



# Laplaza sets, or how to select coherence diagrams for pseudo algebras <sup>☆</sup>

Thomas M. Fiore <sup>a,b</sup>, Po Hu <sup>c</sup>, Igor Kriz <sup>d,\*</sup>

<sup>a</sup> *Department of Mathematics, University of Chicago, 5734 S. University, Chicago, IL 60637, USA*

<sup>b</sup> *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain*

<sup>c</sup> *Department of Mathematics, Wayne State University, 1150 Faculty/Administration Building,  
656 W. Kirby Detroit, MI 48202, USA*

<sup>d</sup> *Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street,  
Ann Arbor, MI 48109-1043, USA*

Received 1 May 2007; accepted 1 May 2007

Available online 5 May 2007

Communicated by Michael J. Hopkins

---

## Abstract

We define a general concept of pseudo algebras over theories and 2-theories. A more restrictive such a notion was introduced in [Po Hu, Igor Kriz, Conformal field theory and elliptic cohomology, *Adv. Math.* 189 (2) (2004) 325–412, <http://www.math.lsa.umich.edu/~ikriz/>], but as noticed by M. Gould, did not capture the desired examples. The approach taken in this paper corrects the mistake by introducing a more general concept, allowing more flexibility in selecting coherence diagrams for pseudo algebras.

© 2007 Published by Elsevier Inc.

*Keywords:* Lawvere theory; Algebraic theory; 2-theory; Pseudo algebra; Coherence; Conformal field theory; Worldsheet

---

---

<sup>☆</sup> Thomas M. Fiore was supported by NSF grant DMS 0501208 at the University of Chicago. At the Universitat Autònoma de Barcelona he was supported by grant SB2006-0085 of the Spanish Ministerio de Educación y Ciencia under the Programa Nacional de ayudas para la movilidad de profesores de universidad e investigadores españoles y extranjeros. Po Hu was supported in part by NSF grant DMS 0503814 and Igor Kriz was supported in part by NSF grant DMS 0305583.

\* Corresponding author.

*E-mail addresses:* [fiore@math.uchicago.edu](mailto:fiore@math.uchicago.edu) (T.M. Fiore), [po@math.wayne.edu](mailto:po@math.wayne.edu) (P. Hu), [ikriz@umich.edu](mailto:ikriz@umich.edu) (I. Kriz).

## 1. Introduction

Generalizing algebras to pseudo algebras is a basic idea which has recently become important in axiomatization of conformal field theory [2,5–7], as well as in other subjects, e.g. [1]. While the exact settings vary, the kind of algebras we are using generally have a set of *operations* and a set of *identities* (equations) the operations are required to satisfy. The corresponding notion of *pseudo algebra* is a category rather than a set. The operations are replaced by functors, and the identities are replaced by natural isomorphisms which we call *coherence isomorphisms*. Generally speaking, however, we now want additional conditions, namely commutative diagrams which are to be satisfied by the coherence isomorphisms. Such diagrams are generally known as *coherence diagrams*. The question of what coherence diagrams one should select is trickier than it may appear, and is the main subject of this note.

In [5], the following scheme was suggested for selecting coherence diagrams: take all diagrams which can be “reasonably expected” to commute. This means, take any word in our algebra which can be formed by using variables represented by formal symbols, and repeated use of operations which apply to them. Now taking such a word, we can use the identities among the operations (and any substitutions) to turn the word successively into other words. It may happen, however, that one word  $a$  can be turned into another word  $b$  in two different ways, using a different sequence of identities. To such a situation, there corresponds in an obvious way a coherence diagram (see, for example, [12]). It was suggested in [5] that *all* such diagrams should be required to commute in a pseudo algebra.

It turns out, however, that such a requirement is unreasonably strong. For example, if the algebras in question are commutative monoids where we denote the operation by  $\oplus$ , then the word  $a \oplus a$  can be turned into the same word either by an empty sequence of identities, or by an application (using substitution) of the identity

$$a \oplus b = b \oplus a. \quad (1)$$

The corresponding coherence diagram would then require that, in a pseudo commutative monoid, the coherence isomorphism

$$\tau_{ab} : a \oplus b \rightarrow b \oplus a$$

corresponding to (1) satisfies

$$\tau_{aa} = Id. \quad (2)$$

This however is unreasonably strong; we would like pseudo commutative monoids to be the same thing as symmetric monoidal categories, and those will not in general be equivalent to categories satisfying (2) (see Proposition 2.5 below). The authors thank M. Gould for this example, see Section 6 of [4].

To correct this, one must generalize the notion of pseudo algebra in a way that allows us to limit the scope of coherence diagrams required, so that “bad diagrams” such as the one mentioned above can be excluded. Surprisingly perhaps, as will be shown in examples given below, there is not a single way to do this which would cover all the examples desired. However, there is a fairly simple and general scheme which includes all the cases needed in [5–7]. This scheme amounts basically to including coherence diagrams coming from processing one word  $a$  to another word  $b$

using identities in the algebra, but with the restriction that the formal variables used in each of the words  $a$  and  $b$  occur exactly once within each word in each identity. This scheme requires an important restriction, namely all the identities in the algebra must be between words which use each variable exactly once. A precise formulation of this for the simplest case of universal algebras modelled on one set (“1-sorted algebras”) involves the language of operads, and its interplay with the language of theories. This idea is also basically due to M. Gould. See [3] and [4], and also [11] for background.<sup>1</sup> The relevant concepts are defined in the next section. The foundational results of [2] remain valid and can be generalized to the new context, and hence all the substantive results of [5–7] remain in effect.

One kind of algebra which is of interest in conformal field theory however is the algebra of “worldsheets,” i.e. Riemann surfaces with analytically parametrized boundary components, and the operations of disjoint union and gluing of boundary components of opposite orientations. Such worldsheets do not form a 1-sorted algebra. This is because gluing requires “dynamically indexed” operations, in the sense that the possible gluings depend on the set of boundary components of the worldsheet and their orientations. Such structures are not axiomatized by theories but by *2-theories*, introduced in [5]. To apply the operad scheme for generating coherence diagrams in this case, one needs to define *2-operads* (which we will do in Section 3 below).

Even this operadic approach, however, is not sufficiently general, since one is interested in algebras whose identities do involve words with repeated symbols. Commutative semi-rings give one such example, where the distributivity axiom involves a word with recurring variables on one side. In this case of pseudo commutative semi-rings, the pseudo algebras should be symmetric bimonoidal categories. The correct condition limiting coherence diagrams was discovered by Laplaza [9]. The condition is somewhat technical and will be explained in Example 2.7 below. It is not obvious what general scheme would select coherence diagrams “correctly” in accordance with what one expects for specific examples of algebraic structures known. However, it is not difficult to axiomatize what general *formal properties* such sets of diagrams must satisfy. Such sets of diagrams, inspired by Laplaza’s diagrams, we call *Laplaza sets* in recognition of his contribution.

## 2. Pseudo algebras with Laplaza sets in theories

Let us recall here the notion of a theory, which was first defined in [10]. We will stick to the “universal algebra” point of view, which is more advantageous for defining pseudo algebras.

