CUSPIDAL COHOMOLOGY CLASSES FOR $\mathrm{GL}_n(\mathbf{Z})$

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To Laurent Clozel, in admiration.

Abstract. We prove the existence of a cuspidal automorphic representation π for $\mathrm{GL}_{79}/\mathbf{Q}$ of level one and weight zero. We construct π using symmetric power functoriality and a change of weight theorem, using Galois deformation theory. As a corollary, we construct the first known cuspidal cohomology classes in $H^*(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C})$ for any $n > 1$.

1. Introduction

It is a well-known fact that there do not exist any cuspidal modular forms of level $N = 1$ and weight $k = 2$. From the Eichler–Shimura isomorphism, this is equivalent to the vanishing of the cuspidal cohomology groups

$$H^i_{\mathrm{cusp}}(\mathrm{GL}_2(\mathbf{Z}), \mathbf{C}) = 0$$

for all i (particularly $i = 1$). It is natural to wonder what happens in higher rank.

Problem A. *Does there exist an $n > 1$ such that $H^i_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C}) \neq 0$ for some i?*

Higher rank analogues of the Eichler–Shimura isomorphism [Bor81, Cor. 5.5] show that Problem A is equivalent to the existence of cuspidal automorphic representations π for $\mathrm{GL}_n/\mathbf{Q}$ which have level one and weight zero. Here level one means that π_p is unramified for all primes p and weight zero means that π_∞ has the same infinitesimal character as the trivial representation.

The work of Fermigier and subsequently of Miller ([Fer96, Cor. 1] for $n \leq 23$, [Mil02, Thm. 1.6] for $n < 27$) showed that the groups $H^*_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C})$ vanish for all $1 < n < 27$; their methods are analytic and are related to the Stark–Odlyzko positivity technique [Odl90] for lower bounds on discriminants of number fields.

Problem A has subsequently been raised explicitly by a number of people, including [Clo16, §2.5], [Kha10], and [CR15, §1.2], where it is referred to as a "well-known" problem. One motivation for this question, emphasized by Khare, is that the vanishing of the $H^i_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C})$ for a given n could provide the base case for an inductive proof of the analogue of Serre's conjecture in dimension n. It was unclear to many people (including some of the authors of this paper) whether it was reasonable to hope for this vanishing for all n, although in recent years the work of Chenevier and Taïbi on self-dual automorphic representations of level 1 (see e.g.

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the introduction to [CT20]) had made this seem unlikely. Another reason to expect an affirmative answer to Problem A is by comparison to the aforementioned discriminant bounds of Odlzyko, which for a number field $K/\mathbf{Q}$ give positive constant lower bounds for the root discriminant $\delta_K = |\Delta_K|^{1/[K:\mathbf{Q}]}$ as the degree of K tends to infinity. One may ask whether there might exist a lower bound which tended to infinity in $[K:\mathbf{Q}]$. The answer to this question is no by the Golod–Shafarevich construction; the existence of class field towers gives an infinite sequence of fields of increasing degree such that δ_K is constant.

Our main theorem resolves Problem A in the affirmative:

Theorem B (Theorem 2.4, Corollary 3.2)**.** *There exist cuspidal automorphic representations for* $\mathrm{GL}_n/\mathbf{Q}$ *of level one and weight zero for* $n = 79$*,* $n = 105$*, and* $n = 106$*. In particular,* $H^*_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C}) \neq 0$ *for these* n*.*

Our argument works for other values of n (presumably infinitely many, although we do not know how to prove this; see Remarks 2.6 and 3.3). In light of Theorem B, there is the obvious variation of Problem A:

Problem C. *What is the smallest* $n > 1$ *such that* $H^i_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C}) \neq 0$ *for some* i*?*

We know from [Mil02] and Theorem B that the answer satisfies $27 \leq n \leq 79$. The work of Chenevier and Taïbi [CT20] suggests that the real answer is much closer to the lower bound than the upper bound.

While the formulation of Problem A makes no reference to motives or Galois representations, according to standard conjectures in the Langlands program it is equivalent to the existence of irreducible rank n pure motives (with coefficients) over $\mathbf{Q}$ with everywhere good reduction and Hodge numbers $0, 1, \ldots, n-1$, or to the existence of irreducible Galois representations $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ unramified away from p and crystalline with Hodge–Tate weights $0, 1, \ldots, n-1$ at p. In fact, we will proceed by producing such Galois representations.

