

The Artin conjecture for some S_5 -extensions

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Abstract. We establish some new cases of Artin’s conjecture. Our results apply to Galois representations over \mathbf{Q} with image S_5 satisfying certain local hypotheses, the most important of which is that complex conjugation is conjugate to (12)(34). In fact, we prove the stronger claim conjectured by Langlands that these representations are automorphic. For the irreducible representations of dimensions 4 and 6, our result follows from known 2-dimensional cases of Artin’s conjecture (proved by Sasaki) as well as the functorial properties of the Asai transfer proved by Ramakrishnan. For the irreducible representations of dimension 5, we encounter the problem of descending an automorphic form from a quadratic extension compatibly with the Galois representation. This problem is partly solved by working instead with a four dimensional representation of some central extension of S_5 . Our modularity results in this case are contingent on the non-vanishing of a certain Dedekind zeta function on the real line in the critical strip. A result of Booker show that one can (in principle) explicitly verify this non-vanishing, and with Booker’s help we give an example, verifying Artin’s conjecture for representations coming from the (Galois closure) of the quintic field K of smallest discriminant (1609).

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1. Introduction

Let $G_{\mathbf{Q}}$ denote the absolute Galois group of \mathbf{Q} , let K/\mathbf{Q} be a number field with Galois closure K^{gal} , and let

$$\rho : G_{\mathbf{Q}} \rightarrow \text{Gal}(K^{\text{gal}}/\mathbf{Q}) \hookrightarrow \text{GL}_n(\mathbf{C})$$

be a continuous irreducible Galois representation. Attached to ρ is an L -function $L(s, \rho)$ which is holomorphic for $\text{Re}(s) > 1$. Artin conjectured that $L(\rho, s)$ had a holomorphic continuation to the entire complex plane, with the possible exception of a pole at $s = 1$ if ρ was the trivial representation. Subsequently, Langlands conjectured that ρ was *modular*, that is, there exists an automorphic representation π for $\text{GL}(n)/\mathbf{Q}$ such that $L(\pi, s) = L(\rho, s)$ and π is cuspidal as long as ρ is non-trivial, which in particular implies Artin’s conjecture. If ρ is a monomial representation (that

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is, induced from a character) then Artin's conjecture was proved by Artin, and if G is nilpotent, then Langlands' conjecture is a consequence of cyclic base change (Theorem 7.1 of [AC89]). Suppose that $G = \text{Gal}(K^{\text{gal}}/\mathbf{Q})$ has a faithful permutation representation of degree ≤ 5 . If G is solvable, then Artin's conjecture follows from the fact that all such G are monomial. Moreover, Langlands' conjectures are also known in these cases. If $G = A_4$ and $G = S_4$, then results of Langlands [Lan80] and Tunnell [Tun81] imply that there exists a cuspidal representation π for $\text{GL}(2)/\mathbf{Q}$ which is associated to a representation with *projective* image G . The faithful representations of G (which are all of dimension three) can then be realized (up to twist) by the symmetric square $\text{Sym}^2\pi$ of Gelbart and Jacquet. If G is *not* solvable, then either G is A_5 or S_5 . If $G = A_5$, then Langlands' conjecture is known providing that complex conjugation is non-trivial. A theorem of Khare–Wintenberger [KW09] guarantees the existence of an automorphic form π for $\text{GL}(2)/\mathbf{Q}$ corresponding to any odd projective A_5 -representation, and then all the faithful representations of A_5 can be reconstructed from π (and $\iota\pi$ for the outer automorphism ι of A_5) via functoriality (see [Kim04]). In this note, we prove some cases of the conjectures of Artin and Langlands for S_5 using functoriality, in the spirit of Tunnell [Tun81]. Recall that the faithful irreducible representations of S_5 have dimensions 4, 5, and 6.

Theorem 1.1 *Let K/\mathbf{Q} be an extension such that $\text{Gal}(K^{\text{gal}}/\mathbf{Q}) = S_5$. Suppose, furthermore, that*

1. *Complex conjugation in $\text{Gal}(K^{\text{gal}}/\mathbf{Q})$ has conjugacy class (12)(34).*
2. *The extension $K^{\text{gal}}/\mathbf{Q}$ is unramified at 5, and the Frobenius element $\text{Frob}_5 \in \text{Gal}(K^{\text{gal}}/\mathbf{Q}) = S_5$ has conjugacy class (12)(34).*

If ρ is irreducible of dimension 4 or 6, then ρ is modular. If ρ is irreducible of dimension 5, there exists a tempered cuspidal ϖ for $\text{GL}(5)/\mathbf{Q}$ such that $\text{WD}(\rho|_{D_v}) \simeq \text{rec}(\varpi_v)$ for a set of places v of density one.

The existence of a weak correspondence between ρ and ϖ , though perhaps an approximation to Langlands' conjecture, is not sufficient even to deduce Artin's conjecture for $L(\rho, s)$ (although one *can* deduce from our arguments that $L(\rho, s)$ is holomorphic for $\text{Re}(s) > 1 - c$ for some ineffective constant $c > 0$). We may, however, remedy this lacuna under a more stringent hypothesis, which shows that the conjectures of Artin and Langlands *can* be established unconditionally for some S_5 -extensions.

Theorem 1.2 *Let K be as in Theorem 1.1. Let E/\mathbf{Q} be the quadratic subfield of K^{gal} , let F/\mathbf{Q} be a subfield of K^{gal} of degree 6 over \mathbf{Q} , and let H be the compositum of F and E . Suppose that $\zeta_H(s)$ does not vanish for real $s \in (0, 1)$. Then ρ is modular.*

Note that the non-vanishing condition on $\zeta_H(s)$ is explicitly verifiable in theory (and in practice, see Theorem 5.1). There is a factorization

$$\zeta_H(s) = \zeta_F(s)L(\eta, E/\mathbf{Q}, s)L(\rho_5, s)$$

for some explicit meromorphic Artin L -function $L(\rho_5, s)$. Our result actually only requires the non-vanishing of $L(\rho_5, s)$ for $s \in (0, 1)$. The non-vanishing of Artin L -functions for $s \in (0, 1)$ is not implied by the GRH, and indeed Armitage found examples of number fields L such that $\zeta_L(\frac{1}{2}) = 0$ (see [Arm72]). Those constructions, however, arose from Artin L -functions ρ with real traces such that the global Artin root number $W(\rho)$ was -1 . Since all the representations of S_5 (indeed of S_n) are self-dual and definable over \mathbf{R} , the the Artin root number $W(\rho)$ of any irreducible representation of these groups is automatically $+1$, and so $\zeta_H(s)$ is never forced to vanish at $s = 1/2$ for sign reasons. Indeed, the non-vanishing of $\zeta_H(s)$ would follow if one assumed that (in addition to the GRH) that Artin L -functions of irreducible representations have only simple zeros. The local condition at 5 is not essential to the method, but the restriction on complex conjugation is completely essential. The reason we can prove anything non-trivial under these assumptions is because we can reduce to known cases of Artin for (projective) two-dimensional A_5 -extensions over quadratic extensions of \mathbf{Q} . The assumption that complex conjugation is conjugate to (12)(34) ensures that these Artin representations are defined over (totally) real quadratic fields, and that the image of complex conjugation in $\mathrm{GL}_2(\mathbf{C})$ has determinant -1 . For such two dimensional representations, one can deduce modularity using results of Sasaki [Sas11a, Sas11b], which generalizes the approach of [BDSBT01].

