CUSPIDAL COHOMOLOGY CLASSES FOR $\text{GL}_n(\mathbb{Z})$

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To Laurent Clozel, in admiration.

Abstract. We prove the existence of a regular algebraic cuspidal automorphic representation $\pi$ for $\text{GL}_{105}/\mathbb{Q}$ of level one and weight zero. We construct $\pi$ using symmetric power functoriality and Galois deformation theory. As a corollary, we construct the first known cuspidal cohomology classes in $H^\ast(\text{GL}_n(\mathbb{Z}), \mathbb{C})$ for any $n > 1$.

1. Introduction

It is a well-known fact that there do not exist any cuspidal modular forms of level $N = 1$ and weight $k = 2$. From the Eichler–Shimura isomorphism, this is equivalent to the vanishing of the cuspidal cohomology groups

$$H^i_{\text{cusp}}(\text{GL}_2(\mathbb{Z}), \mathbb{C}) = 0$$

for all $i$ (particularly $i = 1$). It is natural to wonder what happens in higher rank.

Problem A. Does there exist an $n > 1$ such that $H^i_{\text{cusp}}(\text{GL}_n(\mathbb{Z}), \mathbb{C}) \neq 0$ for some $i$?

Recall that cuspidal cohomology classes are those which are represented by harmonic forms on $\text{GL}_n(\mathbb{Z}) \backslash \text{GL}_n(\mathbb{R})$ coming from cuspidal automorphic forms; see [LS01, §2]. In contrast, the cohomology classes in $\text{GL}_n(\mathbb{Z})$ which are easiest to construct are the stable classes considered by Borel [Bor74], which ultimately come from the trivial automorphic representation, which is at the opposite end of the spectrum to the cuspidal forms of relevance here (at least when $n > 1$). Problem A has been raised explicitly by a number of people, including [Clo16, §2.5]; [CR15, §1.2] refers to it as a “well-known” problem. One motivation emphasised by Khare (see e.g. the penultimate sentence of) [Kha10] is that the vanishing of the $H^i_{\text{cusp}}(\text{GL}_n(\mathbb{Z}), \mathbb{C})$ for some $n$ could provide the base case for an inductive proof of the analogue of Serre’s conjecture in dimension $n$. In [Mil02, Fer96], the authors showed that the groups $H^i_{\text{cusp}}(\text{GL}_n(\mathbb{Z}), \mathbb{C})$ vanish for all $1 < n < 27$; see [Fer96, Cor 1] and [Mil02, Thm 1.6]. Their methods are analytic and are related to the Stark–Odlyzko positivity technique [Odl90] for lower bounds on discriminants of

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1By [Fra98], all of the cohomology of $\text{GL}_n(\mathbb{Z})$ and its congruence subgroups can be understood in terms of automorphic forms.
number fields. In the language of automorphic representations, cuspidal cohomology classes in $H^*_\text{cusp}(\text{GL}_n(\mathbb{Z}), \mathbb{C})$ correspond to cuspidal automorphic representations $\pi$ for $\text{GL}_n/\mathbb{Q}$ of level one and cohomological weight zero, in the sense that the infinitesimal character of $\pi_\infty$ coincides with that of the trivial representation (equivalently, $\pi_\infty$ is cohomological for the trivial representation, i.e. the representation of highest weight zero, whence the terminology).

In [Che20, Thm. A] (see also [CT20]), Chenevier proves a general finiteness statement of a different sort, where $n$ is allowed to vary but the $\pi$ are constrained to both have fixed conductor and bounded motivic weight $w(\pi) \leq 23$; this result interpolates both the vanishing results of the flavour proved in [Mil02, Fer96] but also includes as a special case the Hermite–Minkowski Theorem. (A representation $\pi$ with cohomological weight zero for $\text{GL}_n/\mathbb{Q}$ has motivic weight $w(\pi) = n - 1$.)

