

## HOMEWORK 6 - Math 256, Section 31

Due: Wednesday May 9th.

**Exercise 1.** Fix  $n \in \mathbb{N}$ .

- (1) Determine the set of solutions in  $\mathbb{C}$  of the polynomial  $T^n - 1$ . These are the so called *n-th roots of unity*.
- (2) Prove that the n-th roots of unity form, under multiplication, a cyclic group  $G_n$ . Determine all generators (these are the so-called *primitive n-th roots of unity*).
- (3) Let  $\xi_n$  be a primitive n-th root of unity. Show that the group of automorphisms of the field  $\mathbb{Q}(\xi_n)$  is isomorphic to the group of automorphisms of  $G_n$ .
- (4) Determine explicitly the automorphism groups of  $\mathbb{Q}(\xi_6)$  and  $\mathbb{Q}(\xi_7)$ .
- (5) Determine the automorphism groups of the extensions  $\mathbb{Q}(\xi_{30})/\mathbb{Q}(\xi_6)$ ,  $\mathbb{Q}(\xi_6)/\mathbb{Q}(\xi_3)$  and  $\mathbb{Q}(\xi_3)/\mathbb{Q}$ .

Given a field extension  $K/k$  we say that a nested sequence of subfields

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_r = K$$

is maximal if for all  $i$  there is no field  $L \neq k_i, k_{i+1}$  with  $k_i \subset L \subset k_{i+1}$ .

- (6) Prove that for every finite field extension  $K/k$  there is some maximal nested sequence of subfields.
- (7) Determine some maximal nested sequences of subfields for the extension  $\mathbb{Q}(\xi_{210})/\mathbb{Q}$ .

**Exercise 2.** Prove that  $\mathbb{R}$  has no non-trivial automorphisms. *Hints: prove that every automorphism fixes  $\mathbb{Q}$ , then show that every automorphism is order preserving.*

Another quote, this time from Goethe's *Faust*:

*Habe nun, ach! Philosophie,  
Juristerei und Medizin,  
Und leider auch Theologie  
Durchaus studiert, mit heißem Bemühn.  
Da steh ich nun, ich armer Tor!  
Und bin so klug als wie zuvor;  
Heiße Magister, heiße Doktor gar  
Und ziehe schon an die zehen Jahr  
Herauf, herab und quer und krumm  
Meine Schüler an der Nase herum-  
Und sehe, daß wir nichts wissen können!*

Those among you who speak German should read this aloud. Those among you who don't, should learn German just to be able to read *Faust* aloud. In any case, here is the translation with the important sentence marked.

*I've studied now Philosophy  
And Jurisprudence, Medicine,  
And even, alas! Theology  
All through and through with ardour keen!  
Here now I stand, poor fool, and see  
I'm just as wise as formerly.  
Am called a Master, even Doctor too,  
**And now I've nearly ten years through  
Pulled my students by their noses to and fro  
And up and down, across, about,  
And see there's nothing we can know!***