1.1. Acknowledgments

I would like to thank Andrew Booker for establishing the non-vanishing of $L(\rho_5, s)$ for $s \in (0, 1)$ for an explicit number field K (see Theorem 5.2), demonstrating that one can effectively use the results of this paper to prove the Artin conjecture for particular S_5 -extensions. I would also like to thank Lassima Dembélé for computing the Hilbert modular forms of weight $(2, 2)$ and level one for $\mathbf{Q}(\sqrt{1609})$. Finally, I would like to thank Kevin Buzzard, Peter Sarnak, Dinakar Ramakrishnan, and the referee for useful conversations.

2. Some Group Theory

Let $\mathbf{F}_5^\times \subset \Delta \subset \overline{\mathbf{F}}_5^\times$ be a finite subgroup, necessarily cyclic. Let $N := |\Delta|$ denote the order of Δ — it is divisible by 4. We define the group $N.S_5$ to be $\mathrm{GL}_2(\mathbf{F}_5)\Delta$. There is a tautological map:

$$N.S_5 \rightarrow \mathrm{PGL}_2(\mathbf{F}_5) \simeq S_5$$

which realizes $N.S_5$ as a central extension of S_5 by the cyclic group Δ of order N . Let $N.A_5$ denote the kernel of the composite

$$N.S_5 \rightarrow S_5 \rightarrow \mathbf{Z}/2\mathbf{Z}.$$

Lemma 1 *The group $N.A_5$ is a central extension of A_5 by Δ , and admits a faithful complex representation ϱ of dimension two. Any character of $N.A_5$ is determined by its restriction to Δ .*

Proof. Since the image of $N.A_5$ in $\mathrm{PGL}_2(\mathbf{F}_5)$ is A_5 and the kernel is Δ , it is clear that $N.A_5$ is a central extension. Central extensions of a group G by a cyclic group Δ of order N are classified by $H^2(G, \mu_N)$. It is a result (essentially of Schur) that $H^2(A_5, \mathbf{C}^\times) = \mathbf{Z}/2\mathbf{Z}$. Since A_5 is a perfect group, $H^1(A_5, \mathbf{C}^\times) = 0$. Taking the cohomology of the Kummer exact sequence $1 \rightarrow \mu_N \rightarrow \mathbf{C}^\times \rightarrow \mathbf{C}^\times \rightarrow 1$, we deduce that there is an isomorphism $H^2(A_5, \mu_N) = H^2(A_5, \mathbf{C}^\times)[N]$, and hence that both groups have order two if N is even. It follows that $N.A_5$ is either the unique non-split central extension or $N.A_5 \simeq A_5 \oplus \Delta$. Since A_5 is not a subgroup of $\mathrm{GL}_2(\overline{\mathbf{F}}_5)$, it follows that $N.A_5$ is non-split. Any projective morphism $A_5 \rightarrow \mathrm{PGL}_2(\mathbf{C})$ admits a non-trivial central extension in $\mathrm{GL}_2(\mathbf{C})$ of any any even degree — by uniqueness we identify these covers with $N.A_5$ which therefore admits a faithful representation ϱ . From this description, one may compute that the composite:

$$\Delta \rightarrow N.A_5 \hookrightarrow \mathrm{GL}_2(\mathbf{C}) \rightarrow \mathbf{C}^\times$$

surjects onto the image of $\det(N.A_5)$, and thus characters of $N.A_5$ are determined by their restriction to Δ .

Note that $N.A_5$ (respectively, A_5) admits an outer automorphism which is conjugation by an element of $N.S_5 \setminus N.A_5$ (respectively, $S_5 \setminus A_5$). We denote this automorphism by ι — this is not a particularly egregious abuse of notation since ι is compatible with the natural projection $N.A_5 \rightarrow A_5$. We are interested in representations of $N.S_5$, of $N.A_5$, and of the group S_5 . The character table of S_5 is given as follows:

$\langle g \rangle$	1	2A	2B	3A	4A	5A	6A
$ \langle g \rangle $	1	10	15	20	30	24	20
$ g $	1	2	2	3	4	5	6
1	1	1	1	1	1	1	1
η	1	-1	1	1	-1	1	-1
ρ_4	4	2	0	1	0	-1	-1
$\rho_4 \otimes \eta$	4	-2	0	1	0	-1	1
ρ_5	5	1	1	-1	-1	0	1
$\rho_5 \otimes \eta$	5	-1	1	-1	1	0	-1
ρ_6	6	0	-2	0	0	1	0

Definition 2 *If X is a representation of $N.A_5$ and Y is a representation of S_5 , say that $X \rightsquigarrow Y$ if the action of $N.A_5$ on X factors through the quotient A_5 , and the action of A_5 extends to an action of S_5 which is isomorphic to Y . On the other hand, if X is a representation of $N.A_5$ and Z is a representation of $N.S_5$, say that $X \rightsquigarrow Z$ if the action of $N.A_5$ on X extends to an action of $N.S_5$ which is isomorphic to Z .*

Lemma 3 *We have the following:*

1. $(\varrho \otimes \iota\varrho) \otimes \det(\varrho)^{-1} \rightsquigarrow \rho_4$,
2. $\mathrm{Sym}^4(\varrho) \otimes \det(\varrho)^{-2} \simeq \mathrm{Sym}^4(\iota\varrho) \otimes \det(\varrho)^{-2} \rightsquigarrow \rho_5$,
3. $\wedge^2((\varrho \otimes \iota\varrho) \otimes \det(\varrho)^{-1}) \rightsquigarrow \rho_6$.

Proof. The image of Δ under ϱ lies in the scalar matrices, by Schur's lemma. On the other hand, the involution ι fixes the centre Δ of $N.A_5$, and hence the restriction of $\iota\varrho$ to Δ is the same as the restriction of ϱ to Δ . The product of these restrictions is thus identified with $\det(\varrho)$, and thus the action of the centre Δ on $(\varrho \otimes \iota\varrho) \otimes \det(\varrho)^{-1}$ is trivial, and the action of $N.A_5$ factors through S_5 . On the other hand, the representation is preserved by the automorphism ι , and hence it lifts to a representation of S_5 , from which (1) follows easily.

The action of the centre on $\mathrm{Sym}^2(\varrho) \otimes \det(\varrho)^{-1}$ is trivial, and thus it corresponds to a 3-dimensional representation of A_5 , which must be one of the two irreducible faithful representations of A_5 . There is a plethysm:

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\varrho) \otimes \det(\varrho)^{-1}) = \mathrm{Sym}^4(\varrho) \otimes \det(\varrho)^{-2} \oplus 1$$

Viewing the left hand side as Sym^2 of an irreducible representation of A_5 , we may identify the non-trivial factor on the right hand side as the 5-dimensional irreducible representation of A_5 , which (from the character tables of A_5 and S_5) lifts to a representation of S_5 . We note, moreover, that this identification can equally be applied to $\iota\varrho$. This establishes (2).

The claim (3) follows directly from (1), as $\wedge^2 \rho_4 = \rho_6$.

