CHORDS, LIGHT, AND ANOTHER SYNTHETIC CHARACTERIZATION OF THE ROUND SPHERE

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Abstract. A chord for a closed geodesic γ in a complete Riemannian manifold M is a nontrivial geodesic segment beginning and ending on γ that is not completely contained in γ. We prove the existence of at least one geodesic chord for every closed geodesic in a closed Riemannian manifold. As an application, we give a synthetic characterization of round spheres in terms of blocking light.

The study of closed geodesics in Riemannian manifolds has a long and rich history. In compact manifolds with nontrivial fundamental group, closed geodesics are at least as plentiful as free homotopy classes; namely, homotopically essential curves can be pulled tight to closed geodesics. For compact, simply connected manifolds, more sophisticated techniques are needed to prove the existence of closed geodesics. In the 1930's Lyusternik and Shnirelman (see [Ba78]) proved that every closed simply connected manifold contains at least 3 geometrically distinct closed geodesics and this remains, even today, the best result without genericity assumptions on the metric. For reasons discussed below, we searched the literature for similar existence results for *chords* of closed geodesics but to our surprise we didn't find any.

Definition (Chord)**.** *Let γ be a closed geodesic in a complete Riemannian manifold M. A* chord *for γ is a non-constant geodesic segment η in M with endpoints on γ and which is not contained in γ.*

We obtain the following existence result for chords:

Theorem 1. *Assume that M is a closed Riemannian manifold of dimension at least 2 and that γ is a closed geodesic in M. Then there is a chord for γ.*

Before going further we would like to observe that the condition that the manifold be compact is necessary as one sees in the example of the standard flat cylinder $\mathbb{S}^1 \times \mathbb{R}^1$ where there are no chords. In the same spirit, the example of a bounded flat cylinder $\mathbb{S}^1 \times [0,1]$ shows that chords do not in general exist even if the manifold is compact and has totally geodesic boundary. It should also be remarked that the end-points of the chord cannot in general be taken to be arbitrary. For instance, in the round sphere $\mathbb{S}^n$, chords always join antipodal points. An even more extreme example is given by the real projective space $\mathbb{RP}^n$ where the end-points of every chord coincide.

If the manifold M in Theorem 1 is 2-dimensional, then the existence of chords follows from, for example, Santalo's formula (see e.g. [Sa52] or [Be65]). In higher dimensions, such an argument does not help, but in many cases Theorem 1 is readily established. However, in order to prove the general case we present a quite convoluted, but essentially elementary, Morse theoretic argument.

Again in the case of dimension 2, curve shortening arguments were used by Hass and Scott [HS94] to prove that whenever γ is a homotopically trivial simple closed geodesic in a closed surface, there are at least 2 simple chords which meet γ with right angles. We don't obtain anything in this direction and suspect that in general neither simple nor perpendicular chords need to exist.

Our interest in the existence of chords arose as we were characterizing round, i.e. constant curvature, spheres in terms of light blocking properties.

Definition (Light)**.** *Let X, Y be two nonempty subsets of a Riemannian manifold M, and let $G_M(X,Y)$ denote the set of non-constant unit speed paramaterized geodesics $\gamma : [0, L_\gamma] \to M$ with initial point $\gamma(0) \in X$ and terminal point $\gamma(L_\gamma) \in Y$. The* light from X to Y *is the set*

$$L_M(X,Y) = \{\gamma \in G_M(X,Y) |\, \text{interior}(\gamma) \cap (X \cup Y) = \emptyset\}.$$

A subset $Z \subset M$ blocks *the light from X to Y if the interior of every $\gamma \in L_M(X,Y)$ meets Z.*

Intuitively, we are postulating that X emits light traveling along geodesics, that Y consists of receptors, and that X and Y are opaque while the remaining medium $M \setminus \{X \cup Y\}$ is transparent. From this point of view, $L_M(X,Y)$ is the set of light rays from X to Y and a set Z blocks the light from X to Y if it completely shades X away from Y. This simple model ignores diffraction, the dual nature of light, and all aspects of quantum mechanics.

Remark. Using this notation, it is easy to see that Theorem 1 is equivalent to the following: If γ is a closed geodesic in a closed Riemannian manifold then $L_M(\gamma,\gamma)$ is not empty.

A well known result of Serre [Se51] asserts that for compact M and points $x, y \in M$, the set $G_M(x, y)$ of geodesic segments joining x and y is always infinite. In contrast, $L_M(x, y)$ is sometimes infinite and sometimes not. For instance, if x and y are different points on the standard round sphere $\mathbb{S}^n$ with distance less than π, then $L_{\mathbb{S}^n}(x, y)$ consists of exactly two elements. In particular, we see that, under the same assumptions, it suffices to declare two additional points in $\mathbb{S}^n$ to be opaque in order to block all the light rays from x to y.

Definition (Blocking Number). *Let $x, y \in M$ be two (not necessarily distinct) points in M. The blocking number $b_M(x, y)$ for $L_M(x, y)$ is defined by*

$$b_M(x, y) = \inf\{n \in \mathbb{N} \cup \{\infty\} | L_M(x, y) \text{ is blocked by } n \text{ points}\}.$$

The study of blocking light seems to have originated in the study of polygonal billiard systems and translational surfaces (see e.g. [Fo90], [Gu05a], [Gu05b], [Gu], [HS98], [Mo04], [Mo05], [Mo], and [Mo4]). More recently, blocking light has been studied in Riemannian spaces (see e.g. [GS06], [BG], and [LS]). Here we give a characterization of the round sphere in terms of its blocking properties.

If x, y are two distinct points in the standard round sphere $\mathbb{S}^n$ closer than π then, as remarked above, $b_{\mathbb{S}^n}(x, y) \leq 2$. This property does not characterize the round sphere amongst all closed Riemannian manifolds. In fact, every compact rank one symmetric space, or CROSS for short, has the following property:

> *Cross blocking*: For every distinct pair of points $x, y \in M$ with $d_M(x, y) < \text{diam}(M)$, we have $b_M(x, y) \leq 2$.

Apart from cross blocking, the round sphere also has the following property:

> *Sphere blocking*: For every point $x \in M$, we have $b_M(x, x) = 1$.

The CROSSes are classified and consist of the round spheres $\mathbb{S}^n$, the projective spaces $K\mathbb{P}^n$ where K denotes one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and the Cayley projective plane, each one endowed with its symmetric metric. It is not difficult to check that the round sphere is the only CROSS with sphere blocking.

