HOMEWORK 2 - Math 255, Section 61

Due: Wednesday April 11th.

Exercise 1. Recall that a number $p \neq 1$ in $\mathbb{Z}$ is prime if and only if whenever we have $p = ab$ with $a, b \in \mathbb{Z}$ then either $a = \pm 1$ or $b = \pm 1$.

(1) Using the euclidean algorithm, prove that $\mathbb{Z}$ is a principal ideal domain.
(2) Given $n \in \mathbb{Z}$ let (n) be the ideal generated by n, i.e. (n) is the set of those $m \in \mathbb{Z}$ divisible by n.
(3) Prove that (n) is a proper prime ideal if and only if n is prime.
(4) Let $p \in \mathbb{Z}$ be a prime number. Using (3), prove that whenever $a, b \in \mathbb{Z}$ and p divides ab then it either divides a or b.
(5) Assume that $p_1, p_2, p_3, p_4 \in \mathbb{Z}$ are prime and that $p_1 p_2 = p_3 p_4$. Prove that either $p_1 = \pm p_3$ or $p_1 = \pm p_4$.
(6) Argue as in (5) and prove that $\mathbb{Z}$ is a unique factorization domain.

Exercise 2.
Consider the following subset of $\mathbb{C}$:

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} | a, b \in \mathbb{Z}\}$$

(1) Prove that $\mathbb{Z}[\sqrt{-5}]$ is a subring of $\mathbb{C}$.
(2) Determine the units of $\mathbb{Z}[\sqrt{-5}]$.
(3) Prove that the element 2 is irreducible in $\mathbb{Z}[\sqrt{-5}]$.
(4) Observing that $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ prove that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.
(5) Assume that R and R' are rings and $\phi : R \to R'$ is a surjective ring homomorphism. Assume also that R is a principal ideal domain. Show that R' is also a principal ideal domain.
(6) Let $\mathbb{Z}[T]$ be the ring of polynomials with integer coefficients. Show that the map $\phi : \mathbb{Z}[T] \to \mathbb{Z}[\sqrt{-5}]$ given by $\phi(P(T)) = P(\sqrt{-5})$ is a surjective ring homomorphism.
(7) Use (4), (5) and (6) and deduce that $\mathbb{Z}[T]$ is not a unique factorization domain.

In class I proved that whenever k is a field then $k[T]$ is a unique factorization domain. The only thing I used was that $k[T]$ is a principal ideal domain (compare with exercise 1). In particular, it follows from exercise 2 (7) that $\mathbb{Z}[T]$ is also not a principal ideal domain.