On the other hand, we have the following:

Lemma 4 *There exists an irreducible four dimensional representation ξ of $N.S_5$ such that*

$$\mathrm{Sym}^3(\varrho) \rightsquigarrow N.S_5.$$

Proof. Let us first consider the projective representation:

$$\mathbf{PSym}^3(\varrho) : N.A_5 \rightarrow \mathrm{PGL}_4(\mathbf{C}).$$

The image depends only on the projective image of ϱ , and thus it factors through A_5 . Let us admit, for the moment, that this projective representation extends to a projective representation of S_5 . Then we obtain an identification of projective representations of $N.A_5$:

$$\mathbf{PSym}^3(\varrho) \simeq \mathbf{PSym}^3(\iota\varrho).$$

Any two irreducible representations with isomorphic projective representations are twists of each other. Since any one dimensional character of $N.A_5$ is determined by its restriction to Δ , and since $\mathrm{Sym}^3(\varrho)$ and $\mathrm{Sym}^3(\iota\varrho)$ are the same restricted to Δ , it follows that there must be an isomorphism:

$$\mathrm{Sym}^3(\varrho) \simeq \mathrm{Sym}^3(\iota\varrho).$$

It suffices, therefore, to establish that the fact about projective representations mentioned above. This is a fact which can be determined, for example, by an explicit computation with the Darstellungsgruppe of A_5 , the binary icosahedral group. Note that although the projective representation of ξ factors through S_5 , it is not equivalent to either of the projective representations obtained from the *linear* representations of S_5 .

We now describe, to some extent, the character ξ of $N.S_5$. We start by describing the projective representation associated to ξ . That is, for an element $\sigma \in S_5$, we give a matrix in $\text{GL}_2(\mathbf{C})$ lifting σ . In fact, we only do this for the *odd* permutations, since this is the only information we shall require. We use the same labeling of elements as in the character table of S_5 above. Here $\omega^{12} = 1$ is a primitive 12th root of unity:

$\langle g \rangle$	$2A$	$4A$	$6A$
$ \langle g \rangle $	10	30	20
$ g $	2	4	6
η	-1	-1	-1
$\mathbf{P}\xi$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & i & \\ & & & -i \end{pmatrix}$	$\begin{pmatrix} \omega^3 & & & \\ & \omega & & \\ & & \omega^{-1} & \\ & & & \omega^{-3} \end{pmatrix}$

There is an isomorphism $\mathbf{P}\xi^\vee \simeq \mathbf{P}\xi$, and hence an isomorphism $\xi^\vee \simeq \xi\psi$ for some character ψ . We have, moreover, that $\xi^c \simeq \xi^\vee$.

Lemma 5 *The representation ξ preserves a non-degenerate generalized symplectic pairing:*

$$\xi \times \xi \rightarrow \psi^{-1}.$$

There is an isomorphism

$$(\wedge^2 \chi) \otimes \psi \simeq 1 \oplus \rho_5.$$

Proof. Since $\xi^\vee \simeq \xi\psi$, there is certainly a non-degenerate pairing $\xi \times \xi \rightarrow \psi^{-1}$, it remains to determine whether this pairing is (generalized) symplectic or orthogonal. Yet the restriction of ξ to $N.A_5$ is $\text{Sym}^3(\varrho)$, which is irreducible and symplectic, and thus ξ is symplectic. It follows that $\wedge^2 \xi$ decomposes as ψ plus some 5-dimensional representation of $N.S_5$. Restricting to $N.A_5$, we have the plethysm

$$(\wedge^2 \text{Sym}^3 \varrho) \otimes \det(\varrho)^{-3} = 1 \oplus (\text{Sym}^4 \varrho) \otimes \det(\varrho)^{-2}.$$

Since the latter representation extends to ρ_5 , this shows that $(\wedge^2 \xi) \otimes \psi$ is either $1 \oplus \rho_5$ or $1 \oplus \rho_5 \otimes \eta$. To distinguish between these two representations, let us compute $\wedge^2 \xi$ on some conjugacy class σ in $N.S_5$ which maps to $6A$ in S_5 . We must have

$$\xi(\sigma) = \begin{pmatrix} \omega^3 \zeta & & & \\ & \omega \zeta & & \\ & & \omega^{-1} \zeta & \\ & & & \omega^{-3} \zeta \end{pmatrix}$$

for some root of unity ζ . It follows that $\wedge^2 \xi(\sigma)$ has eigenvalues:

$$\{\zeta^2, \zeta^2, \omega^2 \zeta^2, \omega^{-2} \zeta^2, \omega^4 \zeta^2, \omega^{-4} \zeta^2\}$$

Note that since ξ is (generalized) symplectic with similitude character ψ^{-1} , the eigenvalues of $\xi(\sigma)$ are of the form $\{\alpha, \beta, (\alpha\psi(\sigma))^{-1}, (\beta\psi(\sigma))^{-1}\}$. It follows that $\psi^{-1}(\sigma)$ occurs to multiplicity at least two in the eigenvalues of $\wedge^2 \xi(\sigma)$, and thus $\psi^{-1}(\sigma) = \zeta^2$. In particular, we deduce that

$$\mathrm{Tr}((\wedge^2 \xi) \otimes \psi)(\sigma) = 1 + 1 + \omega^2 + \omega^{-2} + \omega^4 + \omega^{-4} = 2.$$

Yet, if σ has conjugacy class $6A$ in S_5 , then $\mathrm{Tr}((1 \oplus \rho_5)(\sigma)) = 2$ whereas $\mathrm{Tr}((1 \oplus \rho_5 \otimes \eta)(\sigma)) = 0$.

Lemma 6 *Let $\sigma \in N.S_5$ be a lift of $2A$, and suppose that*

$$\xi(\sigma) = \begin{pmatrix} \zeta & & & \\ & \zeta & & \\ & & -\zeta & \\ & & & -\zeta \end{pmatrix}.$$

Then $\psi(\sigma) = -\zeta^{-2}$.

Proof. Suppose instead that $\psi(\sigma) = \zeta^{-2}$. Then $(\wedge^2 \xi)(\sigma)$ has eigenvalues:

$$\{\zeta^2, \zeta^2, -\zeta^2, -\zeta^2, -\zeta^2, -\zeta^2\}$$

If $\psi(\sigma) = \zeta^{-2}$, then the trace of $(\wedge^2 \otimes \psi^{-1})(\sigma)$ is -2 , whereas if $\psi(\sigma) = -\zeta^2$ then this trace is 2 . Yet if σ is of type $2A$, then $\mathrm{Tr}((1 \oplus \rho_5)(\sigma)) = 2$.

3. Irreducible representations of dimension $n = 4$ and $n = 6$

Let $K^{\mathrm{gal}}/\mathbf{Q}$ be an S_5 -extension satisfying the conditions of Theorem 1.1. Let E/\mathbf{Q} denote the quadratic subfield of K^{gal} . By assumption, E is real, and 5 and 3 both split in E .