Our approach to proving Theorem B is ultimately based on the conjecture of Serre [Ser87] predicting the existence of congruences between modular forms of different weights. If f is a cuspidal eigenform of level 1 and weight k and the mod p Galois representation $\overline{\rho}_{f,p} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is irreducible, then Serre predicts that there exists a modular form g of weight 2 and level 1 with $\overline{\rho}_{g,p} \simeq \overline{\rho}_{f,p}$ if and only if $\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ admits a crystalline lift with Hodge–Tate weights 0 and 1. Of course this cannot actually occur as no such g exists! The natural generalization of Serre's conjecture for larger n predicts that if π is a regular algebraic essentially self dual cuspidal automorphic representation for $\mathrm{GL}_n/\mathbf{Q}$ of level 1 and arbitrary weight, and the mod p Galois representation $\overline{\rho}_{\pi,p} : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ has "large" image, then there exists a π' of level 1 and weight 0 with $\overline{\rho}_{\pi',p} \simeq \overline{\rho}_{\pi,p}$ if and only if $\overline{\rho}_{\pi,p}|_{G_{\mathbf{Q}_p}}$ admits a crystalline lift with Hodge–Tate weights $0, 1, \ldots, n-1$. In many instances, these "change of weight" congruences may in fact be produced using automorphy lifting theorems and the Khare–Wintenberger method, as in [Gee07, GG12, BLGGT14].

It remains to explain how we find the π to which the above strategy can be applied. For this, we need a supply of π for which $\overline{\rho}_{\pi,p}|_{G_{\mathbf{Q}_p}}$ may be readily understood. Our idea is to take π to be $\mathrm{Sym}^{n-1} f$ (up to twist) for f a modular form of level 1; this symmetric power lift is now available thanks to the recent work of

Newton–Thorne (see [NT21, Thm. A] for the version we use). If f is a cuspidal eigenform of level 1 and weight $k < p$, then typically f will be ordinary at p and the Galois representation $\overline{\rho}_{f,p}|_{I_p}$ will be a nonsplit extension of $\overline{\varepsilon}^{1-k}$ by 1, where $\overline{\varepsilon}$ denotes the mod p cyclotomic character. In this case no twist of $\mathrm{Sym}^{n-1}\,\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ will have a crystalline lift of Hodge–Tate weights $0,\ldots,n-1$, at least for $n \le p$. On the other hand in the less typical situation that $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ is semisimple (or equivalently tamely ramified) we are sometimes able to succeed. Here there are two possibilities, either f is still ordinary at p but the extension splits and $\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ is a sum of two characters, or f is non-ordinary at p and $\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ is irreducible.

As an illustration, if f is ordinary at p, $\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ splits, and $(k-1,p-1)=1$, then as $\overline{\varepsilon}$ has order $p-1$, we find that

$$\mathrm{Sym}^{p-2}\,\overline{\rho}_{f,p}|_{I_p} = \mathrm{Sym}^{p-2}(1\oplus\overline{\varepsilon}^{1-k}) = \bigoplus_{i=0}^{p-2}\overline{\varepsilon}^{i(1-k)} = \bigoplus_{i=0}^{p-2}\overline{\varepsilon}^{i},$$

and hence $\mathrm{Sym}^{p-2}\,\overline{\rho}_{f,p}|_{G_{\mathbf{Q}_p}}$ has a crystalline lift of Hodge–Tate weights $0,1,\ldots,p-2$ which on inertia is simply a sum of powers of the cyclotomic character. This leads to the case $n=106$ of theorem, taking f to be the cusp form of level 1 and weight 26 and $p=107$, while the case $n=105$ comes from a similar consideration of $\mathrm{Sym}^{104} f$. Our "change of weight" theorem is proved by extending the techniques introduced in [Gee07] and developed further by Gee and Geraghty in [GG12], combining the Khare–Wintenberger method with automorphy lifting theorems for Hida families on unitary groups due to Geraghty [Ger19] (and refined by Thorne [Tho12]). The case $n=79$ comes from considering $\mathrm{Sym}^{78} f$ for a modular form f which is non-ordinary at $p=79$. Here the change of weight theorem is more involved, and closer to the arguments of [BLGGT14], using the Harris tensor product trick.

1.1. **Acknowledgements.** We have been aware of Problem A for some time, but it was most recently brought to our attention at a lecture [Che23] by Gaëten Chenevier at the conference *Arithmétique des formes automorphes* at Orsay in September, 2023, in honour of Laurent Clozel's 70th birthday. In light of this, together with the obvious connections between the methods of this paper and Clozel's work (Galois representations associated to self-dual automorphic representations, modularity lifting theorems for self-dual Galois representations, and symmetric power functoriality for modular forms, to name but three), it is a pleasure to dedicate this paper to him. We would also like to thank James Newton, Will Sawin, Olivier Taïbi and Jack Thorne for helpful comments on earlier versions of this paper.

