably permutation as a Galois module and hence the Galois cohomology group $H^1(G_{\mathbf{Q}}, \mathrm{Pic}_{\overline{\mathbf{Q}}}(X)) = 0$. Therefore, the maps induced by g on the cohomology groups $H^1(P, M)$ and $H^1(P, M^\vee)$ are the zero maps for every subgroup $P \subset H$. Since the map g is multiplication by d, the induced map on cohomology is also multiplication by d, and hence we deduce that the exponent of each of these cohomology groups divides d. □

We give one final statement which can be extracted from `Magma` using the code given in [CC21], but not directly from the table. In order to represent elements of $G = \mathrm{PSp}_4(\mathbf{F}_3)$ by matrices, we follow the conventions of `Magma` by fixing $\mathrm{Sp}_4(\mathbf{F}_3) \subset \mathrm{GL}_4(\mathbf{F}_3)$ to be the matrices preserving the symplectic form

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Lemma 4. *Suppose that the image of $\overline{\rho}$ contains an element conjugate in $\mathrm{PSp}_4(\mathbf{F}_3)$ to*

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{A}_2(\overline{\rho})$ is not rational, and the minimal degree of any rational cover is divisible by 3.

Proof. It suffices to note that this element generates the subgroup labelled as subgroup 6 in the table below, and then to apply Lemma 3. □

2.3. **Other cases where rationality can be established.** The analysis of Baker's parametrization [Bak46] undertaken in [BN18, §4] allows one to deduce the rationality of certain twists of the Burkhardt quartic B (and hence of $\mathcal{A}_2(\overline{\rho})$) in a few more cases. (We thank Nils Bruin for pointing this out to us, as well as explaining the geometric construction below.) The rational parametrization $\mathbf{P}^3 \dashrightarrow B$ over $\mathbf{Q}$ constructed in [BN18] is not equivariant with respect the action of $\mathrm{PSp}_4(\mathbf{F}_3)$. If it were, then the twists $\mathcal{A}_2(\overline{\rho})$ we are considering would all be birational to Brauer–Severi varieties. However, because they are also unirational over $\mathbf{Q}$ by [BCGP18, Prop 10.2.3], they would be rational over $\mathbf{Q}$, which we prove in this paper to be false in general. On the other hand, the parametrization $\mathbf{P}^3 \dashrightarrow B$ *is* equivariant with respect to the (unique up to conjugacy) cyclic group of order 9 [BN18, §4.3], and also with respect to the corresponding group scheme over $\mathbf{Q}$ whose E points are this group of order 9 (c.f. [CCR20, §2.3]), which controls the descent from E to $\mathbf{Q}$. In particular, the same argument implies that $\mathcal{A}_2(\overline{\rho})$ is rational in two further cases, namely, the subgroups labelled $n = 4$ (of order 3) and $n = 24$ (of order 9) in the table below. One can also arrive at this rational parametrization more geometrically, following [BN18, §4], whose notation we now freely follow. The variety of lines L_{J_1,J_2,J_3} incident with 3-distinct planes $J_i \subset \mathbf{P}^4$ is geometrically rational. If these planes are mutually skew and lie on B, there is a dominant map $L_{J_1,J_2,J_3} \dashrightarrow B$ defined by noting that a line will generically intersect B in four points and each J_i in one point, and hence one can send the line to the fourth point of intersection with B. There are 40 Jacobi planes J_i on B, and 2880 triples of mutually skew such planes. The stabilizer under $\mathrm{PSp}_4(\mathbf{F}_3)$ on these 2880 triples is the cyclic group of order 9. The assumption that H is contained inside this group then implies that there exists a triple of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-invariant mutually skew planes on the twist of B corresponding to $\overline{\rho}$. The result then follows after noting that L_{J_1,J_2,J_3} is rational over $\mathbf{Q}$ whenever this triple is defined over $\mathbf{Q}$. (We omit a direct proof of this last claim in light of the alternate argument given above.)