Lemma 7 *There exists an Galois extension $\widetilde{K}^{\mathrm{gal}}/\mathbf{Q}$ with $\mathrm{Gal}(\widetilde{K}^{\mathrm{gal}}/\mathbf{Q}) \simeq N.S_5$ for some N .*

Proof. There is an exceptional isomorphism $\mathrm{PGL}_2(\mathbf{F}_5) \simeq S_5$, giving rise to a representation

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{PGL}_2(\overline{\mathbf{F}}_5)$$

which factors through $\mathrm{Gal}(K^{\mathrm{gal}}/\mathbf{Q})$. The cohomology group $H^2(G_{\mathbf{Q}}, \overline{\mathbf{F}}_5^\times)$ is trivial by a theorem of Tate (see [Ser77]). It follows that the projective representation lifts to a linear representation over $\overline{\mathbf{F}}_5$ with finite image. By a classification of subgroups of $\mathrm{GL}_2(\overline{\mathbf{F}}_p)$, the image must land in $\mathrm{GL}_2(\mathbf{F}_5)\overline{\mathbf{F}}_5^\times$, proving the lemma.

It follows that there exists a corresponding complex representation:

$$\varrho : G_E \rightarrow \mathrm{Gal}(\widetilde{K^{\mathrm{gal}}}/E) \rightarrow \mathrm{GL}_2(\mathbf{C})$$

with projective image A_5 . By assumption, ϱ is odd at both real places of E .

Lemma 8 *The representation ϱ is automorphic for $\mathrm{GL}(2)/E$.*

Proof. This follows immediately from Theorem 1 of [Sas11b], given the assumed conditions on $\varrho(\mathrm{Frob}_5)$.

Assume that ρ is irreducible and of dimension $n = 4$ or $n = 6$. Let π_E denote the automorphic form for $\mathrm{GL}(2)/E$ associated to ϱ by Lemma 8. There exists a conjugate representation $\iota(\varrho)$ given by applying the outer involution to ϱ , which corresponds (by another application of Lemma 8) to an automorphic form which we call $\iota(\pi_E)$.

Suppose that $n = 4$. The Asai transfer $\mathrm{Asai}(\pi_E)$ is automorphic and cuspidal for $\mathrm{GL}(4)/\mathbf{Q}$, as follows by Theorem D of [Ram02] and Theorem B of [PR11]. Yet, by Lemma 3 part (1), we deduce that $\mathrm{Asai}(\pi_E)$ corresponds (by Lemma 3) exactly to a twist of ρ_4 , from which the main result follows.

Suppose that $n = 6$. Then $\rho = \rho_6 = \wedge^2 \rho_4$. We have already shown that ρ_4 is automorphic and corresponds to some cuspidal π for $\mathrm{GL}(4)/\mathbf{Q}$. By a result of Kim [Kim03], the representation $\wedge^2 \pi$ is automorphic for $\mathrm{GL}(6)/\mathbf{Q}$, and by [AR11], it is a simple matter to check that it is cuspidal. Hence ρ_6 corresponds (by Lemma 3) to $\wedge^2 \pi$, from which the main result follows. Alternatively, we may consider the automorphic form $\mathrm{Sym}^2 \pi_E$. The corresponding Galois representation $\mathrm{Sym}^2(\varrho) \otimes \det(\varrho)^{-1}$ has image A_5 , but is not isomorphic to its Galois conjugate (as S_5 has no irreducible representations of dimension three). Thus one may form the automorphic induction of $\mathrm{Sym}^2(\pi_E)$ to $\mathrm{GL}(6)/\mathbf{Q}$, which is cuspidal (by cyclic base change) and corresponds to the Galois representation ρ_6 .

4. Irreducible representations of dimension $n = 5$

Proving Langlands' automorphy conjecture for $\rho = \rho_5$ or $\rho_5 \otimes \eta$ is somewhat harder, for reasons which we now explain. The symmetric fourth power $\mathrm{Sym}^4(\pi_E)$ of π_E is automorphic and cuspidal [Kim03,KS02]. On the other hand, $\mathrm{Sym}^4(\varrho) \otimes \det(\varrho)^{-2}$ is invariant under the involution of $\mathrm{Gal}(E/\mathbf{Q})$, and thus, by multiplicity one [JS81b] and cyclic base change (Theorem 4.2 (p.202) of [AC89]), the corresponding twist $\mathrm{Sym}^4(\pi_E) \times \det(\pi_E)^{-2}$ arises from an automorphic form ϖ for $\mathrm{GL}(5)/\mathbf{Q}$. The Galois representation ρ restricted to G_E corresponds to this automorphic form. We would like to show that ρ (or its quadratic twist) corresponds to ϖ . There is a well known problem, however, that since this descended form is not known a priori to admit a Galois representation, all we can deduce is that the collection of Satake parameters

for ϖ and $\varpi \otimes \eta$ are the same as the collection of Frobenius eigenvalues of ρ and $\rho \otimes \eta$. This implies that $L(\rho, s)L(\rho \times \eta, s)$ is holomorphic, but gives no information about $L(\rho, s)$. This problem is the main obstruction to proving the Artin conjecture for solvable representations.

The usual game in these situations is to play off several representations against each other and use functorialities known in small degrees. It turns out — in this instance — that it is profitable to work instead with the representation ξ of the group $N.S_5$ in dimension four. Let ϱ denote one of the two dimensional representations of $N.A_5$. Since $\text{Sym}^3(\varrho)$ is equal to its conjugate under $\text{Gal}(E/\mathbf{Q})$ (by Lemma 3), we deduce by multiplicity one that the same is true of $\text{Sym}^3(\pi_E)$. It follows by cyclic base change that there exists an automorphic form Π for $\text{GL}(4)/\mathbf{Q}$ such that $\Pi_E \simeq \text{Sym}^3(\pi_E)$. Conjecturally, Π is associated to the Galois representation:

$$\xi : \widetilde{\text{Sym}^3(\varrho)} : \text{Gal}(\widetilde{K^{\text{gal}}}/\mathbf{Q}) \simeq N.S_5 \hookrightarrow \text{GL}_4(\mathbf{C}).$$

Let S_x denote the 4-tuple of the Satake parameters of Π_x . By abuse of notation, we write $\xi(x)$ and $\eta(x)$ for $\xi(\text{Frob}_x)$ and $\eta(\text{Frob}_x)$ respectively.

Lemma 9 *If $\eta(x) = +1$, then S_x consists of the eigenvalues of $\xi(x)$. If $\eta(x) = -1$, then the union $S_x \cup -S_x$ consists of the eigenvalues of $\xi(x)$ together with the eigenvalues of $-\xi(x)$.*

Proof. This follows from the local compatibility between Π_E and $r|G_E$.

Since $\Pi_E^\vee \simeq \Pi_E \otimes \psi_E$ (by multiplicity one) we deduce (again by multiplicity one) that $\Pi^\vee \simeq \Pi \otimes \nu^{-1}$ for some character ν such that $\nu_E \psi_E$ is trivial. It follows that either $\nu = \psi^{-1}$ or $\nu = \eta\psi^{-1}$. Since $\xi^\vee \simeq \xi\psi$, it should be the case that $\nu = \psi^{-1}$, although there does not seem to be any apparent way to prove this a priori. We shall, however, prove that $\nu = \psi^{-1}$ in Lemma 14 below.

