CANNON-THURSTON MAPS FOR THICK FREE GROUPS

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ABSTRACT. We prove the existence of Cannon-Thurston maps for bounded geometry discrete and faithful representations of free groups in $\mathrm{PSL}_2\,\mathbb{C}$.

It is well known that an open, orientable, genus $g \geq 2$ handlebody admits a hyperbolic metric σ. In particular, identifying the $\mathrm{PSL}_2\,\mathbb{C}$ with the group of orientation preserving isometries we obtain a discrete and faithful representation

$$\rho : \mathbb{F}_g \to \mathrm{PSL}_2\,\mathbb{C} \tag{0.1}$$

In fact, it follows from the positive resolution of the tameness conjecture by Agol [Ago] and Calegari-Gabai [CG] that for any such representation the hyperbolic 3-manifold $\mathbb{H}^3/\rho$ is in turn homeomorphic to a handlebody.
The action of $\mathrm{PSL}_2\,\mathbb{C}$ on $\mathbb{H}^3$ extends to an action on the boundary at infinity $\partial_\infty\mathbb{H}^3$ of hyperbolic space and it is well-known that $\partial_\infty\mathbb{H}^3$ can be identified with the complex projective line $\mathbb{C}P^1$ in such a way that the obtained action $\mathrm{PSL}_2\,\mathbb{C} \curvearrowright \mathbb{C}P^1$ is the one by rational transformations. On the other hand, the free group $\mathbb{F}_g$ is hyperbolic in the sense of Gromov and hence admits a compactification by $\partial_\infty\mathbb{F}_g$, its boundary at ∞, which is a homeomorphic to a Cantor set.
The goal of this note is to prove that whenever ρ is as in (0.1) and has bounded geometry then there is a ρ-equivariant continuous map

$$f_{CT} : \partial_\infty\mathbb{F}_g \to \partial_\infty\mathbb{H}^3 = \mathbb{C}P^1 \tag{0.2}$$

Recall that a discrete and faithful representation as in (0.1) has *bounded geometry* if the injectivty radius of the associated hyperbolic manifold $\mathbb{H}^3/\rho$ does not vanish.

Theorem 1. *If $\rho : \mathbb{F}_g \to \mathrm{PSL}_2\,\mathbb{C}$ is a bounded geometry representation then there is a continuous ρ-equivariant map $f_{CT} : \partial_\infty\mathbb{F}_k \to \mathbb{C}P^1$.*

Continuous maps as in (0.2) are called *Cannon-Thurston* maps for these two authors were the first to prove in [CT] their existence for certain bounded geometry representations of the fundamental group Γ_g of a closed genus g surface. Later, Minsky [Min94] proved the existence of Cannon-Thurston maps for every bounded geometry representation of Γ_g. Building on the work of Minsky, Mitra [Mit98] was able to prove the existence of Cannon-Thurston maps for bounded geometry representations of groups Γ that don't split as non-trivial free products[1]. Particular examples of Cannon-Thurston maps for representations without bounded geometry were constructed by Alperin, Dicks and Porti [ADP99]. McMullen [McM01] proved their existence for every representation of a rank 2 free group for which the commutator is parabolic. Recently Brahmachaitanya (formerly M. Mitra), has claimed the existence of Cannon-Thurston maps for every representation without parabolics of freely indecomposable groups [Bra1, Bra2, Bra3].

Theorem 1 above is the first case where Cannon-Thurston maps are constructed in the so called compressible setting. However, this is not the reason that in the opinion of the author justifies writing this paper. The real reason is that the proof is nice and simple. We follow the original approach of Cannon-Thurston [CT] and Minsky [Min94] but, surprisingly, the proof becomes simpler in this setting than in the original situation. The author hopes that the reader also finds that this is the case and that this is justification enough for this note to be writen.
Before concluding this introduction we would like to remark that it is not difficult to combine the results of Mitra [Mit98] with the arguments here to obtain the existence of Cannon-Thurston maps for arbitrary bounded geometry representations.

The proof given here of Theorem 1 was the product of several conversations with Gero Kleineidam and Jean-Pierre Otal while a visit of the author to the U.F.R. de Mathématiques Pures et Appliquées at the Université de Lille 1. They deserve therefore all the credit. The mistakes, faults in exposition and typos are responsibility of the author of this note (and his computer).

Shortly (very shortly), after concluding writing this note the author learned that Theorem 1 was actually due to Miyachi [Miy] who proved the existence of Cannon-Thurston maps for bounded geometry representations in $\mathrm{PSL}_2\,\mathbb{C}$ of arbitrary hyperbolic groups Γ.

[1] In [Mit98], the existence of Cannon-Thurston maps is claimed for tame bounded geometry representations but, at least in the opinion of the author of this note, his argument breaks if Γ splits as a free product. The confusion may be explained remarking that for example in Ohshika's paper [Ohs90] a Kleinian group is *defined* to be tame if it does not split as a free product; we will not follow this convention.