In [LS] it was conjectured that a closed Riemannian manifold M with cross and sphere blocking is a round sphere. We prove that this is the case:

Theorem 2. *A closed Riemannian manifold M has cross and sphere blocking if and only if M is isometric to a round sphere.*

In order to prove Theorem 2 we show that manifolds as in the statements are Blaschke manifolds. Recall that a compact Riemannian manifold M is said to be *Blaschke* if its injectivity radius and diameter coincide. Berger [Be78] proved that a Blaschke manifold diffeomorphic to the sphere is in fact isometric to a round sphere. This was used in [LS] to prove Theorem 2 for Blaschke manifods.

In section 1 we discuss the needed aspects of Morse theory; most of this section is probably superfluous for anyone who has read Milnor's book *Morse theory*. In section 2 we use Morse theoretic arguments to prove Theorem 1. As mentioned above, the proof is elementary but at some points quite convoluted. Finally, in section 3 we prove Theorem 2.

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1. Broken Geodesics

In order to fix notation we start reviewing some well known definitions and results in differential geometry and the space of broken geodesics. Then we review the facts of Morse theory needed below.

1.1. **Basic definitions and notation.** Let M be a closed manifold with a Riemannian metric $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The length and energy of a piecewise smooth curve $\gamma : [0,T] \to M$ are given by

$$\begin{aligned} L_M(\gamma) &= \int \|\dot\gamma(t)\| dt \\ E_M(\gamma) &= \int \|\dot\gamma(t)\|^2 dt. \end{aligned} \tag{1.1}$$

The Cauchy-Schwartz inequality implies that $L(\gamma)^2 \leq T \cdot E(\gamma)$ with equality if and only if γ has constant speed $\|\dot\gamma\|$. A curve with constant speed 1 is said to be parametrized by arc-length. The distance $d_M(x,z)$ between two points in M is the infimum of the lengths of curves joining them and the diameter $\mathrm{diam}(M)$ is the maximal distance between points in M. A parametrized curve $\gamma : (0,T) \to M$ is a geodesic if it is locally distance minimizing. Equivalently, γ fulfills the geodesic differential equation; hence, geodesics are smooth. We will often say that the image of a geodesic is a geodesic as well. Geodesics will usually be denoted by Greek letters $\gamma, \eta, \dots$. A variation of geodesics is a smooth map $(s,t) \to \gamma_s(t)$ where γ_s is a geodesic for all s. The vectorfield $\frac{\partial}{\partial s}\gamma_s(t)$ along the curve γ_0 is said to be a Jacobi field. A vector field along a geodesic is a Jacobi field if and only if it satisfies the so called Jacobi equation, a second order ordinary differential equation. In particular, the space of Jacobi fields along a geodesic is a finite dimensional vector space and every Jacobi field J is determined by its initial value and derivative. Two points x and y in M are conjugate along a geodesic arc γ joining them if there is a nonzero Jacobi field along γ vanishing at x and y.

By the Hopf-Rinow theorem, any two points in M are joined by a geodesic segment whose length realizes the distance between them. Moreover, for every point $p \in M$ and for every direction $v \in T_pM$ there is a geodesic $t \mapsto \exp_p(tv)$ starting at p with direction v. Thus we obtain the so called exponential map

$$\exp_p : T_pM \to M$$

The exponential map is a local diffeomorphism in some small neighborhood of $0 \in T_pM$. The injectivity radius $\mathrm{inj}_p(M)$ is the maximum of those $r > 0$ such that exponential map is injective on the ball $B(0,r) = \{v \in T_pM | \; \|v\| < r\}$. The map $p \mapsto \mathrm{inj}_p(M)$ is continuous and hence attains a minimum, the injectivity radius $\mathrm{inj}(M)$ of the manifold.

For the sake of concreteness we will always assume that the manifolds in question have injectivity radius $\mathrm{inj}(M) \geq 2$ and will simply denote the length and energy functions by L and E instead of L_M and E_M.

1.2. **The space of broken geodesics.** Given $k \in \mathbb{N}$ let $\mathcal{L}_k$ be the set of piecewise geodesic curves consisting of at most k edges of at most length 1. To be more precise, elements $\gamma \in \mathcal{L}_k$ are continuous curves

$$\gamma : [0,k] \to M$$

such that for all $i = 0, 1, \dots, k-1$ the curve $\gamma|_{[i,i+1]}$ is a geodesic segment with length at most 1. When we endow $\mathcal{L}_k$ with the compact open topology, the valuation map

$$\mathcal{L}_k \to M^{k+1}, \quad \gamma \mapsto (\gamma(0), \dots, \gamma(k+1))$$

is continuous. Moreover, the assumption that $\mathrm{inj}(M) \geq 2$ implies that this map is injective and hence a homeomorphism onto its image. The interior $\mathcal{L}_k^\circ$ of $\mathcal{L}_k$, as a subset of M^{k+1}, is the set of those elements γ consisting of geodesic arcs of length stictly less than 1. The tangent space $T_\gamma\mathcal{L}_k^\circ$ at $\gamma \in \mathcal{L}_k^\circ$ is naturally identified with the space of the continuous vectorfields J along γ such that $J|_{[i,i+1]}$ is Jacobi for all $i = 0, 1, \ldots, k-1$. Observe that this identification of $T_\gamma\mathcal{L}_k^\circ$ is consistent with the identification of $\mathcal{L}_k^0$ with an open subset of M^{k+1}. In particular, the later point of view induces a Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{L}_k^\circ$. This metric is incomplete and for this reason we are a bit careful with the discussions about Morse theory that follow.

Given two points $p, q \in M$ set

$$\mathcal{L}_k(p,q) = \{\gamma \in \mathcal{L}_k(p,q) | \gamma(0) = p, \gamma(k) = q\}$$