Lemma 10 *Π is of symplectic type, and ν is the corresponding similitude character. If x is an unramified prime, then the Satake parameters of Π_x are of the form:*

$$\{\alpha, \beta, \nu(x)/\alpha, \nu(x)/\beta\}.$$

Proof. It suffices to show that $\wedge^2 \Pi$ is non-cuspidal, and then we can deduce the result from Theorem 1.1(ii) of [AR11]. Since Π_E is symplectic, we know that $\wedge^2 \Pi_E$ is non-cuspidal. Assume that $\wedge^2 \Pi$ is cuspidal. Then the base change of $\wedge^2 \Pi$ to E is $\wedge^2 \Pi_E$. In other words, $\wedge^2 \Pi$ becomes non-cuspidal after base change. By Theorem 4.2 (p.202) of [AC89], it follows that $\wedge^2 \Pi$ is the automorphic induction of a cuspidal form μ from $\text{GL}(3)/E$. We deduce that $\wedge^2 \Pi_E = \mu \boxplus \mu$, which is incompatible with the fact that the Satake parameters of $\wedge^2 \Pi_E$ do not all have multiplicity two. Thus $\wedge^2 \Pi$ is not cuspidal, and from the classification [AR11], we deduce that Π is symplectic (all other possibilities contradict the known structure of Π_E). If Π has similitude

character $\tilde{\nu}$, then $\Pi \simeq \Pi^\vee \otimes \tilde{\nu}$. It follows that $\Pi \simeq \Pi^\vee \otimes \tilde{\nu} \simeq \Pi \otimes \tilde{\nu}\nu^{-1}$. It is easy to check that Π is not induced from a quadratic subfield, and hence we must have $\tilde{\nu} \simeq \nu$.

The second wedge $\wedge^2 \Pi$ decomposes as an isobaric sum of the similitude character ν together with another isobaric representation, and thus we may write character ν^{-1} , and thus we may write

$$(\wedge^2 \Pi) \otimes \nu^{-1} = \varpi \boxplus 1.$$

By considering Galois representations, we see that ϖ_E corresponds to $\rho|_E$, and ϖ conjecturally corresponds to ρ_5 . It also follows from this that ϖ is cuspidal for $\mathrm{GL}(5)/\mathbf{Q}$.

Let T_x denote the 5-tuple of Satake parameters of ϖ . If the Satake parameters S_x of Π_x are of the form $S_x = \{\alpha, \beta, \nu(x)/\alpha, \nu(x)/\beta\}$, then the Satake parameters T_x of ϖ are of the form

$$\left\{ 1, \alpha\beta/\nu(x), \nu(x)/\alpha\beta, \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right\}$$

Definition 11 *For the automorphic representations Π and ϖ , let*

$$\chi(\Pi, x) = \sum_{S_x} \alpha, \quad \chi(\varpi, x) = \sum_{T_x} \alpha$$

denote the sum of the Satake parameters. For the Galois representations ξ and ρ , let $\chi(\xi, x)$ and $\chi(\rho, x)$ denote the sum of the eigenvalues of $\xi(x)$ and $\rho(x)$ respectively.

Conjecturally, we have $\chi(\Pi, x) = \chi(\xi, x)$ and $\chi(\varpi, x) = \chi(\rho, x)$. Note that $\chi(\rho, x)$ and $\chi(\varpi, x)$ are real valued; the former because ρ is a real representation, and the latter because T_x consists of 1 together with two pairs $\{\zeta, \zeta^{-1}\}$ for a root of unity ζ . We shall consider the following quantity:

Definition 12 *For a prime x , let*

$$\phi(x) = \|\chi(\Pi, x)\|^2 - \|\chi(\xi, x)\|^2 - (\chi(\varpi, x)^2 - \chi(\rho, x)^2).$$

If $\eta(x) = 1$ then S_x consists of the eigenvalues of $\xi(x)$, and T_x consists of the eigenvalues of $\rho(x)$, and hence $\phi(x) = 0$.

We now consider the possible values of $\phi(x)$ as well as the structure of T_x for x with $\eta(x) = -1$. Note that we do not know at this point whether $\nu \simeq \psi^{-1}$ or $\nu \simeq \psi^{-1}\eta$, so we shall have to take both possibilities into account.

1. Suppose that the conjugacy class of Frob_x is $2A$ in S_5 . Then from the (projective) character table of χ , we deduce that

$$S_x \cup -S_x = \{\zeta, \zeta, -\zeta, -\zeta\} \cup \{\zeta, \zeta, -\zeta, -\zeta\}.$$

By Lemma 6, we know that $\psi(x) = -\zeta^{-2}$. Hence either $\nu(x) = -\zeta^2$ if $\nu = \psi$, or $\nu(x) = \zeta^2$ if $\nu = \psi\eta$. Using the known shape of S_x , we deduce that one of the following possibilities holds (we only concern ourselves with identifying S_x up to sign.)

ν^{-1}	$\nu(x)$	$\pm S_x$	$\ \chi(\Pi, x)\ ^2$	T_x	$\chi(\varpi, x)^2$	$\phi(x)$
ψ	$-\zeta^2$	$\{\zeta, \zeta, -\zeta, -\zeta\}$	0	$\{1, 1, 1, -1, -1\}$	1	0
$\psi\eta$	ζ^2	$\{\zeta, \zeta, -\zeta, -\zeta\}$	0	$\{1, -1, -1, -1, -1\}$	9	-8
$\psi\eta$	ζ^2	$\{\zeta, \zeta, \zeta, \zeta\}$	16	$\{1, 1, 1, 1, 1\}$	25	-8

Both here and in the two tables below, the first line of each table represents the (conjectural) reality.

2. Suppose that the conjugacy class of Frob_x is $4A$ in S_5 . Then from the (projective) character table of χ , we deduce that

$$S_x \cup -S_x = \{\zeta, i\zeta, -\zeta, -i\zeta\} \cup \{\zeta, i\zeta, -\zeta, -i\zeta\}.$$

By computing multiplicities in $\wedge^2 S_x$ we deduce that $\psi(x) = i\zeta^2$ or $-i\zeta^2$ but we cannot pin down $\psi(x)$ exactly — indeed, the group $N.S_5$ can (and does) contain distinct conjugacy classes with these same eigenvalues and with values of ψ that differ (up to sign). It follows that the possibilities below are the same regardless whether $\nu = \psi^{-1}$ or $\nu = \psi^{-1}\eta$.

$\nu(x)$	$\pm S_x$	$\ \chi(\Pi, x)\ ^2$	T_x	$\chi(\varpi, x)^2$	$\phi(x)$
$i\zeta^2$	$\{\zeta, i\zeta, -\zeta, -i\zeta\}$	0	$\{1, i, -i, -1, -1\}$	1	0
$i\zeta^2$	$\{\zeta, i\zeta, \zeta, i\zeta\}$	8	$\{1, i, -i, 1, 1\}$	9	0
$-i\zeta^2$	$\{\zeta, i\zeta, -\zeta, -i\zeta\}$	0	$\{1, i, -i, -1, -1\}$	1	0
$-i\zeta^2$	$\{\zeta, -i\zeta, \zeta, -i\zeta\}$	8	$\{1, i, -i, 1, 1\}$	9	0

3. Suppose that the conjugacy class of Frob_x is $6A$ in S_5 . Then, as above, we may write

$$S_x \cup -S_x = \{\omega^3\zeta, \omega\zeta, \omega^{-1}\zeta, \omega^{-3}\zeta\} \cup \{-\omega^3\zeta, -\omega\zeta, -\omega^{-1}\zeta, -\omega^{-3}\zeta\},$$

where $\omega^{12} = 1$. A key point in the computation below is that $\omega^6 = -1$, and so $\omega^{-3} = -\omega^3$. Here $\psi(x) = \zeta^{-2}$, and so $\nu(x) = \zeta^2$ if $\nu = \psi^{-1}$ and $-\zeta^2$ otherwise.

