HOMEWORK 1 - Math 255, Section 61

Due: Wednesday April 4th.

Exercise 1. Let R and R' be rings and $f : R \to R'$ a ring homomorphism.

(1) Prove that the kernel $\ker(f)$ is an ideal of R.
(2) Prove that f is injective if and only if its kernel is trivial, i.e. $\ker(f) = 0$.
(3) Prove that the image $f(R)$ of f is an subring of R'.
(4) Give an example showing that the image of f is in general not an ideal of R'.
(5) Assume that f is surjective. Prove that the rings $R/\ker(f)$ and R' are isomorphic.
(6) Let $\mathfrak{a} \subset R'$ be an ideal. Show that the preimage $f^{-1}(\mathfrak{a})$ is an ideal of R'.

Consider the map

$$\phi : \mathbb{Q}[T] \to \mathbb{R}, \quad \phi : P(T) \mapsto P(\sqrt{2})$$

(7) Prove that ϕ is a ring homomorphism.
(8) Determine the image of ϕ.
(9) Determine the kernel of ϕ.

Exercise 2. Let R a ring.

(1) Let $(R_i)_{i\in I}$ be a family of subrings of R. Prove that $\cap_{i\in I} R_i$ is a subring as well.
(2) Let $S \subset R$ be a subset. Prove that there is some subring R_S of R containing S and with the following property: If $R' \subset R$ is a subring and $S \subset R'$ then $R_S \subset R'$. (Hint: consider the intersection of all subrings of R containing S.)
(3) Let $(\mathfrak{a}_i)_{i\in I}$ be a family of ideals of R. Prove that $\cap_{i\in I} \mathfrak{a}_i$ is an ideal as well.
(4) Let $S \subset R$ be a subset. Prove that there is some ideal $\mathfrak{a}_S$ of R containing S and with the following property: If $\mathfrak{a}' \subset R$ is an ideal and $S \subset \mathfrak{a}'$ then $\mathfrak{a}_S \subset \mathfrak{a}'$.

Exercise 3. Let R be a ring and $\mathfrak{p} \subset R$ an ideal. Prove that $\mathfrak{p}$ is prime if and only if $R/\mathfrak{p}$ has no 0-divisors.