In all of these arguments, the methods break down completely for higher motivic weights. It remains to understand whether this is a limitation of the method or reflects reality. An instructive case is the discriminant bounds of Odlzyko. For a number field $K/\mathbb{Q}$, denote by $\delta_K = |\Delta_K|^1/[K:\mathbb{Q}]$ the root discriminant of $K$. One can prove (assuming GRH) lower bounds of the form

$$\delta_K > 8\pi e^\gamma - \epsilon = 44.7632\ldots$$

as the degree of $K$ tends to infinity. (Without GRH one obtains the weaker lower bound $4\pi e^\gamma - \epsilon = 22.3816\ldots$) One may ask whether there might exist a lower bound which tended to infinity in $[K: \mathbb{Q}]$. The answer to this question is no by the Golod–Shafarevich construction; the existence of class field towers gives an infinite sequence of fields of increasing degree such that $\delta_K$ is constant. (At the moment, the smallest known limit point of $\delta_K$ is approximately 78.4269 [HMR21].) By analogy, this suggests that one might expect the answer to Problem A is positive. Confirming this is the main result of our paper:

**Theorem B** (Theorem 2.2.1). There exist regular algebraic cuspidal automorphic representations for $\text{GL}_n/\mathbb{Q}$ of level one and weight zero for $n = 105$ and $n = 106$. In particular, $H^*_\text{cusp}(\text{GL}_n(\mathbb{Z}), \mathbb{C}) \neq 0$ for these $n$.

Our argument works for other values of $n$ (presumably infinitely many; see §3.1), although $n = 105$ is the smallest value where our methods apply. The representations $\pi$ we construct have the added property of being self-dual, so that associated to $\pi$ is a compatible family of essentially self-dual $p$-adic representations of the absolute Galois group $G_\mathbb{Q}$, each of which is crystalline with Hodge–Tate weights $\{0, 1, \ldots, n - 1\}$, unramified outside $p$, and pure of weight $n - 1$; furthermore, for a positive density set of $p$ (conjecturally, for all $p$), these representations are irreducible. (Other than irreducibility, these properties are a consequence of local-global compatibility for the Galois representations associated to everywhere unramified regular algebraic cuspidal automorphic representations of $\text{GL}_n/\mathbb{Q}$, as initiated by Clozel [Clo91] and completed (in this everywhere unramified situation) by him in [Clo13]; the irreducibility is [PT15, Thm. D].)

When $n = 105$, our Galois representations are (generalized) orthogonal. Since $\text{SO}_{105}$ is the Langlands dual group of $\text{Sp}_{104}$, known transfer results from the theory of automorphic forms allow us to construct a non-endoscopic cusp form for symplectic groups, answering a variant of Problem A in this setting (see Theorem 2.2.1(2)).

In light of Theorem B there is the obvious variation of Problem A.
Problem C. What is the smallest $n > 1$ such that $H^i_{cusp}(GL_n(\mathbb{Z}), \mathbb{C}) \neq 0$ for some $i$?

We know from [Mil02] and Theorem B that the answer satisfies $27 \leq n \leq 105$. In light of some of the papers cited above, there is reason to suspect that the real answer is much closer to the lower bound than the upper bound.

The basic idea for proving Theorem B is as follows. Since symmetric power functoriality for modular forms has been proved by Newton–Thorne (see [NT21 Thm. A] for the version we use here), there is a plentiful supply of cuspidal regular algebraic automorphic representations of $GL_n/\mathbb{Q}$ which are of level one (i.e. are unramified at all primes), namely the symmetric powers $\text{Sym}^{n-1} f$ of (the automorphic representations associated to) modular forms $f$ of weight $k \geq 2$ and level one.

These symmetric powers are of course never of weight zero (even up twist), but since congruences between automorphic representations of different weights exist, one can ask whether there is ever a mod $p$ congruence between a symmetric power of a modular form of level one, and an automorphic representation of level one and weight zero. It turns out that this is indeed possible, and the existence of such congruences (and thus the existence of automorphic representations of level one and weight zero) can be proved using the “change of weight” techniques introduced in [Gee07] (which are based on the Khare–Wintenberger method). We carry this out in Theorem 2.1.1, which we apply in the case $p = 107$ in Theorem 2.2.1 in order to prove Theorem B.