Obviously $\mathcal{L}_k(p,q)$ is a closed subset of $\mathcal{L}_k$ homeomorphic to a closed subset of M^{k-1}. Moreover, from the description above we obtain that the interior $\mathcal{L}_k(p,q)^\circ$ of $\mathcal{L}_k(p,q)$ as a subset of M^{k-1} coincides with the intersection of $\mathcal{L}_k(p,q)\cap\mathcal{L}_k^\circ$. In particular, the tangent space $T_\gamma\mathcal{L}_k(p,q)^\circ$ of $\mathcal{L}_k(p,q)^\circ$ at some curve γ is given by the space of continuous vectorfields J along γ which vanish at 0 and k and such that $J|_{[i,i+1]}$ is Jacobi for all $i = 0, 1, \ldots, k-1$. Observe also that $\mathcal{L}_k(p,q)^\circ$ is totally geodesic with respect to the metric $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{L}_k^\circ$. The length and energy functions are continuous on $\mathcal{L}_k$. The energy function $E(\cdot)$ is actually smooth in $\mathcal{L}_k^\circ$ and the first variation formula asserts that the derivative of $E|_{\mathcal{L}_k(p,q)^\circ}$ at some point γ is given by:

$$d(E|_{\mathcal{L}_k(p,q)^\circ})_\gamma(\cdot) = 2\sum_{i=1}^{k-1}\langle -\Delta\gamma(i), \cdot\rangle \tag{1.2}$$

where $\Delta\gamma(t) = \partial^+\gamma(t) - \partial^-\gamma(t)$ and $\partial^+\gamma(t)$ and $\partial^-\gamma(t)$ are the right and left derivatives at t. Let $\mathcal{X}$ be the negative gradient of $E|_{\mathcal{L}_k(p,q)^\circ}$, i.e.

$$d(E|_{\mathcal{L}_k(\gamma(0),\gamma(k))^\circ})_\gamma(\cdot) = -\langle\langle\mathcal{X}_\gamma, \cdot\rangle\rangle$$

and let ϕ be the associated negative gradient flow

$$\phi'(t) = \mathcal{X}_{\phi(t)}, \quad \phi(0) = \gamma \tag{1.3}$$

Observe that since the vector-field $\Delta\gamma(t)$ is smooth not only on $\mathcal{L}_k(p,q)^\circ$ but on the the whole space $\mathcal{L}_k^\circ$, the vector field $\mathcal{X}$ and the flow (ϕ_t) are also smooth when considered on the whole of $\mathcal{L}_k^\circ$.

In general, gradient lines don't live for all $t \in \mathbb{R}$ but just for some open sub-interval. However we claim that the flow ϕ is defined for all non-negative t. In fact, consider the function

$$\lambda : \mathcal{L}_k(p,q) \to [0,1], \quad \lambda(\gamma) = \text{length of the longest segment in } \gamma$$

It is easy to check that

$$\lim_{t\to 0,\ t>0} \frac{\lambda(\phi_\gamma(t)) - \lambda(\gamma)}{t} \le 0.$$

This implies that λ is non-increasing and hence that flow lines never come close to the boundary in positive times since $\mathcal{L}_k(p,q)^\circ = \{\lambda < 1\}$. Thus, we have:

Lemma 1 ($\mathcal{L}_k^\circ$ is a cage)**.** *There is a semi-flow*

$$\phi : \mathcal{L}_k^\circ \times [0,\infty) \to \mathcal{L}_k^\circ$$

such that $\phi_\gamma(0) = \gamma$ and $\frac{d}{dt}\phi_\gamma(t) = \mathcal{X}_{\phi_\gamma(t)}$ for all γ and t. Moreover, the semi-flow preserves $\mathcal{L}_k(p,q)^\circ$ for all $p,q \in M$. □

1.3. **Morse theory on $\mathcal{L}_k(p,q)^\circ$.** Throughout this section we consider the restriction of the energy function E to $\mathcal{L}_k(p,q)^\circ$ for some pair of points $p,q \in M$. In order to relax notation we write E instead of $E|_{\mathcal{L}_k(p,q)^\circ}$.

It follows directly from the first variation formula (1.2) that the critical points of E are precisely the geodesics of length less than k joining p and q. If γ is such a critical point, then the Hessian of E at γ is given by the second variation formula. Recall that the index of a function at a critical point is the dimension of a maximal subspace on which the Hessian, i.e. the second derivative, is negative definite. A critical point is said to be non-degenerate if the Hessian has trivial kernel. The following well known result is known as Morse's Index Theorem (see e.g. [dC92, Chapter 11]).

Theorem 3. *A curve $\gamma \in \mathcal{L}_k(p,q)^0$ is a critical point of E if and only if it is geodesic. The index of E at γ is equal to the number of $t \in (0,k)$, counted with multiplicity, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ. Moreover, if p and q are not conjugated along γ then the critical point γ is non-degenerate.*

The Morse lemma asserts that if γ is a non-degenerate critical point of the energy function, then there is a small neighborhood U of $T_\gamma\mathcal{L}_k(p,q)^\circ$ and a chart $f : U \to \mathcal{L}_k(p,q)^\circ$ with $f(0) = \gamma$ and with

$$E(f(x)) = E(\gamma) + \mathrm{Hess}(E)|_\gamma(x,x) \tag{1.4}$$

where $\mathrm{Hess}(E)|_\gamma$ is the Hessian of E at γ.

We will often assume the following condition:

(M) The points $p,q \in M$ are not conjugated along any geodesic of length less than k.

By Theorem 3, (M) is equivalent to the assumption that the energy function E is a *Morse function* on $\mathcal{L}_k(p,q)^\circ$, i.e. that all of the critical points of E in $\mathcal{L}_k(p,q)^\circ$ are non-degenerate. In particular, the critical points of E are discrete.

As a first consequence of (M) and Lemma 1 ensuring that the negative gradient flow is defined for all times we obtain:

Lemma 2 (There is death and it is critical)**.** *Assuming (M), then for all* $\gamma \in \mathcal{L}_k(p,q)^\circ$ *the limit* $\lim_{t\to\infty}\phi_\gamma(t)$ *exists and is a critical point.* $\square$

Remark. If in Lemma 2 we remove the assumption that E is a Morse function, then the following weaker statement holds: Every sequence $(\phi_\gamma(t_i))_i$ with $t_i \to \infty$ contains a subsequence $(\phi_\gamma(t_{i_j}))_{i_j}$ which converges to a critical point.

It follows from Lemma 2 that, assuming (M), every point can be connected by a flow line to some critical point with not more energy. Given such a critical point α of index k, its stable manifold

$$\mathcal{W}^s(\alpha) = \{\gamma \in \mathcal{L}_k(p,q)^\circ | \alpha = \lim_{t\to\infty}\phi_\gamma(t)\}$$

is a smooth submanifold of codimension k. In particular, paths in $\mathcal{L}_k(p,q)^\circ$ can be perturbed to avoid the stable manifolds corresponding to non-degenerate critical points with at least index 2.