3. Computation

Let M denote the G-module $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3)) \simeq H^2(\mathcal{A}_2^*(3), \mathbf{Z})$. We have, by Poincaré duality, an isomorphism $M^\vee = H^4(\mathcal{A}_2^*(3), \mathbf{Z})$. Below we present in a table the result of our computation for all 116 conjugacy classes of subgroups $H \subset G$, indicating the following data:

(1) An ordering $n = 1 \ldots 116$ of the conjugacy class of the subgroup H as determined by `Magma`.
(2) The group H in the small groups database [BEO01]. The first element of the pair gives the order of H.
(3) The order of M in the Burnside cokernel of H over $\mathbf{Q}$ (if it is non-trivial). If this is greater than 1, then the corresponding twist is not rational over E (or $\mathbf{Q}$).
(4) The least common multiple of the exponents of $H^1(P, M)$ and $H^1(P, M^\vee)$ as P ranges over subgroups $P \subset H$. If this is greater than 1, then the corresponding twist is not rational over E (or $\mathbf{Q}$). In particular, the fact that this number is 6 for G itself proves Theorem 2.
(5) The pre-image of H in $\mathrm{Sp}_4(\mathbf{F}_3)$ acts on $\mathbf{F}_3^4$. Is this action absolutely irreducible? (That is, is the action on $\overline{\mathbf{F}}_3^4$ irreducible.)
(6) A list of the conjugacy class of maximal subgroups of H (as indexed in the table). This allows one to compute the LCM column directly. The table is separated into blocks to reflect the geometry of the corresponding poset of subgroups. In particular, all maximal subgroups of H occur in blocks before that of H.
(7) The last two columns give $H^1(H, M)$ and $H^1(H, M^\vee)$.

One must be careful while reading the table because the ordering of the conjugacy classes of subgroups is not canonical. The Small Group tag and the indices of the maximal subgroups given in the second and sixth columns of the table *do*, however, determine the ordering uniquely once we distinguish between the conjugacy classes indexed by $n = 2, 3$, $n = 4, 5, 6$, $n = 9, 11$, and $n = 10, 12$. This can be done by considering the length of each of these conjugacy classes (i.e., the number of subgroups in each conjugacy class) as shown in the following table.

n	Length
2	45
3	270
4	40
5	120
6	240

n	Length
9	270
11	405
10	270
12	540

The Magma code available at [CC21] computes G and M directly from the description given by Hoffman and Weintraub [HW01]. This leads to a representation of G as generated by two sparse 61×61 matrices x and y in $\mathrm{GL}_{61}(\mathbf{Z})$ such that the underlying module on which G acts (on the right, by Magma conventions) is M. The matrices x and y are also printed in the output file of our Magma script.

n	SmallGroup	B	LCM	irred	maximal subgroups	$H^1(M)$	$H^1(M^\vee)$
1	<1,1>		1	no			
2	<2,1>		1	no	1		
3	<2,1>		1	no	1		
4	<3,1>		1	no	1		
5	<3,1>		1	no	1		
6	<3,1>		3	no	1	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$
7	<5,1>		1	no	1		
8	<4,1>		1	no	2		
9	<4,2>		1	no	2 3		
10	<4,2>		2	no	3		$(\mathbf{Z}/2\mathbf{Z})^2$
11	<4,2>		2	no	2 3		$\mathbf{Z}/2\mathbf{Z}$
12	<4,2>		1	no	3		
13	<4,1>		1	no	3		
14	<6,1>		3	no	2 6	$\mathbf{Z}/3\mathbf{Z}$	
15	<6,2>		1	no	2 4		
16	<6,2>		3	no	2 6		
17	<6,1>		3	no	3 6		$\mathbf{Z}/3\mathbf{Z}$
18	<6,1>		1	no	3 5		
19	<6,2>		1	no	2 5		
20	<6,2>		1	no	3 5		
21	<9,2>		3	no	5 6		$(\mathbf{Z}/3\mathbf{Z})^2$
22	<9,2>		3	no	4 6		$(\mathbf{Z}/3\mathbf{Z})^2$
23	<9,2>		3	no	4 5 6		
24	<9,1>		1	no	4		
25	<10,1>		1	no	3 7		
26	<8,4>		1	no	8		
27	<8,5>		2	no	11 12		$(\mathbf{Z}/2\mathbf{Z})^2$
28	<8,5>		2	no	10 11		$(\mathbf{Z}/2\mathbf{Z})^2$
29	<8,5>		2	no	9 10 11		
30	<8,2>		2	no	8 11		
31	<8,2>		2	no	11 13	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$