1. Models for the geometry

Let from now on ρ be a discrete and faithful bounded geometry representation as in (0.1) and let $M_\rho = \mathbb{H}^3/\rho$ be the associated hyperbolic 3-manifold. As mentioned above, it follows from the proof of the Tameness Conjecture that M_ρ is homeomorphic to the interior of a genus g handlebody.

If M_ρ is convex-cocompact, i.e. if there is a compact convex submanifold such that the inclusion is a homotopy equivalence, then it is well-known that for all points $p \in \mathbb{H}^3$ the map

$$\mathbb{F}_g \to \mathbb{H}^3, \quad \gamma \mapsto (\rho(\gamma))(p)$$

is a quasi-isometric embedding and hence extends to a continuous, obviously ρ-equivariant, Cannon-Thurston map. In other words one has:

Lemma 1. *If M_ρ is convex-cocompact then there is a continuous ρ-equivariant map $f_{CT} : \partial_\infty \mathbb{F}_k \to \mathbb{C}P^1$.* □

We assume from now on that M_ρ is not-convex cocompact. It follows from the work of Canary [Can96] that the only end of M_ρ is degenerate and hence has an ending lamination λ. See Thurston [Thu] and Canary [Can96] for the definition of the ending lamination. Minsky proved in [Min93] that the geometry of an end E of a bounded geometry hyperbolic 3-manifolds homotopy equivalent to a closed surface Σ_g is essentially determined by its ending lamination λ_E (see [Min, BCM] for the unbounded geometry case). Essentially Minsky proved the following: if q is a quadratic holomorphic differential on Σ_g whose vertical foliation induces λ_E and $(q_t)_{t\in[0,\infty)}$ is the associated Teichmüller ray, then the end E has a neighborhood which is bi-Lipschitz equivalent to

$$(\Sigma_g \times [0,\infty), \sigma_{q_t} \times dt^2)$$

where σ_{q_t} is the flat singular metric on Σ_g induced by the quadratic differential q_t. In local coordinates around a regular point $x \in \Sigma_g$ the metric can be described as follows

$$e^{2t}dx^2 + e^{-2t}dy^2 + dt^2 \tag{1.1}$$

Here x =constant (resp. y =constant) are the horizontal and vertical foliations. Recall that the singular locus of the metric σ_{q_t} consists of a finite collection of points around which it is equally well understood. See [Abi80] for basic facts about Teichmüller space and its geometry.

Minsky's result extends easily to ends of bounded geometry hyperbolic 3-manifolds whose fundamental group does not split as a free product. Moreover, using a trick due to Canary, Ohshika extended Minsky's result in [Ohs98] for bounded geometry (tame) hyperbolic manifolds. With the notation used above we obtain from these results:

Theorem. *The manifold M_ρ contains a compact submanifold C homeomorphic to a closed handlebody of genus g whose complement $M \setminus C$ is bi-Lipschitz equivalent to $(\partial C \times [0,\infty), \sigma_{q_t} \times dt^2)$ where q is a quadratic holomorphic differential on ∂C with vertical foliation inducing λ, $(q_t)_{t\in[0,\infty)}$ the associated Teichmüller ray and σ_{q_t} the corresponding singular flat metric.* □

Observe that the only condition on the quadratic holomorphic differential q is on its vertical foliation. This allows us to choose different quadratic holomorphic differentials to describe the geometry of the end of M_ρ. The key point in the proof of Theorem 1 is to choose q smartly. But before we do so we must describe a little bit more the geometry of the model

$$(\partial C \times [0,\infty), \sigma_{q_t} \times dt^2)$$

Let $\mathcal{F}_v$ and $\mathcal{F}_h$ be the vertical and horizonatal foliations of q. It follows directly from (1.1) that the 2-dimensional foliations $\mathcal{F}_v \times [0,\infty)$ and $\mathcal{F}_h \times [0,\infty)$ are totally geodesic. In fact, every leaf of the foliations obtained by lifting $\mathcal{F}_v \times [0,\infty)$ and $\mathcal{F}_h \times [0,\infty)$ to the universal cover of $\partial C \times [0,\infty)$ in convex.

The existence of a model with two foliations with convex leaves was the key-fact needed in the proof of existence of Cannon-Thurston maps in [CT, Min94]. In our setting we only have these foliations on the complement of the compact submanifold C and we cannot aspire to extend both of them to foliations of the whole manifold M_ρ. However, if we choose q properly we can extend $\mathcal{F}_h \times \mathbb{R}$ and this will suffice.

Let P be a pants decompostion of ∂C consisting of compressible curves in C. It is a theorem of Hubbard and Masur [HM79] that there is a quadratic holomorphic differential q on ∂C whose vertical foliation $\mathcal{F}_v$ induces λ and such that P can be isotoped to, say is, a collection of regular leaves of the horizontal foliation $\mathcal{F}_h$; this implies that every regular leaf of $\mathcal{F}_h$ is a closed curve isotopic to some component of P. We can moreover assume that the length with respect to d_{σ_q} of any two components of P is at least 2π.