2. The ordinary case

We fix once and for all for each prime p an isomorphism $\imath = \imath_p : \mathbf{C} \cong \overline{\mathbf{Q}}_p$, and we will accordingly sometimes implicitly regard automorphic representations as being defined over $\overline{\mathbf{Q}}_p$, rather than $\mathbf{C}$. In particularly we will freely refer to "the" p-adic Galois representation associated to a (regular algebraic) automorphic representation. We write $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ and $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ for the cohomologically normalized representations associated to an eigenform f. Let ε denote the p-adic cyclotomic character and $\overline{\varepsilon}$ its mod-p reduction.

Theorem 2.1. *Let f be an eigenform of level $\mathrm{SL}_2(\mathbf{Z})$ and weight $k \ge 2$, and let $p > 5$ be a prime such that:*

(1) $\overline{\rho}_f(G_{\mathbf{Q}}) \supseteq \mathrm{SL}_2(\mathbf{F}_p)$.
(2) $(p-1, k-1) = 1$.
(3) f *is ordinary at* p.
(4) $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ *is semisimple.*

Then, for both $n = p-1$ *and* $n = p-2$*, there exists a self-dual cuspidal automorphic representation* π *for* $\mathrm{GL}_n/\mathbf{Q}$ *of level one and weight zero whose mod* p *Galois representation* $\overline{\rho}_\pi : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ *is isomorphic to*

$$\mathrm{Sym}^{n-1}(\overline{\rho}_f \otimes \overline{\varepsilon}^{\frac{k-2}{2}}) = \overline{\varepsilon}^{\frac{(n-1)(k-2)}{2}} \otimes \mathrm{Sym}^{n-1} \overline{\rho}_f.$$

Proof. Let $n = p-1$ or $p-2$, and write $G_n = \mathrm{GSp}_n$ if $n = p-1$ (equivalently, if n is even), and $G_n = \mathrm{GO}_n$ if $n = p-2$ (equivalently, if n is odd). Let $\mathbf{F}/\mathbf{F}_p$ be a finite extension such that $\overline{\rho}_f(G_{\mathbf{Q}}) \subseteq \mathrm{GL}_2(\mathbf{F})$, and write

$$\overline{\rho} := \mathrm{Sym}^{n-1}(\overline{\rho}_f \otimes \overline{\varepsilon}^{\frac{k-2}{2}}) = \overline{\varepsilon}^{\frac{(n-1)(k-2)}{2}} \otimes \mathrm{Sym}^{n-1} \overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{F}).$$

Since ρ_f is symplectic with multiplier ε^{1-k}, the twist $\overline{\rho}_f \otimes \varepsilon^{\frac{k-2}{2}}$ is symplectic with multiplier $\overline{\varepsilon}^{-1}$, and so we can and do regard $\overline{\rho}$ as a representation $G_{\mathbf{Q}} \to G_n(\mathbf{F})$ with multiplier $\overline{\varepsilon}^{1-n}$. In particular, we have an isomorphism $\overline{\rho} \simeq \overline{\rho}^\vee \overline{\varepsilon}^{1-n}$.

By the hypotheses that f is ordinary at p and $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ is semisimple, we can write

$$\overline{\rho}_f|_{G_{\mathbf{Q}_p}} \cong \overline{\psi} \oplus \overline{\psi}^{-1} \overline{\varepsilon}^{1-k}$$

for some unramified character $\overline{\psi}$, so that

$$\overline{\rho}|_{G_{\mathbf{Q}_p}} \cong \bigoplus_{j=0}^{n-1} \overline{\psi}^{n-1-2i} \overline{\varepsilon}^{(n-1)(k-2)/2-(k-1)i}.$$

Since $(p-1, k-1) = 1$, either $n = p-1$ or $n = p-2$, and $\overline{\varepsilon}$ has order $(p-1)$, it follows easily that there are unramified characters $\overline{\psi}_i$ for $i = 0, \ldots, n-1$ such that

$$\overline{\rho}|_{G_{\mathbf{Q}_p}} \cong \bigoplus_{i=0}^{n-1} \overline{\psi}_i \overline{\varepsilon}^{-i}; \quad \overline{\psi}_{n-1-i} = \overline{\psi}_i^{-1}. \tag{2.1.1}$$

Since $\mathrm{SL}_2(\mathbf{F}_p) \subseteq \overline{\rho}_f(G_{\mathbf{Q}})$, the representation $\overline{\rho}$ is absolutely irreducible (see also Lemma 2.2.) Let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. Recall that $G_n = \mathrm{GSp}_n$ if n is even, and $G_n = \mathrm{GO}_n$ if n is odd. Write R for the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for G_n-valued deformations of $\overline{\rho}$ which have multiplier ε^{1-n}, are unramified outside p, and whose restrictions to $G_{\mathbf{Q}_p}$ are crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n-1$.