1.1. Acknowledgements. We have been aware of Problem A for some time, but it was most recently brought to our attention at a lecture [Che23] by Gaëtan Chenevier at the conference Arithmétique des formes automorphes at Orsay in September, 2023, in honour of Laurent Clozel’s 70th birthday. In light of this, together with the obvious connections between the methods of this paper and Clozel’s work (Galois representations associated to self-dual automorphic representations, modularity lifting theorems for self-dual Galois representations, and symmetric power functoriality for modular forms, to name but three), it is a pleasure to dedicate this paper to him. We would also like to thank James Newton and Olivier Taïbi for helpful comments on a preliminary version of this paper.

2. The main results

2.1. Change of weight. We fix once and for all for each prime $p$ an isomorphism $\iota_p : \mathbb{C} \cong \overline{\mathbb{Q}}_p$, and we will accordingly sometimes implicitly regard automorphic representations as being defined over $\overline{\mathbb{Q}}_p$, rather than $\mathbb{C}$. In particular, we will freely refer to “the” $p$-adic Galois representation associated to a (regular algebraic) automorphic representation. We write $\rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ and $\overline{\rho}_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ for the cohomologically normalized representations associated to an eigenform $f$. Let $\varepsilon$ denote the $p$-adic cyclotomic character and $\overline{\varepsilon}$ its mod-$p$ reduction.

The following is our main result. While the proof of is somewhat technical, the principle behind it is simple: mod $p$ congruences between self-dual regular algebraic cuspidal automorphic representations of $GL_n/\mathbb{Q}$ obey a “local-global principle”, at least if the images of the corresponding mod $p$ Galois representations are sufficiently large. This principle was applied to the group $\text{GSp}_4/\mathbb{Q}$ in [GG12], and the change
of weight part of our arguments is a straightforward generalization of this to higher rank symplectic and orthogonal groups. In particular, the existence of a congruence between forms of level 1 but of different weights is governed by the restriction to a decomposition group at $p$, and by a careful choice of a symmetric power of a modular form, we can arrange the required local property.

**Theorem 2.1.1.** Let $f$ be an eigenform of level $SL_2(\mathbb{Z})$ and weight $k \geq 2$, and let $p > 5$ be a prime such that:

1. $\overline{\rho}_f(G_\mathbb{Q}) \supseteq SL_2(\mathbb{F}_p)$.
2. $(p-1, k-1) = 1$.
3. $f$ is ordinary at $p$.
4. $\overline{\rho}_f|_{G_\mathbb{Q}_p}$ is semisimple.

Then, for both $n = p-1$ and $n = p-2$, there exists a self-dual regular algebraic cuspidal automorphic representation $\pi$ for $GL_n/\mathbb{Q}$ of level one and weight zero whose mod $p$ Galois representation $\overline{\rho}_\pi : G_\mathbb{Q} \to GL_n(\mathbb{F}_p)$ is isomorphic to

$$Sym^{n-1}(\overline{\rho}_f \otimes \varepsilon^{\frac{k-2}{2}}) = \varepsilon^{\frac{(n-1)(k-2)}{2}} \otimes Sym^{n-1} \overline{\rho}_f.$$ 

**Proof.** Let $n = p-1$ or $p-2$, and write $G_n = GSp_n$ if $n = p-1$ (equivalently, if $n$ is even), and $G_n = GO_n$ if $n = p-2$ (equivalently, if $n$ is odd). Let $\mathbb{F}/\mathbb{F}_p$ be a finite extension such that $\overline{\rho}_f(G_\mathbb{Q}) \subseteq GL_2(\mathbb{F})$, and write

$$\overline{\rho} := Sym^{n-1}(\overline{\rho}_f \otimes \varepsilon^{\frac{k-2}{2}}) = \varepsilon^{\frac{(n-1)(k-2)}{2}} \otimes Sym^{n-1} \overline{\rho}_f : G_\mathbb{Q} \to GL_n(\mathbb{F}).$$

Since $\rho_f$ is symplectic with multiplier $\varepsilon^{1-k}$, the twist $\overline{\rho}_f \otimes \varepsilon^{\frac{k-2}{2}}$ is symplectic with multiplier $\varepsilon^{-1}$, and so we can and do regard $\overline{\rho}$ as a representation $G_\mathbb{Q} \to G_n(\mathbb{F})$ with multiplier $\varepsilon^{1-n}$. In particular, we have an isomorphism

$$\overline{\rho} \simeq \overline{\rho} \varepsilon^{1-n}.$$ 

(2.1.2)

