

HOMEWORK 7 - Math 255, Section 61

Due: Wednesday May 16nd.

Exercise 1. Let k be a field, V and W be k -vector spaces of dimensions n and m respectively, and let $\phi : V \rightarrow W$ be a linear map.

- (1) Let $U \subsetneq V$ be a subspace, let \mathcal{B}_U be a basis of U and $v \in V \setminus U$. Prove that the set $\mathcal{B}_U \cup \{v\}$ is linearly independent.
- (2) Use (1) to prove that there is a basis \mathcal{B} of V with $\mathcal{B}_U \subset \mathcal{B}$.
- (3) Show that $\mathcal{B} \setminus \mathcal{B}_U$ projects to a basis of the quotient space V/U under the canonical projection $\pi : V \rightarrow V/U$, $\pi(x) = [x]$.
- (4) Use (3) to prove that $\dim V = \dim \text{Ker}(\phi) + \dim \phi(V)$.

Exercise 2. Let V be a finite dimensional k -vectorspace and $\phi : V \rightarrow V$ a linear map.

- (1) For $\lambda \in k$, show that the set of vectors $v \in V$ with $\phi(v) = \lambda v$ is a subspace of V .
- (2) Show that for $\lambda, \mu \in k$ different the subspaces $\text{Ker}(\phi - \lambda \text{Id})$ and $\text{Ker}(\phi - \mu \text{Id})$ intersect only at 0.
- (3) Let $E_\lambda = \cup_{i=1}^{\infty} \ker((\phi - \lambda \text{Id})^i)$. Prove that E_λ is a subspace of V .
- (4) Give an example showing that E_λ and $\text{Ker}(\phi - \lambda \text{Id})$ are in general not equal.
- (5) Prove that for $\lambda \neq \mu$ the subspaces E_λ and E_μ intersect only at 0.

Exercise 3 Consider the following map

$$\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \phi(x, y, z) = (y + z, x + z, x + y)$$

- (1) Prove that ϕ is diagonalizable.
- (2) Let A be the matrix of ϕ with respect to the standard basis of \mathbb{R}^3 . Find a matrix B such that BAB^{-1} is diagonal.