Lemma 3 (Sliding around index 2)**.** *Every path* $h : [0,1] \to \mathcal{L}_k(p,q)^\circ$ *can be perturbed to a path* h^* *with the same endpoints, whose interior does not intersect the stable manifold of any non-degenerate critical point with at least index* 2 *and with* $\max_t E(h^*(t)) \leq \max_t E(h(t))$. $\square$

Assume that (M) holds and that α is an index 0 critical point, that β is a critical point contained in the closure $\overline{\mathcal{W}}^s(\alpha)$ of $\mathcal{W}^s(\alpha)$ and let U and f be as in (1.4) at the point β. For any $\gamma \in \mathcal{W}^s(\beta)$ there is some t with $\phi_\gamma(t) \in f(U)$. It follows from the normal form (1.4) that $\phi_\gamma(t)$ is contained in the closure of $\mathcal{W}^s(\alpha)$. Hence we have that $\gamma \in \overline{\mathcal{W}}^s(\alpha)$. We have proved that

$$\overline{\mathcal{W}}^s(\alpha) = \cup_{\beta\in\overline{\mathcal{W}}^s(\alpha)\text{ critical point}}\mathcal{W}^s(\beta) \tag{1.5}$$

Hence, every point $\gamma \in \overline{\mathcal{W}}^s(\alpha)$ can be joined to α by a *broken flow-line* through broken geodesics with not more energy than $E(\gamma)$:

Definition. *A curve* $h : [0,1] \to \mathcal{L}_k(p,q)^\circ$ *is a* broken flow-line *if there is a decomposition* $[0,1]$ *into finitely many subsegments* $[t,t']$ *such that the restriction of* h *to the segment* (t,t') *is, up to reparametrization, a flow line.*

We claim now that every two points in the same connected component of $\mathcal{L}_k(p,q)^\circ$ can be joined by a broken flow-line:

Lemma 4 (Connecting the dots)**.** *Assume that (M) holds and that* γ_1 *and* γ_2 *are the endpoints of a path* $h : [0,1] \to \mathcal{L}_k(p,q)^\circ$*. Then there is a broken flow line* h^* *with* $h^*(0) = \gamma_1$*,* $h^*(1) = \gamma_2$ *and*

$$\max_{t\in[0,1]} E(h^*(t)) \leq \max_{t\in[0,1]} E(h(t))$$

Moreover, we can choose h^* *so that if the maximum of* $E(h^*(t))$ *is attained in the interior of* $[0,1]$*, then it is attained at some time* T *such that* $h^*(T)$ *is a critical point of index* 1*.*

Proof. To begin with, assume that γ_1 and γ_2 are index 0 critical points. By Lemma 3 we can assume without loss of generality that the curve h is contained in the union of the stable manifolds of critical points of index 0 and 1. Thus we find a finite sequence $\alpha_0', \alpha_1, \dots, \alpha_{2n}$ of critical points with the following properties:

(1) $\alpha_0 = \gamma_1$ and $\alpha_{2n} = \gamma_2$,
(2) α_k has index 0 for k even and index 1 for k odd,
(3) $\mathcal{W}^s(\alpha_k) \subset \overline{\mathcal{W}}^s(\alpha_{k-1}) \cap \overline{\mathcal{W}}^s(\alpha_{k+1})$ for k odd, and
(4) the image of h intersects $\mathcal{W}^s(\alpha_k)$ for all k.

For all $k = 0, \dots, 2n-1$ there is a flow-line connecting, up to time reversal, α_k to α_{k+1}. Let h^* be the union of these flow-lines. The maximum of the energy of the broken flow-line h^* is achieved at one of the critical points α_k with k odd, i.e. at an index 1 critical point. Say that this maximum is achieved at α_k and let t be such that $h(t) \in \mathcal{W}^s(\alpha_k)$. Then we have

$$\max_{t\in[0,1]} E(h^*(t)) = E(\alpha_k) \leq E(h(t)) \leq \max_{t\in[0,1]} E(h(t))$$

This concludes the proof if γ_1 and γ_2 are index 0 critical points.

In the general case, let γ_1' and γ_2' be index 0 critical points and let I_i be a broken flow line connecting γ_i and γ_i' through broken geodesics with at most energy $E(\gamma_i)$. The juxtaposition of I_1 traversed in the reverse direction, h, and I_2 defines a curve h' connecting the index 0 critical points γ_1' and γ_2'. Let $(h')^*$ be the curve constructed from h' as above and set h^* to be the juxtaposition of I_1, $(h')^*$ and I_2 traversed in the reverse direction. $\square$

1.4. **Existence of geodesic arcs.** We now use the standard strategy to obtain geodesic loops and arcs, proving two lemmas which we will need later on.

Lemma 5 (Geodesic loop)**.** *If the universal cover* $\tilde{M}$ *of* M *is not contractible then there is a non-trivial geodesic loop in* M *based at* p *which is homotopically trivial in* M*.*

Lemma 5 is likely well-known to experts and non-experts alike; however, we prove it here for the convenience of the reader.

Proof. Let $\tilde{p} \in \tilde{M}$ be a base point over p. And let $\Omega(M, p)$ and $\Omega(\tilde{M}, \tilde{p})$ be the spaces of loops in M and $\tilde{M}$ based at p and $\tilde{p}$ respectively. If, as usual, we endow these loop spaces with the compact open topology, then $\Omega(\tilde{M}, \tilde{p})$ is canonically homeomorphic to the connected component U_Ω of $\Omega(M, p)$ containing the constant curve. The assumption that $\tilde{M}$ is not contractible implies that the space $\Omega(\tilde{M}, \tilde{p})$ is not contractible; hence, there is an r with $\pi_r(U_\Omega) \neq 0$.

For $k \geq 1$, the space $\mathcal{L}_k(p, p)$ is contained in $\Omega(M, p)$. In fact, every loop in $\Omega(M, p)$ can be straightened to a broken geodesic in $\mathcal{L}_k(p, p)^\circ$ for some k large enough. This straightening process is continuous on compact subsets of $\Omega(M, p)$. In particular, for some sufficiently large k, we have $\pi_r(U_{\mathcal{L}_k}) \neq 0$ where $U_{\mathcal{L}_k}$ is the connected component of $\mathcal{L}_k(p, p)^\circ$ containing the constant curve.