By the hypotheses that $f$ is ordinary at $p$ and $\overline{\rho}|_{G_\mathbb{Q}_p}$ is semisimple, we can write

$$\overline{\rho}|_{G_\mathbb{Q}_p} \cong \overline{\psi}$$

for some unramified character $\overline{\psi}.$

Suppose that $n = p-1$. The character $\overline{\psi}$ has order $(p-1)$, so the powers $\overline{\psi}^{-i}$ for $i = 0, \ldots, n-1$ are precisely all the powers of $\overline{\psi}$. Since we are assuming that $(p-1)$ is coprime to $(1-k)$, we deduce that there are unramified characters $\overline{\psi}_i$ for $i = 0, \ldots, n-1$ such that

$$\overline{\rho}|_{G_\mathbb{Q}_p} \cong \bigoplus_{i=0}^{n-1} \overline{\psi}_i \overline{\psi}^{-i}; \quad \overline{\psi}_{n-1-i} = \overline{\psi}^{-1},$$

(2.1.3)

where the isomorphisms follow from (2.1.2). Suppose now that $n = p-2$, so that all but one power of $\overline{\psi}$ appears in $\overline{\rho}$. From our description of $\overline{\rho}_f$, the explicit exponents are given by

$$\left(\frac{k-2}{2}\right)(n-1-i) + \left(-1 - \frac{k-2}{2}\right)i, \quad i = 0, 1, \ldots, n-1.$$

Since this arithmetic progression is cyclic modulo $p-1 = n+1$, the missing exponent can be obtained by letting $i = n \equiv -1 \mod (p-1)$ to give

$$\left(\frac{k-2}{2}\right)(-1) + \left(-1 - \frac{k-2}{2}\right)(-1) \equiv 1 \mod (p-1),$$

where the isomorphisms follow from (2.1.2).
and hence the missing character is $\pi^1$; so we again see that (2.1.3) holds.

Since $\text{SL}_2(F_p) \subseteq \mathfrak{g}(G_Q)$, the representation $\mathfrak{g}$ is absolutely irreducible (see also Lemma 2.1.5). Let $E/Q_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $F$. Recall that $G_n = GSp_n$ if $n$ is even, and $G_n = GO_n$ if $n$ is odd. Write $R$ for the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for $G_n$-valued deformations of $\mathfrak{g}$ which have multiplier $\varepsilon^{1-n}$, are unramified outside $p$, and whose restrictions to $G_{Q_p}$ are crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n - 1$.

By [BG19] Prop. 4.2.6, every irreducible component of $R$ has Krull dimension at least 1. (We are applying [BG19] Prop. 4.2.6 with $l$ equal to our $p$, and the local deformation ring $\mathcal{R}_p$ being the union of those irreducible components of the corresponding crystalline deformation ring which are ordinary, as in [FKP22] Lem. B.4]; this is indeed a nonempty set of components because (2.1.3) shows that $\mathfrak{g}|_{G_{Q_p}}$ admits an ordinary crystalline lift, by lifting the characters $\psi_i$ to their Teichmüller lifts and the $\pi^{-i}$ to $\varepsilon^{-i}$. The remaining hypotheses of [BG19] Prop. 4.2.6 hold because $\mathfrak{g}$ is absolutely irreducible, the multiplier character $\varepsilon^{1-n}$ is odd/even precisely when $G_n$ is symplectic/orthogonal, and the Hodge–Tate weights 0, 1, $\ldots$, $n - 1$ are pairwise distinct.)

Let $F/Q$ be an imaginary quadratic field in which $p$ splits and which is disjoint from $(\mathbb{Q})^\text{gen}(\mathfrak{p})$. As in [CHT08] we let $\mathcal{G}_n$ denote the semi-direct product of $\mathcal{G}_n^0 = \text{GL}_n \times \text{GL}_1$ by the group $\{1, j\}$ where

$$j(g, a)j^{-1} = (ag^{-1}, a),$$

with multiplier character $\nu : \mathcal{G}_n \to \text{GL}_1$ sending $(g, a)$ to $a$ and sends $j$ to $-1$.