By [BG19, Prop. 4.2.6], every irreducible component of R has Krull dimension at least 1. (We are applying [BG19, Prop. 4.2.6] with l equal to our p, and the local deformation ring $\overline{R}_p$ being the union of those irreducible components of the corresponding crystalline deformation ring which are ordinary, as in [FKP22, Lem. B.4]; this is indeed a nonempty set of components because (2.1.1) shows that $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ admits an ordinary crystalline lift, by lifting the characters $\overline{\psi}_i$ to their Teichmüller lifts and the $\overline{\varepsilon}^{-i}$ to ε^{-i}. The remaining hypotheses of [BG19, Prop. 4.2.6] hold because $\overline{\rho}$ is absolutely irreducible, the multiplier character ε^{1-n} is odd/even precisely when G_n is symplectic/orthogonal, and the Hodge–Tate weights $0, 1, \ldots, n-1$ are pairwise distinct.)

Let $F/\mathbf{Q}$ be an imaginary quadratic field in which p splits and which is disjoint from $(\overline{\mathbf{Q}})^{\ker \overline{\rho}}(\zeta_p)$. As in [CHT08] we let $\mathcal{G}_n$ denote the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$ where

$$\jmath(g, a)\jmath^{-1} = (ag^{-t}, a),$$

with multiplier character $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ sending (g, a) to a and sends $\jmath$ to -1. Following [BLGGT14, §1.1], given a homomorphism $\psi : G_{\mathbf{Q}} \to \mathrm{GO}_n(R)$, we have an associated homomorphism $r_\psi : G_F \to \mathcal{G}_n(R)$, whose multiplier character is that of r multiplied by $\delta^n_{F/\mathbf{Q}}$, where $\delta_{F/\mathbf{Q}}$ is the quadratic character corresponding to the extension $F/\mathbf{Q}$. Explicitly, if A_n is the matrix defining the pairing for the group G_n (so $A_n = 1_n$ if n is odd and $A_n = J_n$ if n is even, where J_n is the standard symplectic form), then r_ψ can be defined as the composite

$$G_{\mathbf{Q}} \xrightarrow{\psi \times \mathrm{pr}} G_n(R) \times G_{\mathbf{Q}}/G_F \to \mathcal{G}_n(R),$$

where pr is the projection $G_{\mathbf{Q}} \to G_{\mathbf{Q}}/G_F \cong \{\pm 1\}$, and the second map is the injection

$$G_n \times \{\pm 1\} \hookrightarrow \mathcal{G}_n \tag{2.1.2}$$

given by

$$\begin{aligned} r((g, 1)) &= (g, \nu(g)), \\ r((g, -1)) &= (g, \nu(g)) \cdot (A_n^{-1}, (-1)^{n+1})\jmath. \end{aligned}$$

In particular we can apply this construction to $\overline{\rho}$, and we write $\overline{r} := r_{\overline{\rho}} : G_{\mathbf{Q}} \to \mathcal{G}_n(\mathbf{F})$.

We let R_F be the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for $\mathcal{G}_n$-valued deformations of $\overline{r}$ which have multiplier $\varepsilon^{1-n}\delta^n_{F/F^+}$, are unramified outside p, and whose restrictions to the places above p are crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n-1$. The association $\psi \mapsto r_\psi$ induces a homomorphism $R_F \to R$, which is easily checked to be a surjection. (Indeed, it suffices to show that the map $R_F \to R$ induces a surjection on reduced cotangent spaces. It in turn suffices to see that the induced map of Lie algebras from (2.1.2) is a split injection of $G_{\mathbf{Q}}$-representations, or equivalently (since $p > 2$) a split injection of G_F-representations, which is clear.)

By [Tho12, Thm. 10.1], R_F is a finite $\mathcal{O}$-algebra (see [BLGGT14, Thm. 2.4.2] for a restatement in the precise form we use here; in the notation of that statement, we are taking $l = p$, $n = p - 1$, $S = \{p\}$, $\mu = \varepsilon^{1-n}$, $H_\tau = \{0, 1, \ldots, n-1\}$). Thus R is a finite $\mathcal{O}$-algebra, and since it has dimension at least 1, it has a $\overline{\mathbf{Q}}_p$-valued point. The corresponding lift $\rho : G_{\mathbf{Q}} \to G_n(\overline{\mathbf{Q}}_p)$ of $\overline{\rho}$ is unramified outside p, has multiplier ε^{1-n}, and is crystalline and ordinary with Hodge–Tate weights $0, 1, \ldots, n-1$.