**Definition 2.1.** Let  $\Gamma$  be the category with objects  $0, 1, 2, \dots$  where  $0 = \emptyset$  and  $k := \{1, \dots, k\}$  for  $k \geq 1$ . The morphisms are maps, not necessarily order-preserving. Let  $+$  :  $\Gamma \times \Gamma \rightarrow \Gamma$  be the functor defined by  $k + \ell := \{1, \dots, k + \ell\}$  and by placing maps side by side.

**Definition 2.2.** A *theory* is a functor  $T : \Gamma \rightarrow \mathbf{Sets}$  equipped with *compositions*

$$\gamma : T(k) \times T(n_1) \times \dots \times T(n_k) \rightarrow T(n_1 + \dots + n_k)$$

and a *unit*  $1 \in T(1)$  such that the following hold.

<sup>1</sup> The reference [4] was posted after the initial submission of the present article.

(1) The  $\gamma$ 's are *associative*, i.e.

$$\begin{aligned} &\gamma(w, \gamma(w^1, w_1^1, \dots, w_{n_1}^1), \gamma(w^2, w_1^2, \dots, w_{n_2}^2), \dots, \gamma(w^k, w_1^k, \dots, w_{n_k}^k)) \\ &= \gamma(\gamma(w, w^1, \dots, w^k), w_1^1, \dots, w_{n_1}^1, w_1^2, \dots, w_{n_2}^2, \dots, w_1^k, \dots, w_{n_k}^k). \end{aligned}$$

(2) The  $\gamma$ 's are *unital*, i.e.

$$\gamma(w, 1, \dots, 1) = w = \gamma(1, w)$$

for all  $w \in T(k)$ .

(3) The  $\gamma$ 's are *equivariant* with respect to the *functoriality*  $()_f := T(f)$  in the sense that

$$\gamma(w_f, w_1, \dots, w_\ell) = \gamma(w, w_{f_1}, \dots, w_{f_k})_{\bar{f}}$$

for every function  $f : \{1, \dots, k\} \rightarrow \{1, \dots, \ell\}$  where

$$\bar{f} : \{1, 2, \dots, n_{f_1} + n_{f_2} + \dots + n_{f_k}\} \rightarrow \{1, 2, \dots, n_1 + n_2 + \dots + n_\ell\}$$

is the function that moves entire blocks according to  $f$ .

(4) The  $\gamma$ 's are *equivariant* with respect to functoriality also in the sense that

$$\gamma(w, (w_1)_{g_1}, \dots, (w_k)_{g_k}) = \gamma(w, w_1, \dots, w_k)_{g_1 + \dots + g_k}$$

for all functions  $g_i : \{1, \dots, n_i\} \rightarrow \{1, \dots, n'_i\}$  where  $g_1 + \dots + g_k : \{1, 2, \dots, n_1 + \dots + n_k\} \rightarrow \{1, 2, \dots, n'_1 + \dots + n'_k\}$  is the function obtained by placing  $g_1, \dots, g_k$  next to each other from left to right.

The elements of  $T(k)$  are called *words*.

If  $X$  is a set, then  $End(X)(n) := Map(X^n, X)$  defines the *endomorphism theory* of  $X$ . Composition is the composition of functions. The unit is  $1_X$ . If  $f : k \rightarrow \ell$  is a function and  $w \in End(X)(k)$ , then  $w_f \in End(X)(\ell)$  is defined by  $w_f(x_1, \dots, x_\ell) = w(x_{f_1}, \dots, x_{f_k})$ . This example allows us to define *algebras over theories*. A set  $X$  is a *T-algebra* when it is equipped with a morphism  $T \rightarrow End(X)$  of theories.

**Remark.** Lawvere [10] originally defined theories more elegantly as categories with the set of natural numbers as objects, with the property that  $k$  is a categorical product of  $k$  copies of 1, with given projections. The way this relates to Definition 2.2 is that given a Lawvere theory  $\mathcal{T}$ , we define  $T(k) = \mathcal{T}(k, 1)$ . The axioms are then obviously satisfied. On the other hand, given a theory  $T$  in the sense of Definition 2.2, we set

$$\mathcal{T}(k, \ell) = T(k)^{\times \ell}. \tag{3}$$

(On the right-hand side, we mean the Cartesian product of sets.) Composition of elements of  $\mathcal{T}(m, k)$  and  $\mathcal{T}(k, \ell)$  is defined by applying  $\gamma$ , which will get us to a product of  $\ell$  copies of  $T(mk)$ , and then functoriality with respect to the map  $f : mk \rightarrow m$  which satisfies  $f(i) \equiv i \pmod m$ . To obtain the  $i$ th projection  $k \rightarrow 1$  in  $\mathcal{T}$ , we substitute the injection  $1 \mapsto i \in k$

into the unit 1 in  $T(1)$ . The unit in the category  $\mathcal{T}$  on the object  $k$  is the product of the  $i$ th projections,  $i = 1, \dots, k$ . One needs the equivariance axioms (3), (4) to prove associativity and unitality, although one can show that the axioms have some redundancy (i.e. can be deduced from special cases). (For complete detail, see Chapter 6 of [2].)

Theories model *1-sorted universal algebras*. By this we mean algebras whose definition calls for one set with some operations required to satisfy certain prescribed identities. Then the set of words  $T(n)$  is the set of all operations in  $n$ -symbols that arise as a composite of finitely many basic operations in  $T$ . The symbols in words are allowed to repeat.

More generally, an  $n$ -sorted (or  $I$ -sorted, where  $I$  is an indexing set) algebra calls for  $I$  sets and operations which are allowed to apply to prescribed sets in  $I$ , and produce an element of another prescribed set of  $I$ . Again, prescribed identities (equations) are required to hold. For example, a ring and a module form a 2-sorted algebra.

An important observation about categories  $\mathcal{C}$  of all multi-sorted algebras with given operations and identities is that if we have the category of all multi-sorted algebras  $\mathcal{D}$  whose operations and identities form (possibly empty) subsets of the sets of operations and identities of  $\mathcal{C}$ , then there is a forgetful functor  $\mathcal{C} \rightarrow \mathcal{D}$  which has a left adjoint. We usually refer to the left adjoint as the functor  $F$  taking the *free  $\mathcal{C}$ -algebra on a  $\mathcal{D}$ -algebra*. To prove the existence of these left adjoints, we note two constructions standard in algebra, the construction of a *free algebra* and the construction of a *quotient*. The first is the special case of left adjoint to the forgetful functor to systems of sets. To construct a free multi-sorted algebra of the given kind on a system of sets, take the set of all formal words using the operations on the elements of the applicable sets, and then factor out by the smallest equivalence relation which includes all the required identities and is preserved by operations (an operation on equivalent elements gives equivalent results). The quotient construction, for a multi-sorted algebra  $X$ , and a relation  $\sim$  on its elements, gives a universal *quotient* of  $X$  which is a multi-sorted algebra with the same operations and identities, and in which any pair of elements  $x \sim y$  is identified. Once again, it is constructed by taking the smallest equivalence relation which contains the given relation and is preserved by operations. To construct the left adjoint  $F$  mentioned above, we let  $F(X)$  be the free  $\mathcal{C}$ -algebra on  $X$ , and take the quotient under the relation identifying all  $\mathcal{D}$ -words in elements of  $X$  with their result in  $X$ .

$I$ -sorted algebras are not axiomatized by Lawvere theories, although the formalism can be adapted to them. We do not take this approach here, however, since we will need an even more general context, described in the next section.