32	<8,3>		2	no	8 11		$\mathbf{Z}/2\mathbf{Z}$
33	<8,3>		2	no	10 12 13		$\mathbf{Z}/2\mathbf{Z}$
34	<8,3>		1	no	9 12 13		
35	<12,3>		2	no	5 10		
36	<12,3>		3	no	6 12	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$
37	<12,4>		3	no	9 14 16 17		
38	<12,5>		1	no	9 19 20		
39	<12,1>		1	no	13 20		
40	<12,2>		1	no	8 15		
41	<12,4>		1	no	12 18 20		
42	<18,4>		3	no	17 18 21		$(\mathbf{Z}/3\mathbf{Z})^2$
43	<18,3>		3	no	14 16 21		
44	<18,3>		3	no	14 19 23		
45	<18,3>		3	no	14 15 22		
46	<18,3>		3	no	18 20 21		$\mathbf{Z}/3\mathbf{Z}$
47	<18,3>		3	no	17 20 23		
48	<18,5>		3	no	15 16 19 23		
49	<20,3>		1	yes	13 25		
50	<27,5>		3	no	21 22 23		$\mathbf{Z}/3\mathbf{Z}$
51	<27,3>		3	no	22		$(\mathbf{Z}/3\mathbf{Z})^2$
52	<27,4>		3	no	22 24		$\mathbf{Z}/3\mathbf{Z}$
53	<16,14>		2	yes	28 29		
54	<16,13>		2	no	26 30 32		
55	<16,11>		2	yes	27 28 30 32		$\mathbf{Z}/2\mathbf{Z}$
56	<16,3>		2	no	28 31		$(\mathbf{Z}/2\mathbf{Z})^2$
57	<16,11>		2	yes	27 29 31 33 34		$\mathbf{Z}/2\mathbf{Z}$
58	<16,3>		2	no	29 30 31		
59	<24,3>		1	no	15 26		
60	<24,13>		2	no	20 29 35		
61	<24,3>		3	no	16 26		
62	<24,3>		1	no	19 26		
63	<24,11>	2	1	no	26 40		
64	<24,13>		2	no	19 28 35		
65	<24,13>		6	no	16 27 36		
66	<24,12>		2	no	18 33 35		
67	<24,12>		6	no	17 33 36		$\mathbf{Z}/6\mathbf{Z}$
68	<24,12>		3	no	14 34 36	$\mathbf{Z}/3\mathbf{Z}$	
69	<24,8>		1	no	34 38 39 41		
70	<36,10>		3	no	37 42 43		
71	<36,10>		3	no	41 42 46		$\mathbf{Z}/3\mathbf{Z}$
72	<36,9>		3	no	13 42		$\mathbf{Z}/3\mathbf{Z}$
73	<36,12>		3	no	37 38 44 47 48		
74	<54,8>		3	no	45 51		
75	<54,13>		3	no	42 46 47 50		$\mathbf{Z}/3\mathbf{Z}$
76	<54,12>		3	no	43 44 45 48 50		
77	<60,5>		2	no	18 25 35		
78	<60,5>		3	no	17 25 36		$\mathbf{Z}/3\mathbf{Z}$
79	<81,7>		3	no	50 51 52		$\mathbf{Z}/3\mathbf{Z}$

