HOMEWORK 3 - Math 256, Section 31

Due: Wednesday April 18th.

Exercise 1. Let k be a field.

(1) Let $P(T) \in k[T]$ be an irreducible polynomial (of positive degree). Prove that the ideal $(P(T))$ consisting of all polynomials divisible by $P(T)$ is maximal.
(2) Let $K = k[T]/(P(T))$ and prove that K/k is a finite extension.
(3) Show that $P(T)$ has a solution in K. *Observe that every polynomial divisible by $P(T)$ has also a solution in K.*
(4) Given a second irreducible polynomial $Q(T) \in k[T]$ construct a finite extension K'/K in which $Q(T)$ has a solution. *In principle, you don't know if $Q(T)$ is irreducible in $K[T]$!!!*
(5) Deduce that for every finite subset $S \subset k[T]$ there is a finite extension K_S/k in which every non-constant polynomial in S has a solution.

Assume now that k is either finite or countable.

(6) Show that $k[T]$ is countable.
(7) Let $S_1 \subset S_2 \subset S_3 \subset \ldots$ be a sequence of finite subsets of $k[T]$ with $k[T] = S_1 \cup S_2 \cup S_3 \cup \ldots$. Construct, as above in (5), a sequence of fields $k \subset K_1 \subset K_2 \subset K_3 \subset \ldots$ with the following property: Every non-constant polynomial in S_i has a solution in K_i.
(8) Let $K_1, K_2, K_3, \ldots$ be as in (7) and set $\bar{k} = \cup_i K_i$. Given $a, b \in \bar{k}$ there is some i with $a, b \in K_i$. Define the sum $a + b$ and the product ab of a and b in $\bar{k}$ to be the sum and the product in K_i. Prove that $\bar{k}$ is a field with the property that every polynomial $P(T) \in k[T]$ has a solution in $\bar{k}$.
(8) Prove that every element in $\bar{k}$ is algebraic over k.
(9) Prove that $\bar{k}$ is an algebraically closed field.
(10) Prove that $\bar{k}$ is countable.

As you see, the goal of this exercise is to prove that whenever k is countable then it is contained in some algebraically closed field. If k is uncountable the proof does not work: why?

Exercise 2 Prove that for $n \geq 5$ the group A_n is simple. See exercise 39 in page 153 in Fraleigh's book.