**Definition 2.3.** The notion of *operad* is defined by following verbatim Definition 2.2, except that we replace the morphisms of the category  $\Gamma$  by all bijections, and restrict in the equivariance axioms (3), (4) to all bijective maps. Algebras over an operad are defined precisely analogously as we defined algebras over a theory.

One advantage of Definition 2.2 is that it exhibits the fact that the notion of a theory is itself an  $\mathbb{N}$ -sorted algebra where  $\mathbb{N}$  is the set of all natural numbers. By the above remarks, then, we have forgetful functors from theories to operads, to sequences of sets. We call a sequence of sets  $Z = \{Z(n)\}_{n \geq 0}$  a *collection*. A theory is free on an operad if and only if it is generated by operations and equations where the equations involve no repetition of variables on either side and the exact same variables occur on both sides: then the underlying operad consists of all words in chosen variables  $a_1, \dots, a_n$  which can be written, using the operations, where each variable has to be used exactly once.

These statements are proved easily. The point is, in both cases, we can define the operad generated by the given operations and identities. The free theory on those operations modulo these equations is the same thing as the free theory on the operad free on the operations modulo these equations: maps in both directions are exhibited and proved inverse to each other by the universal properties.

These definitions worked in the category of sets. However, the category  $Cat$  of small categories and functors has properties analogous to those of the category of sets and maps. One therefore immediately gets the analogous notion of *internal theory* in  $Cat$ . We call such an internal theory a *categorical theory*. Explicitly, to define a categorical theory, replace in Definition 2.2 *Sets* by  $Cat$ ; all the axioms (1)–(4) can be rewritten as diagrams, which are the same in  $Cat$  as in *Sets*. In particular, if  $X$  is a category, then we can similarly define a functor  $n \mapsto End_{Cat}(X)(n) = Funct(X^n, X)$  from  $\Gamma$  to  $Cat$  (the morphisms in  $End_{Cat}(X)(n)$  are natural transformations). If  $T$  is a categorical theory, then a category  $X$  is a  $T$ -algebra when it is equipped with a morphism  $T \rightarrow End_{Cat}(X)$  of categorical theories.

It is useful to also note that categorical theories  $T$  have an alternate description: both  $Obj(T)$  and  $Mor(T)$  are theories, while source, target, and identity are morphisms of theories. This makes  $Mor(T) \times_{Obj(T)} Mor(T)$  into a theory, and composition is a morphism of theories. Finally, let us note that we may consider the category of graphical pre-theories. A *graphical pre-theory*  $T$  is a theory  $\{Obj(T)(n)\}_{n \geq 0}$  and a collection  $\{Mor(T)(n)\}_{n \geq 0}$  together with maps *Source*, *Target*, and *Id* which satisfy the usual unital property, but without composition. Equivalently, a graphical pre-theory  $(Obj(T), Mor(T))$  is an internal reflexive graph in the category of collections with the additional property that the object collection  $Obj(T)$  is a theory. From another point of view, however (reinterpreting graphs and categories with a fixed object set as multi-sorted algebras over *Sets*), categorical theories and graphical pre-theories are also multi-sorted algebras in *Sets*. We then have, using our general observations about multi-sorted algebras, a forgetful functor from the category of categorical theories to the category of graphical pre-theories. This functor has a left adjoint, which is the free categorical theory on a graphical pre-theory. Further, both of these functors preserve the object collection.

With the notions of graphical pre-theory and categorical theory in hand, we are now ready to introduce pseudo algebras over a theory with respect to a Laplaza set. The purpose of a Laplaza set  $S$  is to specify which diagrams are required to commute in a pseudo  $T$ -algebra, and to do this we force certain diagrams to commute in an associated categorical theory  $T'_S$ . Roughly speaking, a Laplaza set for a theory  $T$  is a set  $S$  of words in  $T$ , and a diagram of coherence isomorphisms is required to commute whenever the words of the source and target are in the Laplaza set  $S$ . To make this precise, we construct from an ordinary theory  $T$  a categorical theory  $T'_S$  which has an isomorphism between each free composite of words in  $T$  and their composite in  $T$ . Some diagrams of isomorphisms will commute, exactly which ones is decided by the Laplaza set  $S$ . Any morphism  $T'_S \rightarrow End_{Cat}(X)$  of categorical theories then takes the abstract isomorphisms and commutative diagrams to coherence isomorphisms and coherence diagrams for  $X$ , thus, such a morphism is a pseudo  $T$ -algebra with respect to the Laplaza set  $S$ .

**Definition 2.4.** A *Laplaza set* for a theory  $T$  is an arbitrary collection of sets  $S(n) \subseteq T(n)$ . For a given Laplaza set, define a categorical theory  $T'_S$  as follows. First define a graphical pre-theory  $G_S$ . The theory  $Obj(G_S)$  is the free theory on the collection  $\{S(n)\}_{n \geq 0}$ . The set  $Mor(G_S)(n)$  contains one arrow  $\iota_{a,b}$  between each pair of objects in  $Obj(G_S)(n)$  which map

to the same element of  $S(n) \subseteq T(n)$ , and  $Id_a = \iota_{a,a}$ . Now  $T'_S$  is the quotient of the free categorical theory  $F_S$  on the graphical pre-theory  $G_S$  by the relations

$$\iota_{ab} \sim \iota_{ba}^{-1}, \tag{4}$$

If  $\alpha, \beta \in Mor(F_S)(n)$  satisfy  $Source(\alpha) = Source(\beta) = x$ ,  $Target(\alpha) = Target(\beta) = y$  and  $x, y$  project to the same element of  $S(n) \subseteq T(n)$ , then  $\alpha = \beta$ . (5)

A *pseudo  $(T, S)$ -algebra* (or any other permutation of these words, e.g. a pseudo algebra over  $T$  with respect to the Laplaza set  $S$  etc.) is a  $T'_S$ -algebra, i.e. a morphism  $T'_S \rightarrow End_{Cat}(X)$  of categorical theories.

The special case considered in [2,5] is  $S = T$ . The difficulty discovered by M. Gould is expressed very strongly by the following

**Proposition 2.5.** *Let  $T$  be the theory of commutative monoids. Then every pseudo  $(T, T)$ -algebra  $A$  is equivalent to a strictly symmetric strict monoidal category, i.e. a category  $A'$  which is a strict algebra over the theory of commutative monoids.*

**Proof.** Select representatives  $a_i, i \in I$ , of isomorphism classes of  $A$ , and assume  $I$  is a linearly ordered set, with minimum 0. Assume  $a_0$  is the unit. Let the operation be  $\oplus$ . Then define the category  $A'$ , with operation  $+$ , to have objects  $a_i$  with  $i \in I$  as well as formal sums

$$a_{i_1} + a_{i_2} + \dots + a_{i_n} \tag{6}$$

where  $0 < i_1 \leq \dots \leq i_n$ . The operation  $+$  on elements of the form (6) with  $n$  and  $m$  summands, respectively, is the sum of the form (6) which shuffles the elements together so that the indices are again in non-decreasing order. The sum  $x + a_0$  is defined to be  $x$  for any  $x$ . Clearly, this operation is commutative, associative, and unital.