The constant curve p is obviously a non-degenerate global minimum of the energy function on $U_{\mathcal{L}_k}$. If there were no other critical points then the restriction of E to $U_{\mathcal{L}_k}$ is a Morse function and U would be contractible by Lemma 2 contradicting $\pi_r(U_{\mathcal{L}_k}) \neq 0$. □

In the same spirit, assume that $\gamma_0, \gamma_1 \in \mathcal{L}_k(p, q)^\circ$ are the only geodesic segments joining p and q with energy $E(\gamma_i) < C$ for some C. Then by Lemma 2 and the consequent remark we have that for every $\gamma \in \mathcal{L}_k(p, q)^\circ$ with $E(\gamma) < C$ the flow line $\phi_\gamma(t)$ converges to γ_0 or γ_1 when $t \to \infty$. In particular, $\{\gamma_0, \gamma_1\}$ is a deformation retract of $\{\gamma \in \mathcal{L}_k(p, q)^\circ | E(\gamma) < C\}$. This cannot be the case if $\gamma_0 \neq \gamma_1$ is connected by a path in $\mathcal{L}_k(p, q)^\circ$ with maximal energy less than C. We have proved:

Lemma 6 (Third Geodesic)**.** *Assume that $\gamma_0, \gamma_1 \in \mathcal{L}_k(p, q)$ are minimizing geodesics joined by a continuous curve $\gamma : [0, 1] \to \mathcal{L}_k(p, q)$. Then there is a third geodesic $\alpha \in \mathcal{L}_k(p, q)$ joining p to q with $E(\alpha) \leq \max_{t \in [0,1]} E(\gamma(t))$.* □

2. Chords

In this section we prove Theorem 1.

Theorem 1. *Assume that M is a closed Riemannian manifold of dimension at least 2 and that γ is a closed geodesic in M. Then there is a chord for γ.*

Without going any further, recall that we are assuming the metric has been scaled so that $\text{inj}(M) \geq 2$.

Some Easy Cases

We start by considering some simple cases of Theorem 1, ultimately reducing to the case when M is simply connected and γ is a simple closed geodesic.

Reducing to the case of a simple closed geodesic: Assume that the geodesic γ has self-intersections. If $p \in \gamma$ is such a point of self-intersection, let $B_\epsilon(p)$ be a small ball around p with ϵ less than the convexity radius of M. Up to slightly reducing ϵ we can assume that γ is transversal to $\partial B_\epsilon(p)$ and hence that $\gamma \cap B_\epsilon(p)$ is the union of a collection of geometrically distinct properly embedded geodesics segments; the assumption that p is a point of self-intersection implies that there are at least two such segments. Choose points $x, y \in \gamma \cap B_\epsilon(p)$ which are not contained in the same segment. By the choice of ϵ, the ball $B_\epsilon(p)$ is convex. Hence, there is some geodesic segment η joining x and y and contained in $B_\epsilon(p)$. Since x and y are not contained in the same segment in $\gamma \cap B_\epsilon(p)$ we obtain that $\eta \not\subset \gamma$ and hence that it is the desired chord. From now on we assume that γ is simple.

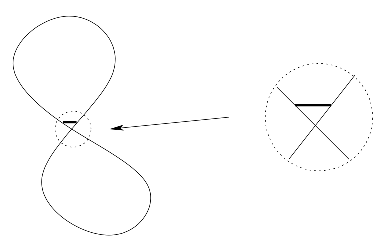

Figure 1. Geodesics with self intersections

Reducing to the case of a simply connected manifold: First assume that γ does not generate $\pi_1(M)$. In this case there is a homotopically essential map of pairs representing an element of $\pi_1(M)$ not in the subgroup generated by γ:

$$\sigma : ([0,1], \{0,1\}) \to (M, \gamma)$$

A minimal length representative of this homotopy class of maps yields the desired geodesic chord. In particular, we can assume that $\pi_1(M)$ is cyclic and generated by γ.

We consider now the case that the fundamental group of M is infinite; hence $\mathbb{Z} = \pi_1(M) = \langle \gamma \rangle$. The assumption that M has dimension at least 2 implies that the universal cover $\tilde{M}$ of M is not contractible. Hence for every $p \in \gamma$ there is by Lemma 5 a non-constant contractible geodesic loop η_p based at p. Since η_p is contractible we have that $\eta_p \not\subset \gamma$ and hence that it is the desired chord.

The case that $\pi_1(M)$ is a finite cyclic group can be reduced to the case that M is simply connected as follows. The universal cover $\tilde{M}$ of M is a closed manifold and the pre-image $\tilde{\gamma}$ of γ is simple closed geodesic. Any chord in $\tilde{M}$ for $\tilde{\gamma}$ projects to a chord in M for γ.

The Harder Case

We have reduced proving Theorem 1 to the case that γ is a simple closed geodesic in a simply connected closed manifold M. Before proceeding, we need some notation. Let $2D$ be the length of γ, and $g : \mathbb{R} \to M$ be a parametrization by arc-length of γ. Denote the reverse orientation by $-g : \mathbb{R} \to M$ so that $-g(t) := g(-t)$. For $s < t$ and $k > t - s$ we can reparametrize $g|_{[s,t]}$ to consider it as an element in $\mathcal{L}_k(g(s), g(t))^\circ$ as

$$\tau \mapsto g(s + \frac{t-s}{k}\tau),$$

for $\tau \in [0, k]$.

Cases for which the harder case is easy: We start again considering some simple cases. If $d_M(0, g(D)) < D$ then both geodesic segments $g|_{[0,D]}$ and $-g|_{[0,D]}$ are longer than the minimizing geodesics joining $g(0)$ and $g(D)$. Hence, any such minimizing geodesic κ provides the desired chord. Therefore, we now assume that $g|_{[0,D]}$ and $-g|_{[0,D]}$ are minimizing.

Next, assume that there are no $s < t \in \mathbb{R}$ such that $g(s)$ is conjugated to $g(t)$ along the segment $g|_{[s,t]}$. We claim that there is a chord joining $g(0)$ and $g(D)$. Assume that this were not the case. The assumption that γ is homotopically trivial implies that for some k the geodesic arcs $g|_{[0,D]}$ and $-g|_{[0,D]}$ can be connected by a path $\eta : [0, 1] \to \mathcal{L}_k(g(0), g(D))^\circ$. The assumption that there is no chord joining $g(0)$ and $g(D)$ implies that the energy function is a Morse function on $\mathcal{L}_k(p, q)^\circ$. The connecting the dots Lemma 4 ensures that there is a broken flow-line $\tau \mapsto \eta^*(\tau)$ connecting $g|_{[0,D]}$ and $-g|_{[0,D]}$ in $\mathcal{L}_k(p, q)^\circ$ which takes its maximum value at a critical point $\kappa := \eta^*(\tau_0)$ with index 1. However, $\kappa = \eta^*(\tau_0)$ cannot be contained in γ because it must contain points which are conjugate to each other. This implies that κ is a chord, contradicting our assumption.

