HOMEWORK 5 - Math 255, Section 61

Due: Wednesday May 2nd.

Exercise 1.
Let k be a field and V a finite dimensional k-vector space.

(1) If X is an arbitrary set let $\mathrm{Map}(X, V)$ be the set of all maps $f : X \to V$. Given $f, g \in \mathrm{Map}(X, V)$ and $\lambda \in k$ define $f + \lambda g$ to be the map with
$$(f + \lambda g)(v) = f(v) + \lambda g(v)$$
for all $v \in X$. Show that $\mathrm{Map}(X, V)$ is a k-vectorspace.

(2) With the same notation as in (1) assume that X is finite. Determine the dimension of $\mathrm{Map}(X, V)$ and give a basis.

Let now W and $\bar{W}$ be two more finite dimensional k-vectorspaces.

(3) Prove that $\mathrm{Hom}(W, V)$ is a subspace of $\mathrm{Map}(W, V)$.

(4) Still assuming that W is a k-vectorspace let $W^n = W \times \cdots \times W$ be the product of n copies of W. Prove that the set $\mathrm{Mult}_n(W, V)$ of all n-multilinear maps $W^k \to V$ is a subspace of $\mathrm{Map}(W^n, V)$.

(5) Let $F : \bar{W} \to W$ be a linear map and $\phi \in \mathrm{Mult}_n(W, V)$. Prove that the map
$$F^*\phi : \bar{W} \to V, \quad (F^*\phi)(x_1, \dots, x_n) = \phi(F(x_1), \dots, F(x_n))$$
is n-multilinear.

(6) With the same notation as in (5), prove that
$$F^* : \mathrm{Mult}_n(W, V) \to \mathrm{Mult}_n(\bar{W}, V), \ \phi \mapsto F^*\phi$$
is a linear map.

(7) With the same notation as in (5) and (6) assume that F is an isomorphism. Prove that F^* is an isomorphism as well.

Given a permutation $\sigma \in S_n$ and $\phi \in \mathrm{Mult}_n(W, V)$ consider the map
$$\phi^\sigma : W^n \to V, \ \phi^\sigma(x_1, \dots, x_n) = \phi(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$
A n-multilinear map ϕ is *symmetric* if $\phi^\sigma = \phi$ for all $\sigma \in S_n$ and it is *alternating* if $\phi^\sigma = \mathrm{signum}(\sigma)\phi$. Let $\mathrm{Sym}_n(W, V)$ and $\mathrm{Alt}_n(W, V)$ be the subsets of $\mathrm{Mult}_n(W, V)$ consisting of symmetric and alternating multilinear forms respectively.

(8) Prove that $\phi^\sigma \in \mathrm{Mult}_n(W, V)$ for all $|phi \in \mathrm{Mult}_n(W, V)$ and $\sigma \in S_n$.

(9) Prove that $\mathrm{Sym}_n(W, V)$ and $\mathrm{Alt}_n(W, V)$ are linear subspace of $\mathrm{Mult}_n(W, V)$.

(10) Determine the dimension and a basis of $\mathrm{Mult}_2(W, k)$, $\mathrm{Sym}_2(W, k)$ and $\mathrm{Alt}_2(W, k)$.