To define morphisms, define a map  $F$  from the proposed objects of  $A'$  to  $Obj(A)$  by sending (6) to

$$a_{i_1} \oplus \dots \oplus a_{i_n}$$

and  $a_0$  to  $a_0$ . Then pull back  $Mor(A)$  via this map, thus promoting  $F$  into an equivalence of categories. It remains to define the operation  $+$  on morphisms. To define  $f + g$ , consider  $F(f) \oplus F(g)$  and compose on both sides with any coherence isomorphisms needed to shuffle the source and target back to order; then pull back via  $F$ . The result is unique by the observation that the switch coherence isomorphism  $a \oplus a \rightarrow a \oplus a$  must be the identity.  $\square$

All the formal results of [2] generalize to pseudo algebras with Laplaza sets, in particular these structures form 2-categories in the obvious way, and enjoy pseudo limits and bicolimits. The proofs of [2] work essentially word by word. From this point of view, pseudo algebras over  $T$  in the sense of [2] are pseudo algebras with respect to the Laplaza set  $S = T$ , i.e. a special case. For the biadjunctions discussed in [2], the appropriate forgetful functor is associated with a morphism of theories

$$\phi : T_1 \rightarrow T_2$$

together with Laplaza sets  $S_i \subseteq T_i$ , satisfying the condition

$$\phi(S_1) \subseteq S_2.$$

Then there is a forgetful 2-functor from pseudo  $(T_2, S_2)$ -algebras to pseudo  $(T_1, S_1)$ -algebras which enjoys a left biadjoint constructed by the same method as in [2].

If  $T$  is the free theory on an operad  $C$ , then there is a canonical example of a Laplaza set associated with  $C$  which is often useful, namely the collection  $\{C(n)\}_{n \geq 0}$  itself. One notes that if we denote by  $\Gamma$  (resp.  $\Sigma$ ) the category whose objects are natural numbers and morphisms  $m \rightarrow n$  are maps (resp. bijections)  $\{1, \dots, m\} \rightarrow \{1, \dots, n\}$ , then the free theory  $T$  on an operad  $C$  is

$$\Gamma \times_{\Sigma} C = \coprod_{n \geq 0} \Gamma(n, ?) \times_{\Sigma_n} C(n), \tag{7}$$

so the canonical map  $C(n) \rightarrow T(n)$  is injective.

If  $C$  is the operad defining commutative monoids and  $T$  is the free theory on this operad, then we do have an inclusion  $C \subset T$ , and pseudo  $(T, C)$ -algebras are unbiased symmetric monoidal categories. An *unbiased* symmetric monoidal category is one in which  $n$ -fold products  $x_1 \otimes x_2 \otimes \dots \otimes x_n$  are chosen in addition to binary products, and there are accompanying coherence isomorphisms satisfying the obvious coherence diagrams. The Laplaza set  $C$  essentially by definition forces precisely the coherence diagrams of [12] and their unbiased counterparts (one diagram in [12], which requires that the left and right unit coherence isomorphisms agree on the unit, is not included in our coherence diagrams, but this diagram was proved to follow from the others in [8]). The equivalence of the category of unbiased symmetric monoidal categories with the category of symmetric monoidal categories is essentially the coherence theorem of Mac Lane.

Let  $C$  be an operad. Then define a *categorical operad*  $C'$  (i.e. operad internal in categories) as follows: the objects are the free operad on the collection  $\{C(n)\}_{n \geq 0}$ , and there is precisely one isomorphism between any two objects of  $C'$  which map to the same element of  $C$ . A *pseudo  $C$ -algebra* is a  $C'$ -algebra, i.e. a morphism  $C' \rightarrow \text{End}_{\text{Cat}}(X)$  of categorical operads. This is the way that pseudo algebras over operads are defined in [3] and [4].

**Proposition 2.6.** *Let  $T$  be a theory which is free on an operad  $C$ , and let  $S$  be the corresponding Laplaza set. Then a pseudo  $(T, S)$ -algebra is the same thing as a pseudo  $C$ -algebra.*

**Proof.** Let  $T_S^{\sharp}$  be the free categorical theory on the categorical operad  $C'$ . There is an obvious morphism of categorical theories

$$T_S^{\sharp} \rightarrow T'_S. \tag{8}$$

Indeed, this map is obtained by universality and the observation that the relations in Definition 2.4 imply the operad relations in  $C'$ . We claim that (8) is an isomorphism. The object theory of  $T_S^{\sharp}$  is the free theory on the free operad  $\text{Obj}(C')$  on the collection  $\{C(n)\}_{n \geq 0}$ . This is the same as the object theory of  $T'_S$ , namely the free theory on the collection  $\{C(n)\}_{n \geq 0}$ . It is easy to see from the definitions that (8) must be full. To show that it is faithful, we claim that we can use the universality of  $T'_S$  to construct a left inverse. This is equivalent to the following statement:

$$\text{In } T_S^{\sharp}, \text{ there is precisely one isomorphism between any two objects which under the canonical map to } T \text{ project to the same element of } C. \tag{9}$$

To prove (9), the objects of  $T_S^\sharp$  are of the form  $(f, u)$  where  $f$  is a function, and  $u$  is an element of the free operad on the collection  $\{C(n)\}_{n \geq 0}$ . Denote by  $\bar{u}$  the image of  $u$  in  $T$ . Note first of all that if  $\bar{u} = \bar{v}$ , there is certainly an isomorphism between  $(f, u)$  and  $(f, v)$  in  $T_S^\sharp$ . Similarly, if  $\sigma$  is a bijection, then  $(f\sigma, u)$  is equal to  $(f, u_\sigma)$  in  $T_S^\sharp$ , so there is an identity isomorphism between them. Therefore, by induction one sees that when  $\bar{u}_f = \bar{v}_g$  in  $T$ , then  $(f, u)$  is isomorphic to  $(g, v)$  in  $T_S^\sharp$ .

Therefore, the only non-trivial statement in (9) is uniqueness, and it suffices to assume that the two objects concerned are the same object of  $C'$ . In other words, we must prove that for  $u \in \text{Obj } C'$ , the only self-map  $u \rightarrow u$  in  $T_S^\sharp$  is the identity. Let, therefore,  $(f, a)$  be another such self-map where  $f$  is a function and

$$a : v \rightarrow w \tag{10}$$

is a morphism in  $C'$ . Then we must have in particular

$$(f, v) = (f, w) = u \tag{11}$$

in  $\text{Obj}(T_S^\sharp)$ . But  $\text{Obj}(T_S^\sharp)$  is a free theory on a collection. Therefore, the elements of  $\text{Obj}(T_S^\sharp)$  are formal words we can write in a given ordered set of variables using the words in  $\{C(n)\}_{n \geq 0}$ . Repetition of variables is allowed, and there is no identification. The word will belong to  $\text{Obj } C'$  (the free operad) if no repetition of variables occurs and each variable is used exactly once. Thus,  $f$  must be a bijection, and hence we may as well assume  $f = \text{Id}$ . It then follows that  $v = w$  and hence  $a$  is the identity.  $\square$

**Example 2.7.** The previous proposition is not sufficient to define symmetric bimonoidal categories. In this case, the theory  $T$  is the theory of commutative semi-rings. This is not a free theory on an operad (the reason being that distributivity involves repetition of symbols on one side of the equation). This is the case [9] where the original Laplaza set was defined: one lets  $S(n)$  consist of all words which, when converted to the form of a sum of monomials using distributivity, identifying a monomial  $m$  with  $1 \cdot m$  and deleting any 0 summands ( $0 \cdot m = 0$ ), reduce to a sum of distinct square free monomials (monomials which are permutations of each other are considered equal). With this choice of Laplaza set  $S$ , a pseudo  $(T, S)$ -algebra is an unbiased symmetric bimonoidal category. The equivalence of the category of unbiased symmetric bimonoidal categories with the category of symmetric bimonoidal categories is essentially Laplaza’s coherence theorem in [9].