The representation ρ is automorphic by [BLGGT14, Thm. 2.4.1] (taking $F = \mathbf{Q}$, $l = p$, $n = p - 1$, $r = \rho$, and $\mu = \varepsilon^{1-n}\delta^n_{F/F^+}$; the hypothesis that $(\overline{r}, \overline{\mu})$ is automorphic is immediate from [NT21, Thm. A] applied to f, and the hypothesis of residual adequacy is immediate from Lemma 2.2). More precisely, there is a self-dual regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$ whose corresponding p-adic Galois representation $\rho_\pi : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is isomorphic to ρ. By local-global compatibility (e.g. [BLGGT14, Thm. 2.1.1]) we see that π has level one and weight zero, as claimed. $\square$

Lemma 2.2. *Let $p > 5$ and let $\overline{r} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ be a representation with $\mathrm{SL}_2(\mathbf{F}_p) \subseteq \overline{r}(G_{\mathbf{Q}})$. Then for $p-2 \leq n \leq p$, the group $(\mathrm{Sym}^{n-1}\overline{r})(G_{\mathbf{Q}(\zeta_p)})$ is adequate in the sense of* [Tho17, Defn. 2.20].

Proof. Since $\mathrm{SL}_2(\mathbf{F}_p)$ is perfect, we have $\mathrm{SL}_2(\mathbf{F}_p) \subseteq \overline{r}(G_{\mathbf{Q}(\zeta_p)})$, so it follows from Dickson's classification that for some power q of p, we have $\mathrm{SL}_2(\mathbf{F}_q) \subseteq \overline{r}(G_{\mathbf{Q}(\zeta_p)})$, and $p \nmid [\overline{r}(G_{\mathbf{Q}(\zeta_p)}) : \mathrm{SL}_2(\mathbf{F}_q)]$. By [GHT17, Rem. 6.1], it suffices to check that for U the standard 2-dimensional $\overline{\mathbf{F}}_p$-representation of $G = \mathrm{SL}_2(\mathbf{F}_q)$, $V := \mathrm{Sym}^{n-1} U$ is adequate. It is absolutely irreducible (because $n \leq p$), and is therefore adequate by [GHT17, Cor. 9.4], noting that since $p > 5$ we have $n \geq p-2 > (p+1)/2$. □

2.3. **The case $p = 107$.** We now prove Theorem B.

Theorem 2.4. *There exist self-dual cuspidal automorphic representations π for $\mathrm{GL}_n/\mathbf{Q}$ of level one and weight zero for $n = 105$ and $n = 106$. In particular, $H^*_{\mathrm{cusp}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{C}) \neq 0$ for these n.*

Proof. Let $f = \Delta E_4^2 E_6 = q - 48q^2 - 195804q^3 + \ldots$ be the unique normalized cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbf{Z})$ of weight $k = 26$. Let $p = 107$, and $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_{107})$ denote the mod 107 Galois representation associated to f (in its cohomological normalization). By [SD73, Cor., p.SwD-31], the image of $\overline{\rho}$ is exactly $\mathrm{GL}_2(\mathbf{F}_{107})$ (note that $(\mathbf{F}_{107}^\times)^{25} = \mathbf{F}_{107}^\times$). Since

$$a_{107}(f) = 358304224654878178133212922 \equiv -1 \bmod 107,$$

f is ordinary at 107.

Certainly $(106, 25) = 1$, so in view of Theorem 2.1 we only need to check that $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ is semisimple. That this is indeed the case is a consequence of a computation of Elkies, recorded in [Gro90, §17]: the form f admits a *companion form* of weight $p + 1 - k = 82$, i.e. an eigenform g of level one and weight 82 with $\overline{\rho}_f \cong \overline{\varepsilon}^{-25}\overline{\rho}_g$. The semisimplicity of $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ is an immediate consequence of the existence of g (see e.g. [Gro90, Prop. 13.8(3)]). By Theorem 2.1 we deduce the existence of the desired automorphic forms π for $\mathrm{GL}_n/\mathbf{Q}$ for $n = 105, 106$ respectively. The existence of such π is then well-known to imply the non-vanishing of the cuspidal cohomology groups, see for example the survey [LS01, §3]. □