We deduce that the following possibilities may occur:

ν^{-1}	$\nu(x)$	$\pm S_x$	$\ \chi(\Pi, x)\ ^2$	T_x	$\chi(\varpi, x)^2$	$\phi(x)$
ψ	ζ^2	$\{\omega^3\zeta, \omega\zeta, \omega^{-1}\zeta, \omega^{-3}\zeta\}$	3	$\{1, \omega^2, \omega^{-2}, \omega^4, \omega^{-4}\}$	1	0
$\psi\eta$	$-\zeta^2$	$\{\omega^3\zeta, \omega\zeta, -\omega^{-1}\zeta, -\omega^{-3}\zeta\}$	9	$\{1, \omega^2, \omega^{-2}, -\omega^4, -\omega^{-4}\}$	9	2
$\psi\eta$	$-\zeta^2$	$\{\omega^3\zeta, -\omega\zeta, \omega^{-1}\zeta, -\omega^{-3}\zeta\}$	1	$\{1, -\omega^2, -\omega^{-2}, \omega^4, \omega^{-4}\}$	1	2

Lemma 13 *The function*

$$\frac{L(\Pi \times \overline{\Pi}, s)}{L(\varpi \times \overline{\varpi}, s)} \cdot \frac{L(\rho_5 \times \overline{\rho_5}, s)}{L(\xi \times \overline{\xi}, s)}$$

is meromorphic for $\operatorname{Re}(s) > 0$, and is holomorphic in some neighbourhood of $s = 1$.

Proof. By a theorem of Jacquet and Shalika ([JS81a]), the Rankin–Selberg L -functions are meromorphic for $\operatorname{Re}(s) > 0$ with a simple pole each at $s = 1$. The same is true of the Artin L -functions by Brauer’s theorem.

Lemma 14 *There is an equality $\nu = \psi^{-1}$.*

Proof. Assume otherwise. From the Euler product, we find that, as $s \rightarrow 1^+$,

$$\log \left| \frac{L(\Pi \times \overline{\Pi}, s)}{L(\varpi \times \overline{\varpi}, s)} \cdot \frac{L(\rho_5 \times \overline{\rho_5}, s)}{L(\xi \times \overline{\xi}, s)} \right| = \sum \frac{\phi(p)}{p^s} + O(1).$$

Then, from the tables above, we compute that $\phi(x) = 0$ unless the projective image of $\xi(\operatorname{Frob}_x)$ is of type $2A$, in which case it is -8 , or $6A$, in which case it is $+2$. By The Chebotarev density theorem, the class $2A$ has density $1/12 = 10/120$ each respectively. Similarly, the class $6A$ has Dirichlet density $1/6$. Thus the RHS is asymptotic to

$$\log |(s-1)| \left(\frac{-8}{12} + \frac{2}{6} \right) + O(1) = -\frac{1}{3} \log |(s-1)| + O(1).$$

This contradicts Lemma 13.

Theorem 4.1 *Let B denote the finite set of places for which ϖ_p is not unramified. For all primes outside a set $\Omega \cup B$ of Dirichlet density zero, T_x consists of the eigenvalues of $\rho(\operatorname{Frob}_x)$. If x lies in Ω , then Frob_x in S_5 has conjugacy class $4A$, and $T_x = \{1, i, -i, 1, 1\}$ rather than $\{1, i, -i, -1, -1\}$.*

Proof. Since $\nu = \psi^{-1}$, the result follows automatically for all conjugacy classes by our computation above except for the density claim concerning Ω . Assume otherwise. Denote the set of such primes in these conjugacy classes with $\|\chi(\Pi, x)\|^2 = 8$ by Ω . Then we compute that

$$-\log \left| \frac{L(\Pi \times \overline{\Pi}, s)}{L(\xi \times \overline{\xi}, s)} \right| \sim \sum_{\Omega} \frac{8}{p^s} + O(1).$$

Once more by Jacquet–Shalika, we deduce that the LHS is bounded, and hence the RHS also has order $o(|\log(s-1)|^{-1})$, and hence that Ω has Dirichlet density zero.

We now summarize what we have shown so far: Namely, in comparing the Artin representation ξ with the automorphic form Π , we have that for a set of x outside the set Ω of density zero, the Satake parameters S_x of Π_x agree with $\xi(\text{Frob}_x)$ up to sign. In particular, outside the same set, the Satake parameters T_x of ϖ_x agree with the eigenvalues of $\rho_5(\text{Frob}_x)$. This completes the proof of Theorem 1.1.

Unfortunately, we do not see an unconditional argument at this point for establishing an equivalence at all places. On the other hand, we know (by construction) precisely the Satake parameters of ϖ at the “troublesome” primes $p \in \Omega$. The special form of these parameters will allow us to prove Theorem 1.2.

Lemma 15 *Let $\mu(s) := \mu_B(s)\mu_\Omega(s)$, where $\mu_\Omega(s)$ denotes the function*

$$\mu_\Omega(s) := \prod_{\Omega} \left(\frac{1 + \frac{1}{p^s}}{1 - \frac{1}{p^s}} \right)^2.$$

and $\mu_B(s) = \frac{L_\infty(\varpi, s)}{L_\infty(\rho_5, s)} \prod_B \frac{L_p(\varpi, p^{-s})}{L_p(\rho_5, p^{-s})}$, where the product is over the finite set of primes of bad reduction of ϖ and ρ_5 . Then the following holds:

1. There is an equality $L(\varpi, s) = L(\rho_5, s)\mu(s)$,
2. $\mu(s)$ extends to a meromorphic function on the complex plane.
3. $\mu_B(s)$ is holomorphic and non-vanishing on the interval $s \in (0, 1)$.
4. Either Ω is finite, or $\mu(s)$ has a pole on the real axis with $s \in (0, 1)$.

Proof. We have that $L(\varpi, s) = L_\infty(\varpi, s) \prod L_p(\varpi, p^{-s})^{-1}$ and $L(\rho_5, s) = L_\infty(\rho_5, s) \prod L_p(\rho \otimes \eta, p^{-s})$, where the local factors agree for $p \notin \{\infty\} \cup B \cup \Omega$. For the exceptional $p \in \Omega$, the corresponding polynomials are:

$$L_p(\varpi, X) = (1 + X^2)(1 - X)^3, \quad L_p(\rho, X) = (1 + X^2)(1 - X)(1 + X)^2.$$

Comparing the two sides leads to the equality (1). We claim that the Gamma factors of both L -functions involve only the standard Γ factors $\Gamma_{\mathbf{R}}(s)$ and $\Gamma_{\mathbf{C}}(s)$. This is true for $L_\infty(\rho_5, s)$ by Artin, and is true for $L_\infty(\varpi, s)$ because of the identity:

$$L_\infty(\varpi, s)L_\infty(\varpi \otimes \eta, s) = L_\infty(\varpi_E, s).$$

Since $\Gamma_{\mathbf{R}}(s)$ and $\Gamma_{\mathbf{C}}(s)$ are both holomorphic and without zeros on $(0, 1)$, so is any ratio of products of such functions. The factors $L_p(\rho, p^{-s})$ and $L_p(\varpi, p^{-s})$ for finite bad primes are of the form $P(p^{-s})$ where $P(T)$ is a polynomial whose roots are roots of unity. In particular, the only poles and zeros of $\mu_B(s)$ occur for complex s such that $p^{ns} = 1$ for some $n \in \mathbf{Z}$, which cannot happen if $s \in (0, 1)$ is real. Hence $\mu_B(s)$ is holomorphic and non-vanishing on $(0, 1)$. It therefore suffices to assume that Ω is infinite, and prove that $\mu_\Omega(s)$ has a pole for $s \in (0, 1)$. The Taylor series coefficients of

$$\frac{(1+x)^2}{(1-x)^2} = 1 + \sum_{n=1}^{\infty} 4nx^n$$

are all positive, and hence the Dirichlet series for $\mu_\Omega(s)$ has non-negative coefficients.