### 3. Laplaza sets in 2-theories

The notion of 2-theory is defined in [5]. The main example of interest here is the 2-theory of commutative monoids with cancellation, which we use to make the symmetric definition of conformal field theory in [13] more precise. We recapitulate these definitions here before turning to Laplaza sets for 2-theories.

**Definition 3.1.** A 2-theory consists of a natural number  $k$ , a theory  $T$ , sets

$$\Theta(w; w_1, \dots, w_n)$$

for all  $w_1, \dots, w_n, w \in T(m)^k, m \geq 0$ , and the following operations.

- (1) For each  $w \in T(m)^k$  there exists a *unit*  $1_w \in \text{End}(X)(w; w)$ .
- (2) For all  $w, w_i, w_{ij} \in T(m)^k$  there is a function called  $\Theta$ -composition.

$$\gamma : \Theta(w; w_1, \dots, w_q) \times \Theta(w_1; w_{11}, \dots, w_{1p_1}) \times \dots \times \Theta(w_q; w_{q1}, \dots, w_{qp_q}) \rightarrow \Theta(w; w_{11}, \dots, w_{qp_q}).$$

- (3) Let  $w, w_1, \dots, w_q \in T(m)^k$ . For any function  $\iota : \{1, \dots, p\} \rightarrow \{1, \dots, q\}$  there is a function

$$()^\iota : \Theta(w; w_{\iota(1)}, \dots, w_{\iota(p)}) \rightarrow \Theta(w; w_1, \dots, w_q)$$

called  $\Theta$ -functoriality.

- (4) Let  $w, w_1, \dots, w_q \in T(m)^k$ . For any function  $f : \{1, \dots, m\} \rightarrow \{1, \dots, \ell\}$  there is a function

$$()_f : \Theta(w; w_1, \dots, w_q) \rightarrow \Theta(w_f; (w_1)_f, \dots, (w_q)_f)$$

where  $w_f$  means to substitute  $f$  in each of the words in the  $k$ -tuple  $w$ . This function is called  $T$ -functoriality.

- (5) For  $u_i \in T(k_i), i = 1, \dots, m$  and  $w, w_1, \dots, w_q \in T(m)^k$  let  $v_j := \gamma^{\times k}(w_j; u_1^{\times k}, \dots, u_m^{\times k})$  for  $j = 1, \dots, q$  and furthermore let  $v := \gamma^{\times k}(w; u_1^{\times k}, \dots, u_m^{\times k})$ . Then there is a function

$$(u_1, \dots, u_m)^* : \Theta(w; w_1, \dots, w_q) \rightarrow \Theta(v; v_1, \dots, v_q)$$

called  $T$ -substitution. Here  $\gamma^{\times k}$  means to use the composition of the theory  $T$  in each of the  $k$  components, which coincides with composition in the theory  $T^k$  with  $T^k(m) := T(m)^k$ .

These operations satisfy the following *relations* (cf. pages 152–154 of [2]):

- (1)  $\Theta$ -composition is associative and unital in an analogous sense as (1) and (2) in the definition of a theory.
- (2)  $\Theta$ -functoriality is functorial in the sense that for functions

$$\{1, \dots, p\} \xrightarrow{\iota} \{1, \dots, q\} \xrightarrow{\theta} \{1, \dots, r\},$$

we have  $(\theta)^\iota = (\theta)^\iota$  and  $()^{Id} = Id$ .

- (3)  $\Theta$ -composition is equivariant with respect to  $\Theta$ -functoriality in two ways, analogously as (3) and (4) in the definition of a theory.
- (4)  $T$ -functoriality is functorial in the sense that for functions

$$\{1, \dots, n\} \xrightarrow{f} \{1, \dots, m\} \xrightarrow{g} \{1, \dots, \ell\},$$

we have  $(g)_f = (g)_f$  and  $()_{Id} = Id$ .

- (5)  $T$ -substitution is compatible with composition and unit in the sense that if  $w, w_1, \dots, w_q \in T(m)^k, t_i \in T(k_i), s_{ij} \in T(k_{ij}), 1 \leq i \leq m, 1 \leq j \leq k_i$ , if we set  $r_i = \gamma^{\times k}(t_i^{\times k}, s_{i1}^{\times k}, \dots, s_{ik_i}^{\times k})$ ,

$$(r_1, \dots, r_m)^* = (s_{11}, \dots, s_{mk_m})^*(t_1, \dots, t_m)^*,$$

and also

$$(1, \dots, 1)^* = Id.$$

- (6)  $\Theta$ -composition is  $T$ -equivariant in the sense that if  $f : \{1, \dots, m\} \rightarrow \{1, \dots, \ell\}$  is a function,  $w, w_i, w_{ij} \in T(m)^k$ ,  $\alpha \in \Theta(w; w_1, \dots, w_q)$ ,  $\alpha_j \in \Theta(w_j; w_{j1}, \dots, w_{jp_j})$  for  $j = 1, \dots, q$ , then

$$\gamma(\alpha_f; (\alpha_1)_f, \dots, (\alpha_q)_f) = \gamma(\alpha; \alpha_1, \dots, \alpha_q)_f.$$

- (7)  $\Theta$ -functoriality and  $T$ -functoriality commute in the sense that for functions  $\iota : \{1, \dots, p\} \rightarrow \{1, \dots, q\}$  and  $f : \{1, \dots, m\} \rightarrow \{1, \dots, \ell\}$ , we have  $(\alpha^t)_f = (\alpha_f)^t$  for all  $\alpha \in \Theta(w; w_{\iota(1)}, \dots, w_{\iota(p)})$ .