Remark 2.5. Combining Theorem 2.4 with the descent result [CKPSS04, Thm. 7.2], we see that there is a globally generic, non-endoscopic, cuspidal automorphic representation for $\mathrm{Sp}_{104}/\mathbf{Q}$ of level one and weight zero. If $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties of dimension g, we deduce that $H^*_{\mathrm{cusp}}(\mathcal{A}_{52}, \mathbf{C}) \neq 0$. However, as Olivier Taïbi explained to us, one can construct cuspidal cohomology classes of $\mathcal{A}_g$ for much smaller g coming from endoscopic representations, and one can even arrange that these endoscopic representations are tempered; see [CR15, § 1.24] for a closely related discussion.

Remark 2.6. Following [CG13], we see that modular forms satisfying the hypotheses of Theorem 2.1 also exist for $p = 139, 151, 173, 179, \ldots$ in weights k with $(p-1, k-1) = 1$, leading to level one weight zero representations π for $\mathrm{GL}_n/\mathbf{Q}$ with $n = p-2$ and $n = p-1$. A naïve heuristic (using Maeda's conjecture, although Sawin pointed out to us an alternate approach based on Bhargava's heuristics which gives answers of the same order) predicts the existence of locally ordinary and split $\mathrm{GL}_2(\mathbf{F}_p)$-representations with image containing $\mathrm{SL}_2(\mathbf{F}_p)$ with probability of order $1/p$ for each weight where cuspidal eigenforms exist. This leads to the expectation that one should expect examples of Theorem 2.1 for a set of primes p of positive density (using that $\varphi(p-1)/p$ has non-zero limiting distribution, see [K\'68]).

3. The non-ordinary case

We now explain how to improve $n = 105$ to $n = 79$, at the cost of a slightly more involved construction. The idea behind the proof is again quite simple: we replace the ordinary eigenform f in Theorem 2.1 by a non-ordinary form, where one can hope to use the change of weight results of [BLGGT14]. It turns out that there is no local obstruction to the existence of a weight zero lift of (a twist of) $\mathrm{Sym}^{n-1}\,\overline{\rho}_f$ if $n = p-1$ or p. However, in the latter case the global representation $\mathrm{Sym}^{n-1}\,\overline{\rho}_f$ is reducible, and we do not know whether to expect a congruence to exist in level one, while in the former case it has dimension p, which is excluded by the hypotheses of [BLGGT14]. Nonetheless, in the case $n = p-1$ we are able to use a simplified version of the arguments of [BLGGT14], since we do not need to change the level and only need to make a relatively simple change of weight, and indeed our arguments are very close to those of [BLGG11].

Theorem 3.1. *Let $p > 5$ be a prime, and let f be an eigenform of level $\mathrm{SL}_2(\mathbf{Z})$ and weight $2 \le k < p$, such that:*

1. $(k-1, p+1) = 1$.
2. *f is non-ordinary at p.*

Then there exists a self-dual cuspidal automorphic representation π for $\mathrm{GL}_p/\mathbf{Q}$ of level one and weight zero whose mod p Galois representation $\overline{\rho}_\pi : G_{\mathbf{Q}} \to \mathrm{GL}_p(\overline{\mathbf{F}}_p)$ is isomorphic to $\mathrm{Sym}^{p-1}\,\overline{\rho}_f$.

Proof. Where possible, we follow the proof of Theorem 2.1. We begin by showing that $\overline{\rho}_f$ has image containing $\mathrm{SL}_2(\mathbf{F}_p)$. Since $(k-1, p+1) = 1$, the projective image of $\overline{\rho}_f(G_{I_{\mathbf{Q}_p}})$ contains a cyclic subgroup of order $p+1 > 5$, so $\overline{\rho}_f$ does not have exceptional image (that is, projective image A_4, S_4, or A_5). Since $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ is absolutely irreducible, so is $\overline{\rho}_f$. Hence it remains to rule out the possibility that $\overline{\rho}_f$ has dihedral image. If this were the case, then since it is unramified outside p, it would have to be induced from $\mathbf{Q}(\sqrt{p^*})$ where $p^* = (-1)^{(p-1)/2}p$. But this would imply that $\overline{\rho}_f|_{G_{\mathbf{Q}_p}}$ is induced from $\mathbf{Q}_p(\sqrt{p^*})$, which would in turn imply that it is invariant under twisting by $\varepsilon^{(p-1)/2} = \omega_2^{(p^2-1)/2}$. Since $\overline{\rho}_f|_{I_p} \simeq \omega_2^{k-1} \oplus \omega_2^{p(k-1)}$, this can only happen if $k \equiv (p+3)/2 \bmod (p+1)$, contradicting the assumption that $(k-1, p+1) = 1$.