80	<32,49>		2	no	54 56		
81	<32,6>		2	yes	55 56		$\mathbf{Z}/2\mathbf{Z}$
82	<32,27>		2	yes	53 55 56 57 58		
83	<48,30>		2	no	39 58 60		
84	<48,49>		2	yes	38 53 60 64		
85	<48,33>		2	yes	40 54 59		
86	<48,48>		2	no	41 57 60 66		$\mathbf{Z}/2\mathbf{Z}$
87	<48,48>		6	yes	37 57 65 67 68		
88	<72,40>		3	no	34 70 71 72		
89	<72,25>	2	3	no	48 59 61 62 63		
90	<80,49>		2	yes	7 53		
91	<108,40>		3	no	71 75		$\mathbf{Z}/3\mathbf{Z}$
92	<108,15>		3	no	40 74		
93	<108,38>		3	no	70 73 75 76		
94	<108,37>		3	no	39 72 75		
95	<120,34>		3	yes	37 49 68 78		
96	<120,34>		2	yes	41 49 66 77		
97	<162,10>		3	no	74 76 79		
98	<64,138>		2	yes	80 81 82		
99	<96,204>		2	no	62 64 80		
100	<96,204>		6	no	61 65 80		
101	<96,201>	2	2	no	63 80 85		
102	<96,195>		2	yes	69 82 83 84 86		
103	<160,234>		2	yes	25 82 90		
104	<216,88>	2	3	no	63 92		
105	<216,158>		3	no	69 88 91 93 94		
106	<324,160>		3	no	36 79 91		$\mathbf{Z}/3\mathbf{Z}$
107	<360,118>		6	no	66 67 72 77 78		$\mathbf{Z}/3\mathbf{Z}$
108	<192,1493>		6	yes	87 98 100		
109	<192,201>		2	yes	84 98 99		
110	<288,860>	2	6	no	89 99 100 101		
111	<648,533>	2	3	no	89 97 104		
112	<648,704>		3	no	68 97 105 106		
113	<720,763>		6	yes	86 87 88 95 96 107		
114	<576,8277>	2	6	yes	73 108 109 110		
115	<960,11358>		2	yes	77 102 103 109		
116	G	2	6	yes	111 112 113 114 115		

References

[Bak46] H. F. Baker. *A Locus with* 25920 *Linear Self-Transformations*. Cambridge Tracts in Mathematics and Mathematical Physics, no. 39. Cambridge, at the University Press; New York, The Macmillan Company, 1946.

[BCGP18] George Boxer, Frank Calegari, Toby Gee, and Vincent Pilloni. Abelian surfaces over totally real fields are potentially modular. preprint, 2018.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[BEO01] Hans Ulrich Besche, Bettina Eick, and E. A. O'Brien. The groups of order at most 2000. *Electron. Res. Announc. Amer. Math. Soc.*, 7:1–4, 2001.

[BN18] Nils Bruin and Brett Nasserden. Arithmetic aspects of the Burkhardt quartic threefold. *J. Lond. Math. Soc. (2)*, 98(3):536–556, 2018.

[CC21] Frank Calegari and Shiva Chidambaram. Auxiliary magma files. `https://github.com/shiva-chid/code_rationality`, 2021.

[CCR20] Frank Calegari, Shiva Chidambaram, and David P. Roberts. Abelian surfaces with fixed 3-torsion. In Steven Galbraith, editor, *Proceedings of the Fourteenth Algorithmic Number Theory Symposium (ANTS-XIV)*, Open Book Series 4, pages 91–108, Berkeley, 2020. Mathematical Sciences Publishers.

[CTS77] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc. La R-équivalence sur les tores. *Ann. Sci. École Norm. Sup. (4)*, 10(2):175–229, 1977.

[HW01] J. William Hoffman and Steven H. Weintraub. The Siegel modular variety of degree two and level three. *Trans. Amer. Math. Soc.*, 353(8):3267–3305, 2001.

[Igu67] Jun-Ichi Igusa. A desingularization problem in the theory of siegel modular functions. *Mathematische Annalen*, 168(1):228–260, Dec 1967.

[Man86] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

Email address: `fcale@uchicago.edu`

The University of Chicago, 5734 S University Ave, Chicago, IL 60637, USA

Email address: `shivac@mit.edu`

Massachusetts Institute of Technology, 77 Massachusetts Ave. Cambridge, MA 02139, USA