Lemma 16 *Let $L(s) := \sum a_n n^{-s}$ be a Dirichlet series with non-negative real coefficients which has a meromorphic continuation to the entire complex plane, and which converges absolutely for at least one point of \mathbf{C} . Then either $L(s)$ converges absolutely for all s , or there exists a $\gamma \in \mathbf{R}$ such that $L(s)$ is absolutely convergent for all $\operatorname{Re}(s) > \gamma$ and has a pole at $s = \gamma$.*

Proof. The abscissa of absolute convergence of any Dirichlet series consists of either \emptyset , \mathbf{C} , or a half plane $\operatorname{Re}(s) > \gamma$ for some $\gamma \in \mathbf{R}$ (Theorem 8 p.8 of [HR15]). By assumption, the abscissa of convergence is non-empty. Hence it suffices to prove that if γ exists, and the coefficients of $L(s)$ are non-negative, then $L(s)$ has a pole at $s = \gamma$. Yet this is Theorem 10 p.10 of [HR15].

If Ω is infinite, then $\mu_\Omega(s)$ is manifestly not absolutely convergent for $s = 0$. On the other hand, $\mu_\Omega(s)$ is absolutely convergent for $s = 1$ as follows from the proof of Theorem 4.1. Hence, by Lemma 16, $\mu_\Omega(s)$ is absolutely convergent for $s > \gamma$ and has a pole at γ for some $\gamma \in [0, 1)$. It suffices to show that $\gamma \neq 0$ — assume otherwise. Choose any finite subset Φ of Ω . Then

$$\mu_\Omega(s) = \prod_{\Phi} \left(\frac{1 + \frac{1}{p^s}}{1 - \frac{1}{p^s}} \right)^2 \prod_{\Omega \setminus \Phi} \left(\frac{1 + \frac{1}{p^s}}{1 - \frac{1}{p^s}} \right)^2 =: \mu_\Phi(s) \mu_{\Omega \setminus \Phi}(s).$$

The Dirichlet series for $\mu_{\Omega \setminus \Phi}(s)$ also has real non-negative coefficients and (by assumption) is holomorphic for $s > 0$. Thus, by Lemma 16, the Dirichlet series for $\mu_{\Omega \setminus \Phi}(s)$ also converges absolutely for $s > 0$. It follows that $\mu_{\Omega \setminus \Phi}(s) \geq 1$ for real $s > 0$, and hence that $\mu_\Omega(s) \geq \mu_\Phi(s)$ for real $s > 0$. Yet $\mu_\Phi(s)$ has a pole of order $2|\Phi|$ at $s = 0$, and thus the pole of $\mu_\Omega(s)$ at $s = 0$ must also be of at least this order. Since Φ was chosen arbitrarily, this is a contradiction, and thus (if Ω is infinite) $\mu_\Omega(s)$ must have a pole in $(0, 1)$.

We now upgrade this lemma to deduce that if Ω is finite, then $L(\varpi, s)$ is equal to $L(\rho_5, s)$ on the nose.

Lemma 17 *If Ω is finite, then $L(\varpi, s) = L(\rho_5, s)$.*

Proof. It suffices to show that the L -factors agree at any place. By assumption, there exists a finite set S of places at which they differ. The proof is similar to Proposition 4.1 of [Ram02], which we follow closely, although it is easier, because we have more explicit information about the Γ -factors at infinity. Indeed, as in *ibid.*, we may find a ramified character ς such that the L -factors of $\varpi \times \varsigma$ and $\rho_5 \otimes \varsigma$ are trivial at finite places dividing S . If we furthermore assume that ς is real, then the corresponding L -factors at infinity do not change. Then, from the functional equations, we deduce that

$$L_\infty(s, \varpi) L_\infty(\rho_5, 1 - s) = L_\infty(1 - s, \varpi) L_\infty(\rho_5, s).$$

Each of these L -factors $L_\infty(s)$ is a product of terms of the form $\Gamma_{\mathbf{R}}(s)$ and $\Gamma_{\mathbf{C}}(s)$, and thus is holomorphic and invertible for $\operatorname{Re}(s) > 0$. It follows that we may identify the polar divisor of $L_\infty(\rho_5, s)$ with the polar divisor of $L_\infty(\varpi, s)$, and then deduce they are equal, by the Baby Lemma of [Ram02]. The equality at the remaining finite places then follows by twisting with a character ζ that is highly ramified at all but one finite place v in S , and split completely at v . Comparing functional equations as in the archimedean case, we deduce an equality of L -factors at v .

We now complete the proof of Theorem 1.2. Since $L(\varpi, s)$ is holomorphic, it follows that if $\mu(s)$ has a real pole for $s \in (0, 1)$, then $L(\rho_5, s)$ has a zero in the same interval. On the other hand,

$$\zeta_H(s) = \zeta_F(s)L(\eta, H/F, s) = \zeta_F(s)L(\eta, E/\mathbf{Q}, s)L(\rho_5, s),$$

and by results of Hecke and Riemann, $\zeta_F(s)$ and $L(\eta, E/\mathbf{Q}, s)$ are holomorphic in $s \in (0, 1)$, and so $\zeta_H(s)$ also has a zero in this interval. If not, then by Lemma 15, Ω is finite, and by Lemma 17, we have equalities $L(\varpi, s) = L(\rho_5, s)$ and $L(\varpi \times \eta, s) = L(\rho_5 \otimes \eta, s)$, implying that the latter functions are automorphic. This completes the proof of Theorem 1.2.

5. An Example

As an example of Theorem 1.2, with help from a custom computation done for us by Andrew Booker, we prove the Artin conjecture for the Galois closure of the quintic field of smallest discriminant.

Theorem 5.1 *Let $K = \mathbf{Q}(x)/(x^5 - x^3 - x^2 + x + 1)$. and let K^{gal} denote the Galois closure of K . Then any complex representation of $\operatorname{Gal}(K^{\text{gal}}/\mathbf{Q})$ is automorphic.*

Proof. The discriminant of K is $\Delta_K = 1609$, there is an isomorphism $G := \operatorname{Gal}(K^{\text{gal}}/\mathbf{Q}) = S_5$, and K has signature $(1, 2)$, so complex conjugation is conjugate to $(12)(34)$. The prime 5 is inert in \mathcal{O}_K , however, so one cannot apply Sasaki's theorem directly. Instead, let us consider the corresponding mod-2 representation

$$\bar{\varrho} : G_{\mathbf{Q}(\sqrt{1609})} \rightarrow \operatorname{SL}_2(\mathbf{F}_4)$$

that is unramified at all finite places. By Theorem 2 of [Sas11a], to establish the modularity of the complex two dimensional representation with projective image $\operatorname{Gal}(K^{\text{gal}}/\mathbf{Q})$, it suffices to prove that $\bar{\varrho}$ is modular. (Note that 2 is totally split in E , and unramified in K^{gal} , and that $\bar{\varrho}$ is 2-distinguished because Frob_2 has order 5 in S_5 .)