The real meat of Theorem 1: Henceforth, we assume that both geodesic segments $g|_{[0,D]}$ and $-g|_{[0,D]}$ are minimizing and that g contains conjugate points. Let T be maximal such that for all $x, y \in \mathbb{R}$ with $|x - y| < T$, $g(x)$ and $g(y)$ are not conjugated along $g|_{[x,y]}$. Up to a time change we may assume that $g(0)$ and $g(T)$ are conjugated along g. Let $\delta > 0$ be such that $\mathcal{N}_\delta(\gamma)$ and $\mathcal{N}_{2\delta}(\gamma)$ are regular neighborhoods of γ. Here,

$$\mathcal{N}_\delta(\gamma) = \{x \in M | d_M(x, \gamma) \leq \delta\}.$$

Observe that any homotopy, relative to endpoints, between any two different geodesic segments contained in γ has to exit $\mathcal{N}_\delta(\gamma)$.

We can choose $\epsilon > 0$ small such that $g(0)$ and $g(T+\epsilon)$ are not conjugated along g. If they are conjugated along any other geodesic then this geodesic is the desired chord. So, assume that $g(0)$ and $g(T+\epsilon)$ are not conjugated at all.

The geodesic arc $g|_{[0,T+\epsilon]}$ is a non-degenerate critical point of the energy functional with positive index. Hence, choosing k large enough, there is a path, say a flow-line,

$$h^\epsilon : [0,1] \to \mathcal{L}_k(g(0), g(T+\epsilon))^\circ$$

starting at $g|_{[0,T+\epsilon]}$, along which the energy is monotonically decreasing and ending in a critical point with less energy than $g|_{[0,T+\epsilon]}$. In particular, if every curve in the image of h^ϵ is contained in $\mathcal{N}_\delta(\gamma)$, then we have that the critical point $h^\epsilon(1)$ cannot be contained in γ and hence is the desired chord. So, we may assume that there is some t with $h^\epsilon(t)$ not contained in the interior of $\mathcal{N}_\delta(\gamma)$. Let s be minimal with this property and set $\eta_\delta^\epsilon = h^\epsilon(s)$. By construction we have that $E(\eta_\delta^\epsilon) < T+\epsilon$ and that η_δ^ϵ is homotopic to $g|_{[0,T+\epsilon]}$ through curves with energy at most $T+\epsilon$.

Up to passing to a subsequence, we may assume that the curves η_δ^ϵ converge to some curve η_δ as ϵ tends to 0. The image of the limit curve η_δ still intersects $\partial\mathcal{N}_\delta$ and hence is not contained in γ. In particular, if η_δ is a geodesic arc, then it is a chord and we are done.

Before considering the remaining case, we suggest to the reader to recognize in figure 2 how we constructed η_δ.

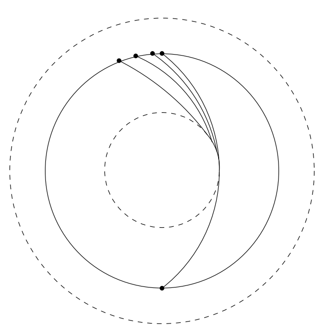

FIGURE 2. The circle is the geodesic γ, the dotted lines represent the boundary of $\mathcal{N}_\delta(\gamma)$ and the variation represents the curves η_δ^ϵ converging to η_δ.

Remark. The picture in figure 2 can be seen as the stereographical projection of $\mathbb{S}^2$ to the plane from the north pole. The closed geodesic is then the equator and the neighbrhood is a belt around it.

With the same notation as above, we consider now the case that η_δ is not a geodesic. By construction one has

$$E(\eta_\delta) \leq \liminf E(\eta_\delta^\epsilon) \leq T$$

and η_δ is homotopic to $g|_{[0,T]}$ through curves of energy at most $T+\epsilon$ for all $\epsilon > 0$. Since η_δ is not a geodesic we can homotope it within $\mathcal{N}_{2\delta}(\gamma)$ to a curve of less energy. Say that we homotope it to a curve of energy $T-3\epsilon_0$ for some small positive ϵ_0. Then the curve obtained by juxtaposition of this curve and the segment $g|_{[T-\epsilon_0,T]}$ traversed backwards has at most energy $T-2\epsilon_0$. Up to replacing 2δ by δ we are in the following situation:

> (*) For all δ there is a positive ϵ_0 such that for all $T' \in [T-\epsilon_0, T]$ and $\epsilon > 0$ there is a curve $\eta_{T'}$, joining $g(0)$ and $g(T')$ with energy less than T' and homotopic relative to endpoints to $g|_{[0,T']}$ within $\mathcal{N}_\delta(\gamma)$ through paths of energy at most $T+\epsilon$.

Now for some $\delta > 0$ small let notation be as in (*). By the connecting the dots Lemma 4 we find a curve $\alpha_{T'} : [0,1] \to \mathcal{L}_k(p,q)^\circ$ joining $g|_{[0,T']}$ and $\eta_{T'}$ consisting of flow lines and with $\max_t E(\alpha_{T'}(t)) \leq T+\epsilon$. Observe that, since $T' < T$, the geodesic $g|_{[0,T']}$ does not contain conjugate points and hence is a local minimum of the energy function. In particular, the maximum of the energy of the curves $\alpha_{T'}(t)$ is achieved for some $t \neq 0, 1$.

Let $[0, \tau_{T'}]$ be the domain of definiton for the first flow line in $\alpha_{T'}$. Then $\alpha_{T'}(\tau_{T'})$ is a critical point of the energy function with index at least 1 and energy $E(\alpha_{T'}(\tau_{T'})) \leq T+\epsilon$. If $\alpha_{T'}(\tau_{T'})$ is not contained in γ then we are done. Therefore, we assume that it is and note that since $\alpha_{T'}(\tau_{T'})$ contains conjugate points, $E(\alpha_{T'}(\tau_{T'})) \geq T$. It follows that $\alpha_{T'}(\tau_{T'}) = -g|_{[0,E(\alpha_{T'}(\tau_{T'}))]}$. Observe that since ϵ is arbitrary this implies that the points $g(0)$ and $g(T)$ either coincide or are antipodal in γ; hence

$$\alpha_{T'}(\tau_{T'}) = -g|_{[0,2T-T']}$$

Since the paths $g|_{[0,T']}$ and $\alpha_{T'}(\tau_{T'})$ are not homotopic within $\mathcal{N}_\delta(\gamma)$ there is a first time $t_{T'}$ such that $\alpha_{T'}(t_{T'}) \cap \partial\mathcal{N}_\delta(\gamma) \neq \emptyset$. Observe that we have:

$$\begin{aligned} E(g|_{[0,T']}) &= E(\alpha_{T'}(0)) \leq E(\phi_{\alpha_{T'}(t_{T'})}(1)) \\ &\leq E(\alpha_{T'}(t_{T'})) \leq E(\alpha_{T'}(\tau_{T'})) = E(-g[0, 2T-T']) \end{aligned}$$

Let α be the limit of the curves $\alpha_{T'}(t_{T'})$ for some sequence $T' \to T$. The curve α still intersects $\partial\mathcal{N}_\delta(\gamma)$ and the last equation implies that

$$E(\phi_\alpha(1)) = E(\alpha).$$

Therefore α is a geodesic connecting $g(0)$ and $g(T)$ which is not contained in γ. This concludes the proof of Theorem 1. □

Before going further, we would like to observe that in the case that the geodesic γ is simple the argument can be easily modified to show that there is a chord starting at every point of γ. The example of $\mathbb{S}^2$ shows that the second endpoint cannot be freely chosen.