Let $\mathbf{F}/\mathbf{F}_p$ be a finite extension such that $\overline{\rho}_f(G_{\mathbf{Q}}) \subseteq \mathrm{GL}_2(\mathbf{F})$, and write $\overline{\rho} := \mathrm{Sym}^{p-1}\,\overline{\rho}_f$, so that $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GO}_p(\mathbf{F})$ has multiplier $\overline{\varepsilon}^{1-p} = 1$, and $\overline{\rho}(G_{\mathbf{Q}(\zeta_p)})$ is adequate by Lemma 2.2.

Let $\varepsilon_2, \varepsilon_2' : G_{\mathbf{Q}_{p^2}} \to \overline{\mathbf{Z}}_p^\times$ be the two Lubin–Tate characters trivial on $\mathrm{Art}_{\mathbf{Q}_{p^2}}(p)$, and write ω_2 for the reduction modulo p of ε_2. For any $n, m \ge 1$ we let $\rho_{n,m}$ denote the representation

$$\mathrm{Sym}^{n-1}\,\mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \varepsilon_2^m : G_{\mathbf{Q}_p} \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p),$$

which is crystalline with Hodge–Tate weights $0, m, \ldots, (n-1)m$.

We have

$$\overline{\rho}_{p,m} \cong \overline{\varepsilon}^{m(p-1)/2} \oplus \bigoplus_{i=1}^{(p-1)/2} \mathrm{Ind}_{G_{\mathbf{Q}_{p^2}}}^{G_{\mathbf{Q}_p}} \omega_2^{m(1-p)i}.$$

Suppose that $(m, p+1) = 1$ (so that in particular m is odd). Then $\omega_2^{m(1-p)}$ has order exactly $p+1$, and the $\mathrm{Gal}(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$ Galois conjugate of $\omega_2^{m(1-p)i}$ is $\omega_2^{-m(1-p)i}$. It follows, under this assumption on m, that $\overline{\rho}_{p,m}$ does not depend on m, so there is an isomorphism of orthogonal representations $\overline{\rho}_{p,m} \cong \overline{\rho}_{p,1}$. Our assumptions that f is non-ordinary, that $k < p$, and that $(k-1, p+1) = 1$ therefore imply that $\overline{\rho}|_{G_{\mathbf{Q}_p}} \cong \overline{\rho}_{p,1}$, which admits the weight 0 crystalline lift $\rho_{p,1}$.

Write R for the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for GO_p-valued deformations of $\overline{\rho}$ which have multiplier ε^{1-p}, are unramified outside p, and whose restrictions to $G_{\mathbf{Q}_p}$ are crystalline of weight 0, and lie on the same component of the corresponding local crystalline deformation ring as $\rho_{p,1}$. By [BG19, Prop. 4.2.6], every irreducible component of R has Krull dimension at least 1.

Let $F^+/\mathbf{Q}$ and F/F^+ be quadratic extensions, with F^+ real quadratic and F imaginary CM, such that p is inert in F^+, the places of F^+ above p split in F, and $F/\mathbf{Q}$ is disjoint from $(\overline{\mathbf{Q}})^{\ker \overline{\rho}}(\zeta_p)$. As in the proof of [BLGGT14, Prop. 4.1.1], using [BLGGT14, Cor. A.2.3, Lem. A.2.5] we can find a cyclic CM extension M/F of degree $(k-1)$, and characters $\theta, \theta' : G_M \to \overline{\mathbf{Q}}_p^\times$ with $\overline{\theta} = \overline{\theta}'$, such that the representation $\overline{s} := \mathrm{Ind}_{G_M}^{G_F}(\theta \otimes \overline{\rho}|_{G_F})$ is absolutely irreducible. Furthermore we choose θ, θ' so that $\theta\theta^c = \varepsilon^{2-k}$, $\theta'(\theta')^c = \varepsilon^{p(2-k)}$, and $\mathrm{Ind}_{G_M}^{G_F} \theta, \mathrm{Ind}_{G_M}^{G_F} \theta'$ both are crystalline, with all sets of labelled Hodge–Tate weights respectively equal to $\{0, 1, \ldots, k-2\}$, $\{0, p, \ldots, p(k-2)\}$.