- (8)  $\Theta$ -functoriality and  $T$ -substitution commute:

$$(u_1, \dots, u_m)^* ()^t = ()^t (u_1, \dots, u_m)^*.$$

- (9)  $T$ -functoriality and  $T$ -substitution commute in the sense that for  $u_i \in T(k_i)$ , and  $f_i : \{1, \dots, k_i\} \rightarrow \{1, \dots, k'_i\}$ , if we denote by  $f : \{1, \dots, \sum k_i\} \rightarrow \{1, \dots, \sum k'_i\}$  the juxtaposition of the functions  $f_i$ , then

$$()_f (u_1, \dots, u_m)^* = ((u_1)_{f_1}, \dots, (u_m)_{f_m})^*.$$

- (10)  $T$ -functoriality and  $T$ -substitution also commute in the sense that if  $f : \{1, \dots, m\} \rightarrow \{1, \dots, \ell\}$  is a function, then

$$(u_1, \dots, u_\ell)^* ()_f = ()_{\bar{f}} (u_{f(1)}, \dots, u_{f(m)})^*.$$

- (11)  $T$ -substitution and  $T$ -composition commute in the sense that if  $u_i \in T(k_i)$ ,  $i = 1, \dots, m$ ,  $\alpha \in \Theta(w; w_1, \dots, w_q)$ ,  $\alpha_\ell \in \Theta(w_\ell; w_{\ell 1}, \dots, w_{\ell p_\ell})$  for  $\ell = 1, \dots, q$ , and  $\beta = (u_1, \dots, u_m)^* \alpha$ ,  $\beta_i = (u_1, \dots, u_m)^* \alpha_i$ , then

$$(u_1, \dots, u_m)^* \gamma(\alpha; \alpha_1, \dots, \alpha_q) = \gamma(\beta; \beta_1, \dots, \beta_q).$$

Again, there is an alternate “Lawvere-style” categorical description (cf. [5]). More precisely, in that sense, a 2-theory consists of a natural number  $k$ , a theory  $T$ , and a (strict) contravariant functor  $\Theta$  from  $T$  to the category of small categories (and functors) with the following properties. Let  $T^k$  denote the category with the same objects as  $T$  (natural numbers) and such that  $Hom_{T^k}(m, n) = (Hom_T(m, n))^{\times k}$ . Then for every natural number  $m$ ,

$$Obj(\Theta(m)) = \coprod_{n \geq 0} Hom_{T^k}(m, n),$$

for every morphism  $\phi : m \rightarrow n$  in  $T$  the map  $Obj(\Theta(n)) \rightarrow Obj(\Theta(m))$ , which is a part of  $\Theta(\phi)$ , is given by precomposition with  $(\phi, \dots, \phi)$ , and lastly every

$$\psi \in Hom_{T^k}(m, n)$$

is the product, in  $\Theta(m)$ , of the  $n$ -tuple

$$w_1, \dots, w_n \in Hom_{T^k}(m, 1)$$

with which it is identified by the fact that  $T$  is a theory.

This “categorical” definition is shown to be equivalent with the “algebraic” Definition 3.1 as follows. Operations (1), (2), (3) and relations (1), (2), (3) are equivalent to saying that  $\Theta(m)$  is a category, similarly as in the case of theories. The key point is to identify

$$Hom_{\Theta(m)}\left(\prod_{i=1}^n w_i, w\right) = \Theta(w; w_1, \dots, w_n).$$

Operations (4) and (5) are the morphism part of the strict 2-functor  $\Theta : T^{op} \rightarrow Cat$ . Relations (4), (5), (9), and (10) are then equivalent to saying that the morphism part of  $\Theta$  is a functor into sets. Relations (6), (7), (8), (11) are then equivalent to promoting  $\Theta$  to a functor into the category of small categories and functors.

Roughly speaking, the point of 2-theories is to index algebras with “dynamically indexed” operations. We have an algebra  $I$  over a certain theory and sets  $X_{(i_1, \dots, i_k)}$  where  $i_1, \dots, i_k \in I$ . The kind of  $n$ -ary operations we allow on the  $X$ ’s take as input tuples of elements

$$x_j \in X_{(w_{j1}, \dots, w_{jk})}$$

and produce an element of

$$X_{(w_1, \dots, w_k)}$$

where  $w_{ji}, w_i$  are certain specified words, all in the same given set of variables. Relations can also be specified on these operations, leading to the above definition.

More formally, if  $I$  is a set and  $X : I^k \rightarrow Sets$  is a map, then we have a 2-theory  $End(X)$  where  $End(I)$  is the underlying theory, and  $End(X)(w; w_1, \dots, w_n)$  is the set of maps

$$X \circ w_1 \circ d^m \times \dots \times X \circ w_n \circ d^m \rightarrow X \circ w \circ d^m$$

for maps  $w_1, \dots, w_n, w : (I^m)^k \rightarrow I^k$ . Here  $d^m : I^m \rightarrow (I^m)^k$  is the diagonal map. Using this example, a map  $X : I^k \rightarrow Sets$  is a  $(\Theta, T)$ -algebra when it is equipped with a morphism

$$(\Theta, T) \rightarrow (End(X), End(I))$$

of 2-theories.

It is useful to note again that the notions in the last paragraph are defined in the category of sets and maps, but can be defined in  $Cat$  when we replace “sets” by “categories” and “maps” by “functors.” In particular, associated to any category  $I$  and any strict 2-functor  $X : I^2 \rightarrow Cat$  there is a *categorical 2-theory*

$$(End_{Cat}(X), End_{Cat}(I)) \tag{12}$$

and an algebra over a categorical 2-theory is defined as a morphism from a given categorical 2-theory to (12). It is also useful to note that, again, we can alternately define categorical 2-theories as consisting of an object 2-theory and morphism 2-theory, which satisfy the axioms of a category, but in the category of 2-theories.

2-theories are not multi-sorted algebras in the usual sense. However, if we have already fixed a theory  $T$ , then 2-theories over  $T$  are multi-sorted algebras (sorted over  $(n + 1)k$ -tuples of elements of  $T(m)$  for all  $m$ ). Therefore, we can speak of a free 2-theory on a system of sets  $\mathcal{E}(\gamma; \gamma_1, \dots, \gamma_n)$  over a given theory  $T$ . We can also, once  $T$  and  $\mathcal{E}$  are fixed, impose equivalence relations  $\sim$  on each of the sets  $\mathcal{E}(\gamma; \gamma_1, \dots, \gamma_n)$ . There exists a universal quotient of  $\mathcal{E}$  which forms a 2-theory over  $T$  and on which  $\sim$  will turn into equality.

Given a categorical theory  $T$ , we may define the notion of graphical pre-2-theory  $(\mathcal{E}, T)$ . This consists of a 2-theory  $(Obj(\mathcal{E}), Obj(T))$  with underlying theory  $Obj(T)$ , a set  $Mor(\mathcal{E})(\gamma; \gamma_1, \dots, \gamma_n)$  for all  $\gamma; \gamma_1, \dots, \gamma_n$   $k$ -tuples of words of the theory  $Mor(T)$ , as well as *Source*, *Target*, *Id* maps satisfying the usual unitality axioms, but no composition. There are no 2-theory axioms on the sets  $Mor(\mathcal{E})(\gamma; \gamma_1, \dots, \gamma_n)$ . The free categorical 2-theory on the graphical pre-2-theory  $(\mathcal{E}, T)$  has the same underlying categorical theory  $T$  and the same object 2-theory  $(Obj(\mathcal{E}), Obj(T))$ .

**Definition 3.2.** A Laplaza set  $(\Sigma, S)$  for a 2-theory  $(\mathcal{E}, T)$  consists of an arbitrary collection of sets  $S(n) \subseteq T(n)$  and an arbitrary system of sets  $\Sigma(\gamma; \gamma_1, \dots, \gamma_n) \subseteq \mathcal{E}(\gamma; \gamma_1, \dots, \gamma_n)$  where  $\gamma; \gamma_1, \dots, \gamma_n$  are words in  $S(m)^k$  for some  $m$ . For a given Laplaza set, define a categorical 2-theory  $(\mathcal{E}'_\Sigma, T'_S)$  as follows. First,  $T'_S$  was already defined in Definition 2.4. Next, define  $Obj(\mathcal{E}'_\Sigma)$  as the free 2-theory on the system of sets  $\Theta(\delta; \delta_1, \dots, \delta_n)$  indexed by words  $\delta; \delta_1, \dots, \delta_n$  in  $Obj(T'_S)(m)^k$  that project to some  $\gamma; \gamma_1, \dots, \gamma_n$  in  $S(m)^k$ . Here the set  $\Theta(\delta; \delta_1, \dots, \delta_n)$  is equal to  $\Sigma(\gamma; \gamma_1, \dots, \gamma_n)$  where  $\gamma; \gamma_1, \dots, \gamma_n$  is the projection of  $\delta; \delta_1, \dots, \delta_n$  to  $S(m)^k$ .