Following [BLGGT14] §1.1, given a homomorphism $\psi : G_Q \to GO_n(R)$, we have an associated homomorphism $r_\psi : G_F \to \mathcal{G}_n(R)$, whose multiplier character is that of $r$ multiplied by $\delta_{\psi/Q}$, where $\delta_{\psi/Q}$ is the quadratic character corresponding to the extension $F/Q$. Explicitly, if $A_n$ is the matrix defining the pairing for the group $G_n$ (so $A_n = 1_n$ if $n$ is odd and $A_n = J_n$ if $n$ is even, where $J_n$ is the standard symplectic form), then $r_\psi$ can be defined as the composite

$$G_Q \xrightarrow{\psi \times pr} G_n(R) \times G_Q/G_F \to \mathcal{G}_n(R),$$

where $pr$ is the projection $G_Q \to G_Q/G_F \cong \{\pm 1\}$, and the second map is the injection

$$G_n \times \{\pm 1\} \to \mathcal{G}_n$$

given by

$$r((g, 1)) = (g, \nu(g)),
$$

$$r((g, -1)) = (g, \nu(g)) \cdot (A_n^{-1}, (-1)^{n+1})j.$$

In particular we can apply this construction to $\mathfrak{g}$, and we write $\mathfrak{r} := r_{\mathfrak{g}} : G_Q \to \mathcal{G}_n(F)$.

We let $R_F$ be the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for $\mathcal{G}_n$-valued deformations of $\mathfrak{r}$ which have multiplier $\varepsilon^{1-n}$, are unramified outside $p$, and whose restrictions to the places above $p$ are crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n - 1$. The association $\psi \mapsto r_\psi$ induces a homomorphism $R_F \to R$, which is easily checked to be a surjection. (Indeed, it suffices to show that the map $R_F \to R$ induces a surjection on reduced cotangent spaces. It in turn suffices to see that the induced map of Lie algebras...
from [2.1.4] is a split injection of $G_\mathbb{Q}$-representations, or equivalently (since $p > 2$) a split injection of $G_\mathbb{F}$-representations, which is clear.)

By [Tho12] Thm. 10.1, $R_F$ is a finite $O$-algebra (see [BLGGT14] Thm. 2.4.2) for a restatement in the precise form we use here: in the notation of that statement, we are taking $l = p$, $n = p - 1$, $S = \{p\}$, $\mu = \varepsilon^{1-n}$, $H_\tau = \{0, 1, \ldots, n - 1\}$). Thus $R$ is a finite $O$-algebra, and since it has dimension at least 1, it has a $\mathbb{Q}_p$-valued point.

The corresponding lift $\rho : G_\mathbb{Q} \to G_n(\mathbb{Q}_p)$ of $\overline{\rho}$ is unramified outside $p$, has multiplier $\varepsilon^{1-n}$, and is crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n - 1$.

The representation $\rho$ is automorphic by [BLGGT14] Thm. 2.4.1 (taking $F = \mathbb{Q}$, $l = p$, $n = p - 1$, $\mu = \varepsilon^{1-n} \delta_{F/F}$); the hypothesis that $(\pi, \overline{\rho})$ is automorphic is immediate from [NT21] Thm. A] applied to $f$, and the hypothesis of residual adequacy is immediate from Lemma [2.1.5]. More precisely, there is a self-dual regular algebraic cuspidal automorphic representation $\pi$ of $GL_n(A_{\mathbb{Q}})$ whose corresponding $p$-adic Galois representation $\rho_\pi : G_\mathbb{Q} \to GL_n(\mathbb{Q}_p)$ is isomorphic to $\rho$.

By local-global compatibility (e.g. [BLGGT14] Thm. 2.1.1) we see that $\pi$ has level one and weight zero, as claimed.