3. Light

In this section we prove Theorem 2. The bulk of the work lies in proving the following technical result.

Proposition 1. *Suppose that M is a closed Riemannian manifold with cross blocking. If M is not a Blaschke manifold, then there is simple closed geodesic $\gamma \subset M$ of length $2\,\mathrm{inj}(M)$.*

Proof. We assume that M has been scaled so that $\mathrm{inj}(M) = 2$. Choose $p \in M$ with $\mathrm{inj}_p(M) = \mathrm{inj}(M)$ and let $\mathrm{cut}(p) \subset T_pM$ be its cut-locus. Choose $\theta \in \mathrm{cut}(p)$ with $||\theta|| = 2$ realizing the injectivity radius. For $r > 0$ and $v \in T_pM$ denote by $B(v, r) \subset T_p(M)$, the open ball with radius r and center v. We first argue that there is an open neighborhood $U \subset T_pM$ of θ for which the restriction of $\exp_p : T_pM \to M$ to $U \cap \mathrm{cut}(p)$ is one-to-one.

Indeed, if this were not the case, then the restriction of $\exp_p$ to $B(\theta, r) \cap \mathrm{cut}(p)$ is not one-to-one for each $r > 0$. Fix a positive ϵ' smaller than $\frac{1}{2}$. By continuity of the exponential map and the distance function in M, there is a sufficiently small $r_0 > 0$ so that for all $\theta_0, \theta_1 \in B(\theta, r_0)$ we have that

$$d_M(\exp_p(\frac{\theta_0}{2}), \exp_p(\frac{\theta_1}{2})) < \frac{\epsilon'}{2}$$

Let $\epsilon < \min\{\epsilon', r_0, \mathrm{diam}(M) - 2\}$ and choose $\theta_0, \theta_1 \in B(\theta, \epsilon) \cap \mathrm{cut}(p)$ with $\exp_p(\theta_0) = \exp_p(\theta_1) := q$. Define $\gamma_i : [0, 4] \to M$ by $\gamma_i(t) := \exp_p(t\frac{\theta_i}{4})$ for $i = 0, 1$. Note that both γ_0 and γ_1 are minimizing geodesics between p and q with $L(\gamma_i) \leq 2 + \epsilon$ for $i = 0, 1$. We consider the curve

$$\sigma_p : [0, 1] \to T_pM, \quad \sigma_p(s) = (1 - s)\frac{\theta_0}{2} + s\frac{\theta_1}{2}$$

in the tangent space to M at p and its image under the exponential map

$$\sigma : [0, 1] \to M, \quad \sigma(s) = \exp_p(\sigma(s)).$$

For each $s \in [0,1]$, we have that

$$d_M(q, \sigma(s)) \leq d_M(q, \sigma(0)) + d_M(\sigma(0), \sigma(s)) \leq \frac{2+\epsilon}{2} + \frac{\epsilon'}{2} < 1 + \epsilon' < 2.$$

Therefore, there is a unique curve $\sigma_q : [0,1] \to B(0,2) \subset T_qM$ with $\exp_q(\sigma_q(s)) = \sigma(s)$. For $s \in [0,1]$, define the one paramater family of curves $s \mapsto \gamma_s$ by

$$\gamma_s(t) = \begin{cases} \exp_p(t\frac{\sigma_p(s)}{2}), & \text{for } t \in [0,2] \\ \exp_q((4-t)\frac{\sigma_q(s)}{2}), & \text{for } t \in [2,4]. \end{cases}$$

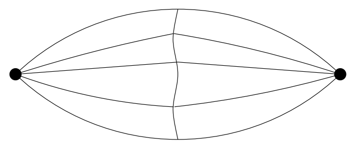

FIGURE 3. The variation γ_s interpolating by not much longer curves between the geodesics γ_0 and γ_1.

It is easy to check that $L(\gamma_s|_{[i,i+1]}) < 1$ for all $s \in [0,1]$ and $i = 0, \ldots, 3$ so that this family defines a continuous curve $\gamma : [0,1] \to \mathcal{L}_4(p,q)^\circ$ connecting γ_0 and γ_1. One also checks easily that the curve γ_s has at most energy $(1+\epsilon')^2$ so that by Lemma 6, there is a third geodesic α joining p to q with $E(\alpha) \leq (1+\epsilon')^2$. It follows that $L(\alpha) \leq 2 + 2\epsilon' < 3$. Note that since each of α, γ_0, and γ_1 have length strictly less than 4, no two can intersect in their interiors without contradicting $\text{inj}(M) = 2$. Hence, $b_M(p,q) \geq 3$, a contradiction to cross blocking since $d_M(p,q) \leq 2 + \epsilon < \text{diam}(M)$.

We have proved that there is some open neighborhood $U \subset T_pM$ of θ such that the restriction of $\exp_p$ to $U \cap \text{cut}(p)$ is one-to-one. From now on, let U be such a neighborhood.