To define  $Mor(\mathcal{E}'_\Sigma)$ , first define (as in the theory case) a graphical pre-2-theory  $\Gamma_{(\Sigma, S)}$  over the categorical theory  $T'_S$  whose objects are  $Obj(\mathcal{E}'_\Sigma)$ , and morphisms are one  $\iota_{a,b}$  between each  $a, b \in Obj(\mathcal{E}'_\Sigma)$  indexed over tuples  $\delta; \delta_1, \dots, \delta_n, \epsilon; \epsilon_1, \dots, \epsilon_n$ , respectively, indexed over the  $\iota_{\delta_i, \epsilon_i}$  for  $i = \emptyset, 1, \dots, n$ , when, additionally,  $a, b$  project to the same word in  $\Sigma(\gamma; \gamma_1, \dots, \gamma_n) \subseteq \mathcal{E}(\gamma; \gamma_1, \dots, \gamma_n)$ . Again, we impose that  $\iota_{a,a} = Id$  to get a reflexive graph.

Now we define  $\mathcal{E}'_\Sigma$  as the quotient of the free categorical 2-theory  $\Phi_{(\Sigma, S)}$  on  $\Gamma_{(\Sigma, S)}$  by the relations

$$\iota_{ab} \sim \iota_{ba}^{-1}, \tag{13}$$

$$\begin{aligned} &\text{If } \alpha, \beta \in Mor(\Phi_{(\Sigma, S)}) \text{ satisfy } Source(\alpha) = Source(\beta) = x, \\ &Target(\alpha) = Target(\beta) = y \text{ and } x, y \text{ project to the same element} \\ &\text{of } \Sigma, \text{ then } \alpha = \beta. \end{aligned} \tag{14}$$

(Note that since elements of  $\Sigma$  are only allowed to be indexed over tuples of words in  $S^k$ ,  $\alpha$  and  $\beta$  are necessarily indexed over the same tuple of morphisms.) A pseudo  $(\mathcal{E}, T, \Sigma, S)$ -algebra (or any other permutation of these words, e.g. a pseudo algebra over  $(\mathcal{E}, T)$  with respect to the Laplaza set  $(\Sigma, S)$  etc.) is an algebra over the categorical 2-theory  $(\mathcal{E}'_\Sigma, T'_S)$ , i.e. a morphism  $(\mathcal{E}'_\Sigma, T'_S) \rightarrow (End_{Cat}(X), End_{Cat}(I))$  of categorical 2-theories.

Again, pseudo algebras over a 2-theory  $(\mathcal{E}, T)$  with respect to a given Laplaza set  $(\Sigma, S)$  enjoy pseudo limits; the proofs of [2] generalize easily. Therefore, we can speak of stacks of pseudo  $(\mathcal{E}, T, \Sigma, S)$ -algebras.

**Definition 3.3.** One can define the notion of 2-operad by repeating Definition 3.1 with the following changes:

- $T$  is the free theory on an operad  $C$ .
- The word “function” is replaced by “bijection” in  $\Theta$ -functoriality and also in axioms pertaining to  $\Theta$ -functoriality as appropriate.

In particular, the indexing words of a 2-operad can be theory words.

By an analogous argument as before, the forgetful functor from 2-theories to 2-operads has a left adjoint, the free 2-theory  $(\mathcal{E}, T)$  on a 2-operad  $(\Delta, C)$ . In this case,  $(\Delta, C)$  provides a canonical choice of Laplaza set in  $(\mathcal{E}, T)$ .

It is worth commenting again that a 2-theory  $(\mathcal{E}, T)$  is free over a 2-operad when  $T$  is free over an operad  $C$ , and  $\mathcal{E}$  can be expressed as a quotient of a free 2-theory on a set of given generating operations in tuples of words in  $T$  by equations both sides of which have the exact same indeterminates, which do not repeat. The proof is analogous to the case of the adjunction between theories and operads.

The main example of a 2-operad of interest here is the 2-operad of commutative monoids with cancellation. In this example,  $k = 2$ , and  $T$  is the theory of commutative monoids. We describe this 2-operad via its algebras.

**Definition 3.4.** A strict 2-functor  $X : I^2 \rightarrow \text{Cat}$  is an algebra over the 2-operad of commutative monoids with cancellation if  $I$  is an algebra over the operad of commutative monoids, and  $X$  is equipped with natural functors

$$\begin{aligned}
 + : X_{a,b} \times X_{c,d} &\rightarrow X_{a+c,b+d}, \\
 \check{?} : X_{a+c,b+c} &\rightarrow X_{a,b}, \\
 0 &\in X_{0,0}
 \end{aligned}$$

satisfying the following axioms.

- (1) The operation  $+$  is commutative.

$$\begin{array}{ccc}
 X_{a,b} \times X_{c,d} & \xrightarrow{+} & X_{a+c,b+d} \\
 \downarrow & & \parallel \\
 X_{c,d} \times X_{a,b} & \xrightarrow{+} & X_{c+a,d+b}
 \end{array}$$

(2) The operation  $+$  is *associative*.

$$\begin{array}{ccc}
 (X_{a,b} \times X_{c,d}) \times X_{e,f} & \xrightarrow{+\times 1_{X_{e,f}}} & X_{a+c,b+d} \times X_{e,f} \\
 \downarrow & & \downarrow + \\
 X_{a,b} \times (X_{c,d} \times X_{e,f}) & & X_{(a+c)+e,(b+d)+f} \\
 \downarrow 1_{X_{a,b}} \times + & & \parallel \\
 X_{a,b} \times X_{c+e,d+f} & \xrightarrow{+} & X_{a+(c+e),b+(d+f)}
 \end{array}$$

(3) The operation  $+$  has *unit*  $0 \in X_{0,0}$ .

$$\begin{array}{ccc}
 X_{a,b} \times \{0\} & \xrightarrow{+} & X_{a+0,b+0} \\
 \searrow pr_1 & & \parallel \\
 & & X_{a,b}
 \end{array}$$

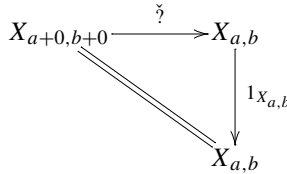
(4) The operation  $\check{?}$  is *transitive*.

$$\begin{array}{ccc}
 X_{(a+c)+d,(b+c)+d} & \xrightarrow{\check{?}} & X_{a+c,b+c} \\
 \parallel & & \downarrow \check{?} \\
 X_{a+(c+d),b+(c+d)} & \xrightarrow{\check{?}} & X_{a,b}
 \end{array}$$

(5) The operation  $\check{?}$  *distributes* over the operation  $+$ .

$$\begin{array}{ccc}
 X_{a+c,b+c} \times X_{e,f} & \xrightarrow{+} & X_{(a+c)+e,(b+c)+f} \\
 \downarrow \check{?} \times 1_{X_{e,f}} & & \parallel \\
 X_{a,b} \times X_{e,f} & \xrightarrow{+} & X_{(a+e)+c,(b+f)+c} \\
 & & \downarrow \check{?} \\
 & & X_{a+e,b+f}
 \end{array}$$

(6) Trivial cancellation is trivial.