Lemma 2.1.5. Let $p > 5$ and let $\tau : G_\mathbb{Q} \to GL_2(\mathbb{F}_p)$ be a representation with $SL_2(\mathbb{F}_p) \subseteq \tau(G_\mathbb{Q})$. Then for $n = p - 1$ or $n = p - 2$, the group $\langle \text{Sym}^{n-1} \tau(p_\mathbb{Q}(\zeta_p)) \rangle$ is adequate in the sense of [Tho12] Defn. 2.3.

Proof. Since $SL_2(\mathbb{F}_p)$ is perfect, we have $SL_2(\mathbb{F}_p) \subseteq \tau(G_\mathbb{Q}(\zeta_p))$, so it follows from Dickson’s classification that for some power $q$ of $p$, we have $SL_2(\mathbb{F}_q) \subseteq \tau(G_\mathbb{Q}(\zeta_p))$, and $p \nmid [\tau(G_\mathbb{Q}(\zeta_p)) : SL_2(\mathbb{F}_q)]$. By (for example) [BLGGT13] Lem. A.1.3 that it suffices to check that for $U$ the standard 2-dimensional $\mathbb{F}_p$-representation of $G = SL_2(\mathbb{F}_q)$, $V := \text{Sym}^{n-1} U$ is adequate.

Since $\dim V = n < p$, it follows from [GHT15] Cor. 1.4 that we only need to check that we are not in any of the exceptional cases of [GHT15] Thm. 1.3]. Since $SL_2(\mathbb{F}_q)$ is generated by the unipotent matrices, the group $G^+$ of loc. cit. is equal to $G$, and so (in the notation of [GHT15]) $W = V$, and $H = G$.

We now rule out the exceptional cases of [GHT15] Thm. 1.3] in turn. We are not in case (a), because $G$ is not solvable, and we are not in case (b)(i) because $n \geq p - 2 > (p + 1)/2$ by our assumption that $p > 5$. We are not in any of cases (b)(ii), (iv), (v) or (vi) by our assumptions that $p > 3$ and $G = SL_2(\mathbb{F}_q)$. Finally in case (b)(iii) we would need $q = 2p - 1$ to be a power of $p$, which is impossible. □

Corollary 2.1.6. Under the hypotheses of Theorem 2.1.1, there exists a globally generic non-endoscopic cuspidal automorphic representation of $Sp_{p-3}(\mathbb{Q})$ of level one and weight zero.

Proof. From Theorem 2.1.1 we deduce the existence of a self-dual regular algebraic cuspidal automorphic representation $\pi$ for $GL_{p-2}(\mathbb{Q})$ of level one and weight zero. Since $\pi$ is self-dual, either $L(s, \pi, \wedge^2)$ or $L(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$. Since $p - 2$ is odd, the former case does not occur (see the remarks following [CKPSS04] Thm. 7.1) and hence the result follows from [CKPSS04] Thm. 7.2] (or alternatively from [Art13] Thm. 1.4.1, 1.5.3)]. □

2.2. The case $p = 107$. We now prove Theorem 2.2.1] and a variant for symplectic groups.

Theorem 2.2.1.
(1) There exist self-dual regular algebraic cuspidal automorphic representations \( \pi \) for \( \text{GL}_n / \mathbb{Q} \) of level one and weight zero for \( n = 105 \) and \( n = 106 \). In particular, \( H^*_\text{cusp}(\text{GL}_n(\mathbb{Z}), \mathbb{C}) \neq 0 \) for these \( n \).

(2) There exists a globally generic, non-endoscopic, regular algebraic cuspidal automorphic representation for \( \text{Sp}_{104} / \mathbb{Q} \) of level one and weight zero. In particular, \( H^*_\text{cusp}(\text{Sp}_{104}(\mathbb{Z}), \mathbb{C}) \neq 0 \).

Remark 2.2.2. If \( A_g \) is the moduli space of principally polarized abelian varieties of dimension \( g \), we deduce from Theorem 2.2.1 that \( H^*_\text{cusp}(A_{52}, \mathbb{C}) \neq 0 \). However, as Olivier Taibi explained to us, one can construct cuspidal cohomology classes of \( A_g \) for much smaller \( g \) coming from endoscopic representations, and one can even arrange that these endoscopic representations are tempered; see [CR15, §1.24] for a closely related discussion.