We argue next that there are at least two distinct unit speed minimizing geodesics $\gamma_0, \gamma_1 : [0,2] \to M$ joining p and $q := \exp_p(\theta)$ (and hence exactly two by the cross blocking condition). Define

$$r_p : \partial\overline{B}(0,1) \to (0, \text{diam}(M)]$$
$$r_p(v) = \sup\{t \in (0, \text{diam}(M)] | d_M(p, \exp_p(tv)) = t\}$$

It is well-known that the function r_p is continuous. Hence, the function

$$i_p : \overline{B}(0,1) \setminus \{0\} \to T_pM, \quad i_p(x) = r_p(\frac{x}{||x||})x$$

is continuous as well. Therefore, $i_p^{-1}(U)$ is an open subest of $\frac{\theta}{2}$ in $\overline{B}(0,1)$. Choose $\delta > 0$ sufficiently small so that the set $V_\delta := \overline{B}(\frac{\theta}{2}, \delta) \cap$

$\overline{B}(0,1)$ is contained in $i_p^{-1}(U_\theta)$. Note that V_δ is homeomorphic to a basic closed set of 0 in the upperhalf space $\mathbb{R}^n_{x_n\geq 0}$ and that the map $\exp_p \circ i_p$ is continuous and one-to-one on V_δ. Hence, $\exp_p(i_p(V_\delta))$ does not cover an entire neighborhood of q so that we find a sequence of points $q_i \in M - \exp_p(i_p(V_\delta))$ converging to q. For each, i, let

$$\eta_i : [0,2] \to M$$

be a minimizing geodesic joining p to q_i and define $\gamma_0 : [0,2] \to M$ by $\gamma_0(t) = \exp_p(t\frac{\theta}{2})$. Up to passing to a subsequence the minimizing geodesics η_i converge to a second unit speed geodesic $\gamma_1 : [0,2] \to M$ joining p to q.

Next we argue that γ_0 and γ_1 together form a closed geodesic. If not, then either $\dot\gamma_0(0) \neq -\dot\gamma_1(0)$ or $\dot\gamma_0(2) \neq -\dot\gamma_1(2)$. We assume the latter, the former case being handled symmetrically. Fix a positive $\epsilon < 1$ and choose $v \in T^1_q M$ making obtuse angle with both $\dot\gamma_0(2)$ and $\dot\gamma_1(2)$. Note that for all sufficiently small s, the distance between the points $\gamma_i(2-\epsilon)$ and $\exp_q(sv)$ is less than one and in particular they are connected by a unique minimizing geodesic segment $\sigma^i_s : [0,1] \to M$. By the first variation formula the energy $E(\sigma^i_s)$ is strictly decreasing for sufficiently small s. Fix $s_0 < \epsilon$ positive and small enough such that $E(\sigma^i_{s_0}) < E(\sigma^i_0) = \epsilon^2$.

For $i = 0, 1$ we define broken geodesics $\alpha_i : [0,3] \to M$ by

$$\alpha_i(t) = \begin{cases} \gamma_i(\frac{2-\epsilon}{2}t), & \text{for } t \in [0,2] \\ \sigma^i_{s_0}(t-2) & \text{for } t \in [2,3] \end{cases}$$

The curves α_0, α_1 belong to $\mathcal{L}_3(p, \exp_q(s_0 v))^\circ$ and have at most energy

$$E(\alpha_i) \leq \frac{(2-\epsilon)^2 + 2\epsilon^2}{2} < 2.$$

Since $d_M(p, \exp_q(s_0 v)) < 2$ is less than the injectivity radius, the points p and $\exp_p(s_0 v)$ are connected by a unique geodesic segment α shorter than 2. The uniqueness of α implies that the flow lines $\tau \mapsto \phi_{\alpha_i}(\tau)$ of the flow provided by Lemma 1 and starting in α_0 and α_1 respectively converge to α with $\tau \to \infty$ (compare with the remark following Lemma 2). We conclude that α_0 and α_1 are homotopic through piecewise geodesics with three segments having energy not more than $\frac{(2-\epsilon)^2+2\epsilon^2}{2}$. See figure 4.

Similarly, the once broken geodesics joining $\gamma_i(2-\epsilon)$ to q defined by concatenating $\sigma^i_{s_0}$ with $\sigma|_{[0,s_0]}$ traversed in the opposite direction are homotopic to $\gamma_i|_{[2-\epsilon,2]}$ through once broken geodesics of total energy not more than $2\epsilon^2$. Combining these homotopies with those between α_0 and α_1 yields a continuous curve $\gamma : [0,1] \to \mathcal{L}_4(p,q)^\circ$ joining γ_0 and

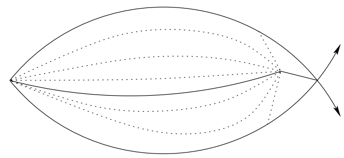

FIGURE 4. Flowing α_0 and α_1 to the geodesic α.

γ_1 with $\max_{t\in[0,1]} E(\gamma(t)) \leq \frac{(2-\epsilon)^2+4\epsilon^2}{2}$. By Lemma 4, there is a third geodesic $\beta : [0,4] \to M$ joining p to q with $E(\beta) < \frac{(2-\epsilon)^2+4\epsilon^2}{2}$. One easily checks that $L(\beta) < 4$. Therefore, β cannot intersect γ_0 or γ_1 in their interiors without contradicting $\mathrm{inj}(M) = 2$. Hence $b_M(p,q) \geq 3$, contradicting cross blocking since $d(p,q) = 2 < \mathrm{diam}(M)$.

We obtain that $\dot{\gamma_0}(0) = -\dot{\gamma_1}(0)$ and $\dot{\gamma_0}(2) = -\dot{\gamma_1}(2)$, completing the proof of Proposition 1. □

We can now prove Theorem 2.

Theorem 2. *A closed Riemannian manifold M has cross and sphere blocking if and only if M is isometric to a round sphere.*

Proof. We first scale the metric on M so that $\mathrm{inj}(M) = 2$. To begin with we claim that M is a Blaschke manifold. Otherwise there is simple closed geodesic $\gamma \subset M$ with

$$L(\gamma) = 2\,\mathrm{inj}(M) < 2\,\mathrm{diam}(M) \tag{3.1}$$

by Proposition 1. By Theorem 1, there is a chord $\eta : [0,1] \to M$ with end-points in γ. Up to replacing η by a subsegment whose end-points are again in γ we can assume that the interior of η is disjoint from γ. Let x and y be the end-points of η. If $x = y$ then γ and η are two light rays from x to itself with disjoint interior. Hence one needs at least two points to block x from itself contradicting the assumption that M has sphere blocking. Assume now that $x \neq y$. Then η and the two subsegments of γ connecting x and y are three light rays with disjoint interior. This implies that x and y have at least blocking number $b_M(x,y) \geq 3$. Since M is assumed to have cross blocking we obtain that x and y are at distance $\mathrm{diam}(M)$ and hence γ has at least length $2\,\mathrm{diam}(M)$ contradicting (3.1).

We have proved that M is Blaschke. As mentioned in the introduction, Theorem 2 follows now from [LS, Corollary 3.7] where it was shown that Blaschke manifolds with sphere blocking are isometric with round spheres. □

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