By construction, after possibly replacing F^+ by a solvable extension, we can and do assume that for each place $v|p$ of F, we have

$$\big(\mathrm{Ind}_{G_M}^{G_F} \theta\big)|_{G_{F_v}} \sim \rho_{k-1,1}|_{G_{F_v}}, \ \big(\mathrm{Ind}_{G_M}^{G_F} \theta'\big)|_{G_{F_v}} \sim \rho_{k-1,p}|_{G_{F_v}},$$

where $\sim$ is the notion "connects to" of [BLGGT14, §1.4]. We let R_F be the complete local Noetherian $\mathcal{O}$-algebra which is the universal deformation ring for $\mathcal{G}_{(k-1)p}$-valued deformations of (the usual extension of) $\overline{s}$, which have multiplier $\varepsilon^{1-(k-1)p}\delta_{F/F^+}$, are unramified outside p, and whose restrictions to the places above p are crystalline with Hodge–Tate weights $0, 1, \ldots, (k-1)p-1$, and lie on the same irreducible components of the local crystalline deformation rings as

$$(\rho_{k-1,p} \otimes \rho_{p,1})|_{G_{F_v}} \cong \rho_{(k-1)p,1}|_{G_{F_v}} \cong (\rho_{p,k-1} \otimes \rho_{k-1,1})|_{G_{F_v}}.$$

We have a finite map $R_F \to R$, taking a lifting ρ of $\overline{\rho}$ to $\mathrm{Ind}_{G_M}^{G_F}(\theta \otimes \rho|_{G_F})$.

We claim that the conclusions of [Tho17, Prop. 7.2] apply in our setting, so that R_F is a finite $\mathcal{O}$-algebra by [Tho12, Thm. 10.1]. Admitting this claim for a moment, we deduce that R is a finite $\mathcal{O}$-algebra, and since it has dimension at least 1, it has a $\overline{\mathbf{Q}}_p$-valued point. The corresponding lift $\rho : G_{\mathbf{Q}} \to \mathrm{GO}_p(\overline{\mathbf{Q}}_p)$ of $\overline{\rho}$ is unramified outside p, has multiplier ε^{1-p}, and is crystalline with Hodge–Tate weights $0, 1, \ldots, p-1$. By [Tho17, Thm. 7.1], $\mathrm{Ind}_{G_M}^{G_F}(\theta \otimes \rho|_{G_F})$ is automorphic, so ρ itself is automorphic by [BLGGT14, Lem. 2.2.1, 2.2.2, 2.2.4].

It remains to show that we can apply [Tho17, Thm. 7.1, Prop. 7.2]. To this end, we note that the notion of adequacy in [Tho17, Defn. 2.20] can be relaxed to assume only that $H^1(H, \mathrm{ad}) = 0$, rather than assuming that $H^1(H, \mathrm{ad}_0) = 0$; more precisely, the proof of [Tho17, Prop. 2.21] only uses this weaker assumption. Now, since $\overline{\rho}(G_{\mathbf{Q}(\zeta_p)})$ is adequate, and since $p \nmid (k-1)$, we see that $\overline{s}(G_{F(\zeta_p)})$ is adequate by [BLGG13, Lem. A.3.1] (whose proof goes over unchanged in this setting), as required. ☐

Corollary 3.2. *There exists a self-dual cuspidal automorphic representation π for $\mathrm{GL}_{79}/\mathbf{Q}$ of level one and weight zero.*

Proof. There exists ([Gou01, CG13]) a modular eigenform f of level 1 and weight $k = 38$ which is non-ordinary at $p = 79$, and $(37, 79+1) = 1$. ☐

Remark 3.3. The prime $p = 79$ is the second smallest prime for which there exists a non-ordinary form f of weight $k < p$. The smallest is $p = 59$ for which there exists a non-ordinary eigenform of weight $k = 16$. However, $(k-1, p+1) \neq 1$ in this case, so the construction fails in a number of places. Following [CG13], we see non-ordinary eigenforms of weight $k < p$ with $(p+1, k-1) = 1$ exist for $p = 151, 173, 193, \ldots$. As in Remark 2.6, we expect that they exist for a positive density set of primes p.

Remark 3.4. If π is cuspidal automorphic of level one and weight zero for $\mathrm{GL}_n/\mathbf{Q}$ with n odd, then for each $m \geq 1$ there is conjecturally a cuspidal automorphic representation of level one and weight zero for $\mathrm{GL}_{nm}/\mathbf{Q}$. Indeed, for each level one cuspidal eigenform f of weight $n+1$ (such an f exists because $n > 26$), the conjectural tensor product $\pi \boxtimes \mathrm{Sym}^{m-1} f$ should be automorphic and cuspidal of level one and weight zero.

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