Proof. Let \( f = \Delta E_4^2 E_6 = q - 48q^2 - 195804q^3 + \ldots \) be the unique normalized cuspidal Hecke eigenform for \( \text{SL}_2(\mathbb{Z}) \) of weight \( k = 26 \). Let \( p = 107 \), and \( \bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(F_{107}) \) denote the mod 107 Galois representation associated to \( f \) (in its cohomological normalization). By [SD73, Cor., p. SwD-31], the image of \( \bar{\rho} \) is exactly \( \text{GL}_2(F_{107}) \) (note that \( F_{107}^{107} = F_{107}^{25} = F_{107}^{4} \)). Since

\[ a_{107}(f) = 35830422465487817813321292 \equiv -1 \mod 107, \]

\( f \) is ordinary at 107.

Certainly \((106, 25) = 1\), so in view of Theorem 2.2.1 and Corollary 2.1.6, we only need to check that \( \bar{\rho}_f|_{G_{F_{107}}} \) is semisimple. That this is indeed the case is a consequence of a computation of Elkies, recorded in [Gro90, §17]: the form \( f \) admits a companion form of weight \( p + 1 - k = 82 \), i.e. an eigenform \( g \) of level one and weight 82 with \( \bar{\rho}_f \cong \varepsilon^{-25} \bar{\rho}_g \). The semisimplicity of \( \bar{\rho}_f|_{G_{F_{107}}} \) is an immediate consequence of the existence of \( g \) (see e.g. [Gro90, Prop. 13.8(3)]; this is the easy direction of Gross’ “companion forms” theorem). By Theorem 2.1.1 and Corollary 2.1.6, we deduce the existence of the desired automorphic forms \( \pi \) for \( \text{GL}_n / \mathbb{Q} \) and \( \text{Sp}_{2n} / \mathbb{Q} \) for \( n = 105, 106 \) and \( 2n = 104 \) respectively. The existence of such \( \pi \) is then well-known to imply the non-vanishing of cuspidal cohomology of the appropriate arithmetic group, see for example the survey [LS01, §3].

3. Complements

3.1. Other examples. Following [CG13], we see that (in level one) companion forms also exist for \( p = 139, 151, 173, 179, \ldots \) in weights \( k \) with \( (p - 1, k - 1) = 1 \), leading to level one weight zero representations \( \pi \) for \( \text{GL}_n / \mathbb{Q} \) with \( n = p - 2 \) and \( n = p - 1 \). We present a naïve heuristic that predicts that level one forms with companion forms will exist for a positive density of primes \( p \). In each even weight \( k \leq p \), one expects\(^2\) (on average) approximately one mod-\( p \) representation with image contained in \( \text{GL}_2(F_p) \). One further expects the image of such a representation will contain \( \text{SL}_2(F_p) \) with probability one. One then further

\(^2\)This expectation comes from Maeda’s conjecture; given a single Galois orbit of newforms of some fixed weight defined over a number field \( K / \mathbb{Q} \), the average number of embeddings \( K \to \mathbb{Q}_p \) as \( p \) varies is equal to one. (This is equivalent to the claim that the expected number of fixed points of a transitive action of a group on a finite set is equal to one.) Hence if \( \mathcal{O} \subset K \) is any order, one expects on average at least one prime above \( p \) with residue field \( F_p \). If Maeda’s conjecture fails, then the average only increases.
expects the restriction to $G_{Q_p}$ to be ordinary and split with probability of order $1/p$. Hence one might guess that mod-$p$ companion forms exist with probability $1 - (1 - 1/p)^{p/2} \sim 1 - e^{-1/2}$. (Minor variations of the heuristic may change the relevant constants, but not the overall prediction.)

Once one has such a $\pi$ for $\text{GL}_n$ with $n$ odd, then for each $m \geq 1$ there is conjecturally a regular algebraic cuspidal automorphic representation of level one and weight zero for $\text{GL}_{nm}/Q$. Indeed, for each level one cuspidal eigenform $f$ of weight $n + 1$ (such an $f$ exists because $n > 26$), the conjectural tensor product $\pi \boxtimes \text{Sym}^{m-1} f$ should be automorphic and cuspidal of level one and weight zero.

References


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