Research Statement

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1 Introduction

My primary area of research is the study of dynamics on moduli spaces. The first part of my thesis is on the recurrence behavior of the *Teichmuller geodesic flow*. The second part applies the ergodic theory of this flow to obtain asymptotics for the number of closed trajectories for right-angled billiards.

The subject of dynamics on moduli spaces lies at the intersection of low-dimensional topology, ergodic theory, and dynamics of group actions. Currently, I am using the ideas developed and knowledge acquired while writing my thesis to study several problems, in particular, to understand the $SL(2, \mathbb{R})$ action on the cotangent bundle of the Riemann moduli space and its interactions with the geometry of the space, with application to the study of 4-manifolds. In addition, I am studying dynamics of Lie group actions on trees, and applications to diophantine approximation in fields of positive characteristic.

2 Research Summary

Throughout, let Σ_g denote a compact topological surface of genus $g \geq 0$. Let T_g and M_g denote the Teichmuller space and Riemann moduli space of surfaces of genus g respectively. Let $\tilde{Q}_g$ and Q_g denote the respective cotangent bundles. The fiber over a Riemann surface S is the vector space of unit-area holomorphic quadratic differentials $\Omega(S)$. Letting $\Gamma_g = \pi_0(Homeo^+(\Sigma_g))$, we have $M_g = T_g/\Gamma_g$ and $Q_g = \tilde{Q}_g/\Gamma_g$. There is an $SL(2, \mathbb{R})$ action on $\tilde{Q}_g$, such that the projections of orbits to T_g are isometrically embedded (with respect to the Teichmuller metric on T_g) copies of the hyperbolic plane $\mathbb{H}$, known as Teichmuller disks. The action commutes with that of Γ_g and thus descends to Q_g. While Q_g (and M_g) are not compact, there is an absolutely continuous ergodic $SL(2, \mathbb{R})$-invariant probability measure μ on Q_g. We are particularly interested in the action of the diagonal subgroup $A = \left\{ g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$. This action is known is as the *Teichmuller geodesic flow*. Other subgroups of interest are the maximal compact subgroup $K = \left\{ r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : 0 \leq \theta < 2\pi \right\}$

(circle flow) and $N = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ (horocycle flow). The geodesic and horocycle flows are known to be ergodic and even mixing with respect to the measure μ, by results of Masur [15] and Veech [17].

2.1 Quantitative recurrence and large deviations

Motivated by the question of the *rate of mixing* of Teichmuller geodesic flow, in [1], we prove the following large deviations result:

Theorem 1. *Let $q \in Q_g$. There is a compact set C and constants $c_1, c_2 > 0$ and $\lambda < 1$ such that*

$$|\{\theta : \frac{1}{T}|\{0 \leq t \leq T : g_t r_\theta q \in C\}| < 1 - \lambda\}| \leq c_1 e^{-c_2 T},$$

for all T sufficiently large.

In intuitive terms, this result states that for almost all quadratic differentials $q \in Q_g$, if we pick a rotation $r_\theta q$ at random (according to Lebesgue measure on the circle $[0, 2\pi)$), the likelihood of picking a trajectory that has spent a large proportion of its life up to time T outside a compact set decays exponentially in T.

While ergodicity guarantees that $\frac{1}{T}|\{0 \leq t \leq T : g_t q \in C\}| \to \nu(C)$ for μ-almost every $q \in Q_g(P)$, our result gives explicit information about the likelihood of bad trajectories, and in particular, follwing Forni in [9] information about hyperbolicity of geodesic trajectories along circle orbits through *every* point.

In [5], Avila, Gouezel, and Yoccoz use similar results to show that the rate of mixing for the geodesic flow is exponential, making it one of the first examples of an exponentially mixing flow outside of surfaces of negative curvature. This also yields the existence of a spectral gap for the $SL(2, \mathbb{R})$ action on Q_g.

2.2 Rectangular Billiards

In [4], I have collaborated with my advisor Alex Eskin and Anton Zorich to obtain asymptotics for the number of closed trajectories for billiards in tables with right angles.

Precisely, we have the following: let S denote a billiard table with all angles multiples of $\pi/2$, and let $N(S, r)$ denote the number of closed billiard trajectories on S of length less than or equal to r.

Theorem 2. *There is a constant c depending only on the shape of the table and a function $e(R)$ such that for almost every rectangular billiard S,*

$$N(S, R) = cR^2 + e(S, R),$$

where

$$\lim_{R \to \infty} \frac{1}{\log R} \int_1^R \frac{e(S, r)}{r^3} dr = 0.$$

The proof of this result follows techniques developed by Eskin and Masur in [6], which relate the counting problem to a problem in the ergodic theory of the Teichmuller flow on a certain subset of Q_g, for appropriate g. In this case, $g = 0$, and the subset is a collection of meromorphic quadratic differentials with simple poles.

In order to calculate the constants, one must calculate volumes of certain moduli spaces. For this, we follow techniques of Eskin and Okounkov [8], and use a generalization of a counting formula of Kontsevich [12].