Let now $\phi : [0,\infty) \to [0,\infty)$ be a smooth monotonous function such that

$$\phi(t) = \sinh(t) \text{ for } t < \frac{1}{10} \text{ and } \phi(t) = e^t \text{ for all } t \geq 10.$$

We consider now the following metric σ' on $\partial C \times (0,\infty)$

$$\sigma' = \phi(t)^2 dx^2 + e^{-2t} dy^2 + dt^2 \tag{1.2}$$

Both metrics $\sigma_{q_t} \times dt^2$ and σ' coincide on $\partial C \times [10, \infty)$ but they differ when t is close 0. For example, if l is a regular leaf of $\mathcal{F}_h$ then the length with respect to σ' of $l \times \{t\}$ tends to 0 when $t \to 0$. In particular, the σ'-metric completion of $l \times (0, \infty)$ is homeomorphic to $\mathbb{R}^2$. On the other hand the length of a segment contained in $\mathcal{F}_v \times (0, \infty)$ is the same with respect to $\sigma_{q_t} \times dt^2$ and σ'. We deduce that the metric completion of $(\partial C \times (0, \infty), \sigma')$ is a manifold homeomorphic to M_ρ (by a homeomorphism in the correct homotopy class). Moreover, it follows from the expression in coordinates (1.2) of σ' that the metric completion of $l \times (0, \infty)$ is convex for all regular leaves l in $\mathcal{F}_h$.
From all that, choosing the pants decomposition P which contains a sub-collection P' with $\partial C \setminus P'$ homeomorphic to a punctured sphere, we obtain:

Proposition 1. *There is a metric σ_0 on M_ρ which is bi-Lipschitz to the hyperbolic metric and such that (M_ρ, σ_0) contains a collection Δ of convex disks cutting M_ρ into an open ball.* □

2. Existence of the map

With the same notation as in the last section, we prove now the existence of a ρ-equivariant Cannon-Thurston map $f_{CT} : \partial_\infty \mathbb{F}_g \to \mathbb{C}P^1$.
When we pull back the metric σ_0 provided by Proposition 1 via the covering $\mathbb{H}^3 \to M_\rho$ then we obtain a metric on $\mathbb{H}^3$ which we denote again by σ_0. The first important observation is that since the hyperbolic metric and the metric σ_0 are bi-Lipschitz on M_ρ the same holds on $\mathbb{H}^3$. In particular, $(\mathbb{H}^3, \sigma_0)$ is Gromov-hyperbolic and its boundary coincides with the one of $\mathbb{H}^3$. If $\Delta \subset M_\rho$ is the collection of disks provided by Proposition 1 then the pre-image $\tilde{\Delta}$ of Δ in $\mathbb{H}^3$ is a locally finite collection of disks which are convex with respect to σ_0.
The manifold M contains a graph G with a single vertex which meets every component of Δ exactly once and such that the inclusion $G \hookrightarrow M$ is a homotopy equivalence. The pre-image $\tilde{G}$ of G to $\mathbb{H}^3$ is then a locally finite tree which meets every disk of $\tilde{\Delta}$ exactly once. In particular, every component of $\tilde{G} \setminus \tilde{\Delta}$ is a connected subtree whose image has diameter at most D where D is some constant independent of the component in question.
The graph G has fundamental group $\mathbb{F}_g$ and hence we can identify the boundary at infinite $\partial_\infty \mathbb{F}_g$ with the boundary $\partial_\infty \tilde{G}$ at infinity of the tree $\tilde{G}$. Points in $\partial_\infty \tilde{G}$ correspond to infinite injective paths in $\tilde{G}$. Any such path

$$\eta : [0, \infty) \to \tilde{G}$$

is coded by the sequence $(v_1, v_2, v_3, \dots)$ of vertices that it traverses. For all $j \geq 1$ there is a component D_j pf $\tilde{\Delta}$ separating v_j from v_{j+1}. Observe that the disk D_j separates also v_j of v_{j+k} for all $k \geq 1$. Let $\bar{U}_j$ be the closure in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$ of the component U_j of $\mathbb{H}^3 \setminus D_j$ which does not contain v_1. By the above we have that $\bar{U}_i \subset \bar{U}_j$ for all $i \geq j$. Since $\bar{U}_j$ is compact for all j we have that

$$\cap_{j=1}^{\infty} \bar{U}_j \tag{2.1}$$

is not empty. In order to prove that the path η has an endpoint in $\partial_\infty \mathbb{H}^3$ it suffices to prove that (2.1) consists of a single point. But this is immediate because $(\mathbb{H}^3, \sigma_0)$ is Gromov-hyperbolic, U_j is convex and $\cap_{j=1}^{\infty} U_j = \emptyset$. This proves that every infinite path in $\tilde{G}$ has a well-defined end point in $\partial_\infty \mathbb{H}^3$. In other words, the map $\tilde{G} \to \mathbb{H}^3$ extends to a map

$$(\tilde{G} \cup \partial_\infty \tilde{G}) \to (\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3)$$

which is obviously equivariant by the action of $\rho(\mathbb{F}_g)$. The continuity of the extension follows directly from (2.1). This concludes the proof of Theorem 1.

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