Lemma 18 *The representation ϱ is modular of weight $(2, 2)$ and level one for $\operatorname{GL}(2)/E$.*

Proof. Using a magma program written by Lassina Dembélé, one may verify the following facts:

1. There exists a form f with coefficients in \mathbf{F}_4 which is an eigenform for all the Hecke operators $T_{\mathfrak{p}}$ with $N(\mathfrak{p})$ odd.
2. The image of $\bar{\varrho}_f$ surjects onto $\mathrm{SL}_2(\mathbf{F}_4)$.
3. If $\sigma(f)$ denotes the conjugate of f by $\mathrm{Gal}(E/\mathbf{Q})$, and $\mathrm{Frob}(f)$ denotes the image of f under the Frobenius automorphism acting on the coefficient field \mathbf{F}_4 , then $f \neq \sigma(f)$, but

$$\sigma(f) = \mathrm{Frob}(f).$$

Since the kernel of $\bar{\varrho}_{\mathrm{Frob}(f)}$ is the same as the kernel of $\bar{\varrho}_f$, we deduce that this kernel defines an A_5 -extension of E which is Galois over \mathbf{Q} . Since $f \neq \sigma(f)$, it follows that this must be a non-split extension, and thus define an S_5 -extension of \mathbf{Q} . Using Fontaine's bounds [Fon85] for root discriminants of finite flat group schemes, we deduce that the discriminant of quintic subfield divides $1609 \cdot 2^9$ (the 2-adic valuation of the root discriminant must be strictly less than $1 + 1/(2-1) = 2$). The only quintic subfield with this property is K (see [SPDyD94]), and so $\bar{\varrho}_f = \bar{\varrho}$, which is thus modular. (Note that Frob acting on $\mathrm{SL}_2(\mathbf{F}_4)$ is the outer automorphism ι of A_5 discussed previously coming from the inclusion of A_5 in S_5 .)

As in the proof of Theorem 1.2, to complete the proof of Theorem 5.1 it suffices to prove that $L(\rho_5, s)$ does not vanish for $s \in (0, 1)$. In [Boo06], Booker found an algorithm that allows one to unconditionally verify the Artin conjecture and the GRH for Artin L -functions of S_5 -representations (and many other groups) in any bounded range within the critical strip. Booker has carried out his algorithm in this case, and one finds the following:

Theorem 5.2 (Booker) *The lowest lying zero of $L(\rho_5, s)$ occurs at*

$$s = \frac{1}{2} + \gamma i, \quad \text{where } \gamma = 1.624\dots > 0.$$

(Moreover, all the zeros of $L(\rho_5, s)$ with $|\Im(s)| < 100$ lie on the critical line.)

In particular, $L(\rho_5, s)$ does not vanish for real $s \in (0, 1)$, proving Theorem 5.1.

References

- [AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR MR1007299 (90m:22041)
- [AR11] Mahdi Asgari and A. Raghuram, *A cuspidality criterion for the exterior square transfer of cusp forms on $\mathrm{GL}(4)$* , On certain L -functions, Clay Math. Proc., vol. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 33–53. MR 2767509
- [Arm72] J. V. Armitage, *Zeta functions with a zero at $s = \frac{1}{2}$* , Invent. Math. **15** (1972), 199–205. MR 0291122 (45 #216)

-
- [BDSBT01] Kevin Buzzard, Mark Dickinson, Nick Shepherd-Barron, and Richard Taylor, *On icosahedral Artin representations*, *Duke Math. J.* **109** (2001), no. 2, 283–318. MR 1845181 (2002k:11078)
- [Boo06] Andrew R. Booker, *Artin’s conjecture, Turing’s method, and the Riemann hypothesis*, *Experiment. Math.* **15** (2006), no. 4, 385–407. MR 2293591 (2007k:11084)
- [Fon85] Jean-Marc Fontaine, *Il n’y a pas de variété abélienne sur \mathbf{Z}* , *Invent. Math.* **81** (1985), no. 3, 515–538. MR 807070 (87g:11073)
- [HR15] G. H. Hardy and M. Riesz, *The general theory of Dirichlet’s series*, *Cambridge Tracts in Mathematics and Mathematical Physics*, No. 18, Cambridge University Press, 1915.
- [JS81a] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, *Amer. J. Math.* **103** (1981), no. 3.
- [JS81b] ———, *On Euler products and the classification of automorphic representations. II*, *Amer. J. Math.* **103** (1981), no. 4, 777–815.
- [Kim03] Henry H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* , *J. Amer. Math. Soc.* **16** (2003), no. 1, 139–183 (electronic), With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR 1937203 (2003k:11083)
- [Kim04] ———, *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type*, *Invent. Math.* **156** (2004), no. 3, 495–502. MR 2061327 (2005f:11101)
- [KS02] Henry H. Kim and Freydoon Shahidi, *Cuspidality of symmetric powers with applications*, *Duke Math. J.* **112** (2002), no. 1, 177–197. MR 1890650 (2003a:11057)
- [KW09] Chandrashekhara Khare and Jean-Pierre Wintenberger, *Serre’s modularity conjecture. I*, *Invent. Math.* **178** (2009), no. 3, 485–504. MR 2551763 (2010k:11087)
- [Lan80] Robert P. Langlands, *Base change for $GL(2)$* , *Annals of Mathematics Studies*, vol. 96, Princeton University Press, Princeton, N.J., 1980. MR 574808 (82a:10032)
- [PR11] Dipendra Prasad and Dinakar Ramakrishnan, *On the cuspidality criterion for the Asai transfer to $GL(4)$* , preprint, 2011.
- [Ram02] Dinakar Ramakrishnan, *Modularity of solvable Artin representations of $GO(4)$ -type*, *Int. Math. Res. Not.* (2002), no. 1, 1–54. MR 1874921 (2003b:11049)
- [Sas11a] Shu Sasaki, *On Artin representations and nearly ordinary Hecke algebras over totally real fields. I*, preprint, 2011.
- [Sas11b] ———, *On Artin representations and nearly ordinary Hecke algebras over totally real fields. II*, preprint, 2011.
- [Ser77] J.-P. Serre, *Modular forms of weight one and Galois representations*, *Algebraic number fields: L -functions and Galois properties* (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, pp. 193–268.
- [SPDyD94] A. Schwarz, M. Pohst, and F. Diaz y Diaz, *A table of quintic number fields*, *Math. Comp.* **63** (1994), no. 207, 361–376. MR 1219705 (94i:11108)
- [Tun81] Jerrold Tunnell, *Artin’s conjecture for representations of octahedral type*, *Bull. Amer. Math. Soc. (N.S.)* **5** (1981), no. 2, 173–175.