3 Research Proposal

The overarching theme of my research is the use of techniques from probability theory and dynamics to obtain quantitative results in a variety of different contexts: moduli spaces, billiards, and diophantine approximation, in particular. We outline some specific areas of study below:

3.1 $SL(2, \mathbb{R})$ and moduli space

There are many more open problems exploring the connections between dynamics of the $SL(2, \mathbb{R})$ action and the geometry of moduli space. In particular, there are 3 problems that I wish to study:

3.1.1 Brownian motion and random walks

The following question was posed to me by G. Forni:

Question 1. *Does the brownian motion on Q_g have an exponential rate of mixing with respect to a general harmonic measure?*

Another version of this question is to replace brownian motion by certain discrete-time random walks.

The results mentioned in section 2.1 all have analogues for the case of random walks, which were also proved in the first part of my thesis [1]. I have done extensive research in this area, and am ideally suited to attack this problem.

The solution of this problem will also likely yield an alternative (and likely simpler) proof for existence of a spectral gap for the $SL(2, \mathbb{R})$-action on Q_g, and one which will hold for *all* harmonic measures on Q_g.

3.1.2 A lattice point problem

Let $x \in T_g$, and consider the orbit $\Gamma_g x$. Consider the number of points in this orbit within distance R of another point y in the Teichmuller metric, i.e., $N(x, y, R) = |B(x, R) \cap \Gamma_g x|$. What are the asymptotics of $N(x, y, R)$ as $R \to \infty$? We have the following conjecture:

Conjecture 1. *For all $x, y \in T_g$, $N(x, y, R)/vol(B(x, R)) \to vol(M_G)$ as $R \to \infty$.*

Here, *vol* of a set denotes its measure when lifted to Q_g or $\tilde{Q}_g$. This is a natural generalization of similar questions for hyperbolic manifolds, or more generally, symmetric spaces. For results in those cases, see, for example [7].

3.1.3 Veech groups and 4-manifolds

Given a $q \in Q_g$, we denote $SL(q)$ to be the stabilizer of q in $SL(2, \mathbb{R})$. This is known as the *Veech group*, or affine diffeomorphism group, of the quadratic differential q. While $SL(q)$ is never co-compact, it can be a lattice, in which case q is said to be a Veech surface. In this case, the orbit of q in Q_g is closed, and forms an embedded non-compact Riemann surface. In this case, let $g(q)$ denote the genus of the orbit.

The following problem was posed by Benson Farb:

Question 2. *Can you find Veech surfaces $q_1, q_2 \in Q_g$, $g(q_1) \neq g(q_2)$ and a parabolic element $\gamma \in SL(2, \mathbb{R})$ such that γ is the unique parabolic element in each $SL(q_i)$?*

A positive answer to this question would yield new constructions of surface bundles overs surfaces with nonzero signature.

3.1.4 Affine realization

Kerckhoff has shown [11] that any finite subgroup G of Γ_g can be realized as a group of automorphisms of a Riemann surface, thus, in particular, that it fixes a point $x \in T_g$. A similar question can be asked with affine diffeomorphisms in place of automorphisms, and, more generally, for arbitarary subgroups of the mapping class group. Geometrically, this question asks if every finite subgroup G fixes (setwise) a Teichmuller disk.

More generally, one can ask which subgroups G of Γ_g stabilize Teichmuller disks. Partial progress can be found in [3], and can be summarized as follows:

- Every finite cyclic subgroup stabilizes an entire family of Teichmuller disks through its fixed point x, and acts by rotations.
- Every cyclic subgroup generated by a pseudo-Anosov element stabilizes the Teichmuller disk containing its axis, and acts via a hyperbolic tranformation.
- Every cyclic subgroup generated by a Dehn twist stabilizes a Teichmuller disk, and acts parabolically ([14]).
- Almost all subgroups generated by two postitive multi-twists about curve systems that fill the surface stabilize a Teichmuller disc ([13]).

For a finite subgroup G, the question boils down to understanding the induced representation $\rho : G \to \Omega(x)$, where $\Omega(x)$ is the vector space of quadratic differentials on the fixed point $x \in T_G$. The behavior of this represenation also yields information on sub-Teichmuller spaces with prescribed symmetries.

3.2 Diophantine approximation in positive characteristic

In collaboration with Anish Ghosh, I have been studying metric diophantine approximation in the field $K = \mathcal{F}_q((t^{-1}))$, where $q = p^n$ is a prime power, and $\mathcal{F}_q$ denotes the finite field of order q.

Our program is to extend dynamical techniques which have been successful in characteristic zero to the case of positive characteristic. Prelminary results include using a logarithm law for the action of the diagonal subgroup on quotients T/Γ of the Bruhat-tits tree T for $G = SL(2, K)$, Γ a non-uniform lattice to obtain a Khintchin-type result. Precisely, we have:

Theorem 3. *For every $x, y \in T/\Gamma$, and for almost every direction $\theta \in \mathbb{P}K$,*

$$\limsup_{n\to\infty} \frac{d(g_n(x,\theta), y)}{\log n} = 1/q,$$

where $\{g_n(x, \theta)\}$ denotes the geodesic ray from x in the direction θ.

Applying this result to the modular group, $\Gamma = SL(2, A)$, where $A = k[t]$, we obtain our diophantine result:

Theorem 4. *For almost all $f \in K$, there are infinitely many solutions $P, Q \in k[t]$ to*

$$|f - P/Q| \leq a(|Q|)/|Q|^2$$

if and only if $\int_1^\infty \frac{a(x)}{x} dx$ diverges.

Other partial results can be found in [2, 10].

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