**Remark.** The reader should note that characterizing commutative monoids with cancellation in terms of its algebras is purely a matter of language. In terms of Definition 3.1,  $T$  is the theory of commutative monoids,  $k = 2$ , and the  $\Theta$ 's are all the operations we can express by iterating the operations  $+$ ,  $\checkmark$ , and  $0$  using iteration and substitution without repetition of variables. Such operations are identified subject to the relations (1)–(6).

In this example,

$$+ \in \text{End}(X)((pr_1 + pr_3, pr_2 + pr_4); (pr_1, pr_2), (pr_3, pr_4))$$

for  $pr_i \in \text{End}(I)(4)$  and

$$\checkmark \in \text{End}(X)((pr_1, pr_2); (pr_1 + pr_3, pr_2 + pr_3))$$

for  $pr_i \in \text{End}(I)(3)$ . Projection to the  $i$ th coordinate is the same as substitution by the injective map  $t_i : \{1\} \rightarrow \{1, \dots, k\}$ ,  $t_i(1) = i$ . This is the reason why theory words are permitted as indexing words in the definition of 2-operad.

It is clear that the theory of commutative monoids is given by an operad. The full force of the  $T$ -functoriality of 2-operads as we defined it is used in the transitivity axiom (4): In the composition of cancellations

$$X_{(a+c)+d,(b+c)+d} \rightarrow X_{a+c,b+c} \rightarrow X_{a,b}, \tag{15}$$

the first map is obtained from the cancellation

$$X_{(u+d),(v+d)} \rightarrow X_{u,v}$$

by  $T$ -substituting  $a + e$  for  $u$ ,  $b + h$  for  $v$ , but then applying  $T$ -functoriality with respect to the map identifying the variables  $e$  and  $h$  into  $c$ .

**Example 3.5.** Recall that a *worldsheet* is a real, compact, not necessarily connected, two-dimensional manifold with complex structure and analytically parametrized boundary components.

**Proposition 3.6.** *Worldsheets form a pseudo algebra over the 2-theory of commutative monoids with cancellation (with Laplaza set corresponding to the 2-operad described above, on which the 2-theory is free).*

**Proof.** Let  $I$  denote the symmetric monoidal category of finite sets and bijections with  $+=\coprod$ . For finite sets  $A$  and  $B$ ,  $X_{A,B}$  is the category of worldsheets with inbound components labelled by  $A$  and outbound components labelled by  $B$ . A morphism in  $X_{A,B}$  is a holomorphic diffeomorphism that preserves boundary parametrizations and boundary component labellings. If  $f$  and  $g$  are bijections the functor  $X_{f,g}$  corresponds to boundary relabellings. The operation  $+$  is the disjoint union of worldsheets,  $\check{\gamma}: X_{a+c,b+c} \rightarrow X_{a,b}$  is the self-gluing of boundary components with the same label in  $c$ , and  $0 \in X_{0,0}$  is the empty manifold. The coherence isomorphisms from the previous definition are defined by noting that we have canonical embeddings  $X, Y \rightarrow X \coprod Y$  and a canonical map  $X \rightarrow \check{X}$  which is an embedding on the interior  $X - \partial X$  of  $X$ . We see then for  $n$  distinct worldsheets  $X_1, \dots, X_n$ , and any worldsheet  $X$  obtained by repeated use of  $+$  and  $\check{\gamma}$  on  $X_1, \dots, X_n$  where we use each  $X_i$  exactly once, there are canonical maps  $X_i \rightarrow X$  which are embeddings on the interior of  $X_i$  with disjoint images, whose union is dense in  $X$ . Further, these embeddings commute with the coherence isomorphisms corresponding to the identities in Definition 3.4. Therefore, any coherence diagram corresponding to two ways of processing a word on distinct variables  $X_1, \dots, X_n$  into another word using identities in Definition 3.4 will commute on the union of images of the  $X_i$ 's in the result  $X$  of the composite operation. But this union is dense in  $X$ .  $\square$

More strongly, worldsheets actually form a *stack* of pseudo algebras over the 2-operad of commutative monoids with cancellation: the construction of the stack structure given in [5] is correct in this new definition.

**Definition 3.7.** A *conformal field theory* (in the most abstract sense) is a morphism of stacks of pseudo algebras over the 2-operad of commutative monoids with cancellation.

## Acknowledgment

The authors thank Michael Shulman for some comments on an earlier draft.

## References

- [1] R. Blackwell, G.M. Kelly, A.J. Power, Two-dimensional monad theory, *J. Pure Appl. Algebra* 59 (1) (1989) 1–41.
- [2] Thomas M. Fiore, Pseudo limits, biadjoints, and pseudo algebras: Categorical foundations of conformal field theory, *Mem. Amer. Math. Soc.* 182 (860) (2006), x+171, <http://arxiv.org/abs/math.CT/0408298>.
- [3] Miles Gould, Coherence for categorified operadic theories, <http://arxiv.org/abs/math.CT/0607423>.
- [4] Miles Gould, The categorification of a symmetric operad is independent of signature, <http://arxiv.org/abs/0711.4904>.
- [5] Po Hu, Igor Kriz, Conformal field theory and elliptic cohomology, *Adv. Math.* 189 (2) (2004) 325–412, <http://www.math.lsa.umich.edu/~ikriz/>.
- [6] Po Hu, Igor Kriz, Closed and open conformal field theories and their anomalies, *Comm. Math. Phys.* 254 (1) (2005) 221–253, <http://www.math.lsa.umich.edu/~ikriz/>.
- [7] Po Hu, Igor Kriz, On modular functors and the ideal Teichmüller tower, *Pure Appl. Math. Q.* 1 (3) (2005) 665–682, <http://www.math.lsa.umich.edu/~ikriz/>.
- [8] A. Joyal, R. Street, Braided tensor categories, *Adv. Math.* 102 (1) (1993) 20–78.
- [9] Miguel L. Laplaza, Coherence for distributivity, in: *Coherence in Categories*, in: *Lecture Notes in Math.*, vol. 281, Springer, Berlin, 1972, pp. 29–65.
- [10] F. William Lawvere, Functorial semantics of algebraic theories, *Proc. Nat. Acad. Sci. USA* 50 (1963) 869–872.

- [11] Tom Leinster, *Higher Operads, Higher Categories*, London Math. Soc. Lecture Note Ser., vol. 298, Cambridge Univ. Press, Cambridge, 2004, <http://arxiv.org/abs/math.CT/0305049>.
- [12] Saunders Mac Lane, *Categories for the Working Mathematician*, second ed., Grad. Texts in Math., vol. 5, Springer-Verlag, New York, 1998.
- [13] Graeme Segal, The definition of conformal field theory, in: *Topology, Geometry and Quantum Field Theory*, in: London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577.