HEEGAARD SPLITTINGS AND PSEUDO-ANOSOV MAPS

HOSSEIN NAMAZI AND JUAN SOUTO

1. Introduction

Let M^+ and M^- be oriented 3-dimensional handlebodies whose boundary is identified with an oriented surface S of genus $g > 1$ in such a way that the orientation of S agrees with the orientation of ∂M^+ and does not with the one of ∂M^-. Given a mapping class $f \in \mathcal{MCG}(S)$ we consider the closed, oriented 3-manifold

$$N_f = M^+ \cup_f M^-$$

obtained by identifying the boundaries of M^+ and M^- via f. In this note we study geometric and topological properties of N_{f^n} where f^n is a sufficiently large power of a generic pseudo-Anosov mapping class; here, a pseudo-Anosov mapping class $f \in \mathcal{MCG}(S)$ is *generic* if its stable lamination λ^+ is not a limit, in the space $\mathcal{PML}$ of projective classes of measured laminations on S, of meridians of M^+ and its unstable lamination λ^- is not a limit of meridians of M^-. Recall that a meridian in $M^\pm$ is an essential simple closed curve in $\partial M^\pm$ which is homotopically trivial in $M^\pm$. The term *generic* is appropriate because Kerckhoff [Ker90] proved that the closure in $\mathcal{PML}$ of the set of meridians of M^+ and M^- have zero measures with respect to the canonical measure class of $\mathcal{PML}$.

We should point out that the above construction is due to Feng Luo by using an idea of Kobayashi (cf. Hempel [Hem01]). These were constructed as examples of Heegaard splittings where the minimum distance in the complex of curves of the Heegaard surfaces between a pair of meridians induced by the two handlebodies is sufficiently large.

Our first result is that the manifold N_{f^n} admits, for n large enough, a negatively curved metric:

Theorem 1.1. *Let M^+ and M^- be 3-dimensional handlebodies whose boundary is identified with a surface S of genus $g > 1$ and let $f \in$*

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$\mathcal{MCG}(S)$ be a generic pseudo-Anosov mapping class. Then, for arbitrary $\epsilon > 0$ there is n_ϵ such that the manifold $N_{f^n} = M^+ \cup_{f^n} M^-$ admits a Riemannian metric ρ_n with all sectional curvatures pinched by $-1-\epsilon$ and $-1+\epsilon$ for all $n \geq n_\epsilon$. Moreover, the injectivity radius of the metric ρ_n is bounded from below independently of n and ϵ.

If not otherwise stated we will assume from now on that M^+, M^- and f are as in the statement of Theorem 1.1. Moreover by abusing notation, we denote (N_{f^n}, ρ_n) by N_{f^n}.

It follows from Gromov's $C^{1,1}$-compactness theorem that for any choice of base points $p_n \in N_n$ the sequence of pointed manifold (N_{f_n}, p_n) has a subsequence which converges in the pointed Gromov-Hausdorff topology, also called geometric topology, to a complete $C^{1,1}$-Riemannian manifold (N_G, p_G). More precisely, this means that for every large R and small ϵ, there is some i_0 such that for all $i \geq i_0$, there are $(1+\epsilon)$-bi-Lipschitz, base points preserving, embeddings $\kappa_i^R : (B_R(p_G, N_G), p_G) \to (N_{f^{n_i}}, p_{n_i})$ of the ball $B_R(N_G, p_G)$ in N_G of radius R and center p_G. Taking R and ϵ in a suitable way, we obtain better and better embeddings of larger and larger balls and we will refer in the sequence to these maps as the *almost isometric embeddings* provided by geometric convergence. If a Riemannian manifold N_G is isometric to the geometric limit of some subsequence $(N_{f^{n_i}}, p_{n_i})$ for some choice of base points $p_n \in N_{f^n}$ then we say that N_G is a geometric limit of the sequence $(N_{f^n})_n$.

An important feature of the proof of Theorem 1.1 is that the negatively curved metric ρ_n is explicitly constructed from known hyperbolic manifolds. As a consequence we obtain the following classification of the possible geometric limits of the sequence $(N_{f^n})_n$ up to homeomorphism:

Theorem 1.2. *Every geometric limit of the sequence $(N_{f^n})_n$ is hyperbolic and either homeomorphic to a genus g handlebody or to the trivial interval bundle $S \times (-1, 1)$.*

In fact, we obtain a classification of the possible geometric limits of the sequence (N_{f^n}) up to isometry as well.

Theorem 1.1 and Theorem 1.2 are the keys which allow us to obtain a variety of topological results about the manifolds N_{f^n}. For example, an immediate corollary from Theorem 1.1 is the following:

Corollary 1.1. *There is n_f such that $\pi_1(N_{f^n})$ is infinite and word hyperbolic for all $n \geq n_f$.* □

It is well-known that word hyperbolic groups have solvable word problem and many other highly desirable algorithmic properties. However, Corollary 1.1 does not say much about the specific group $\pi_1(N_{f^n})$. For example, it is still unclear at this point if there is some non-trivial $\gamma \in \pi_1(M^+)$ which lies in the kernel of the homomorphism $\pi_1(M^+) \to \pi_1(N_{f^n})$ induced by the inclusion $M^+ \hookrightarrow N_{f^n}$ for infinitely many n. The following result implies that this cannot be the case:

Theorem 1.3. *If $\Gamma \subset \pi_1(M^+)$ is a finitely generated subgroup of infinite index, then there is some n_Γ such that for all $n \geq n_\Gamma$ the map $\Gamma \to \pi_1(N_{f^n})$ given by the inclusion $M^+ \hookrightarrow N_{f^n}$ is injective.*

Continuing with the study of the algebraic properties of $\pi_1(N_{f^n})$ recall that the rank of a finitely generated group is the minimal number of elements needed to generate it. Both $\pi_1(M^+)$ and $\pi_1(M^-)$ surject onto $\pi_1(N_{f^n})$ yielding thus the upper bound

$$\mathrm{rank}(\pi_1(N_{f^n})) \leq \mathrm{rank}(\pi_1(M^\pm)) = g$$

Generating sets of $\pi_1(N_{f^n})$ which arise as the image of minimal generating sets of $\pi_1(M^\pm)$ are said to be *standard*.

Theorem 1.4. *There is n_f with $\mathrm{rank}(\pi_1(N_{f^n})) = g$ for all $n \geq n_f$. Moreover, every minimal generating set of $\pi_1(N_{f^n})$ is Nielsen equivalent to a standard generating set for all sufficiently large n. In particular $\pi_1(N_{f^n})$ has at most 2 Nielsen equivalence classes of minimal generating sets.*

Recall that two (ordered) generating sets $\mathcal{S} = (g_1, \ldots, g_r)$ and $\mathcal{S}' = (g_1', \ldots, g_r')$ of a group are *Nielsen equivalent* if they belong to the same class of the equivalence relation generated by the following three moves:

$$\begin{array}{ll}
\text{Inversion of } g_i & \begin{cases} g_i' = g_i^{-1} & \\ g_k' = g_k & k \neq i \end{cases} \\
\text{Permutation of } g_i \text{ and } g_j \text{ with } i \neq j & \begin{cases} g_i' = g_j & \\ g_j' = g_i & \\ g_k' = g_k & k \neq i,j \end{cases} \\
\text{Twist of } g_i \text{ by } g_j \text{ with } i \neq j & \begin{cases} g_i' = g_i g_j & \\ g_k' = g_k & k \neq i \end{cases}
\end{array}$$

The authors suspect that for sufficiently large n the two possible Nielsen equivalence classes of standard generating sets, the class given by generators of $\pi_1(M^+)$ and that given by those of $\pi_1(M^-)$, are in fact different. However, we were not able to prove that this is the case.

It is well-known that every Gromov-hyperbolic group contains many free subgroups. In fact, it is in some sense true that most subgroups of a Gromov hyperbolic group are free. However, in general it is difficult to give conditions ensuring that the group generated by a given finite set of elements $\mathcal{S}$ is free. The following result asserts that, for sufficiently large n, every set of at most $2g-2$ elements in $\pi_1(N_{f^n})$ which generate a proper subgroup do in fact generate a free subgroup:

Theorem 1.5. *There is n_f such that for every $n \geq n_f$ the following holds: every proper subgroup $\Gamma \subset \pi_1(N_{f^n})$ with $\mathrm{rank}(\Gamma) \leq 2g-2$ is free.*

In section 7 we will construct examples showing that the bound given in Theorem 1.5 is sharp. We will also show that, again for sufficiently large n, proper finite index subgroups of $\pi_1(N_{f^n})$ are the only non-free subgroups of $\pi_1(N_{f^n})$ with small rank:

Theorem 1.6. *For all g' there is $n_{g'}$ such that for every $n \geq n_{g'}$ the following holds: every infinite index subgroup $\Gamma \subset \pi_1(N_{f^n})$ with $\mathrm{rank}(\Gamma) \leq g'$ is free.*

While Theorem 1.5 follows using arguments related to the proof of Theorem 1.4, Theorem 1.6 follows from a simple computation of the Euler-characteristics and from the following result:

Theorem 1.7. *For every g' there is $n_{g'}$ such that $\pi_1(N_{f^n})$ does not contain any subgroup isomorphic to the fundamental group of a closed surface of genus g' for all $n \geq n_{g'}$.*

Theorem 1.7 plays also an important role in our investigation of the possible Heegaard splittings of N_{f^n}. Recall that a Heegaard splitting is a decomposition $N_{f^n} = U \cup V$ as the union of two handlebodies with disjoint interiors; the genus of the splitting being the genus of the involved handlebodies. Denote by $g(N_{f^n})$ the Heegaard genus of N_{f^n}, i.e. the minimal possible genus of a Heegaard splitting. The manifold N_{f^n} has by construction the Heegaard splitting $N_{f^n} = M^+ \cup_{f^n} M^-$ of genus g to which we refer as being *standard*; in particular $g(N_{f^n}) \leq g$ for all n. On the other hand, the Heegaard genus of a manifold is an upper bound for the rank of its fundamental group. In particular we derive from Theorem 1.4:

Corollary 1.2. *There is n_f with $g(N_{f^n}) = g$ for all $n \geq n_f$.* □

The following much more precise result asserts that the standard Heegaard splitting of N_{f^n} is, up to isotopy and for sufficiently large n, the unique Heegaard splitting of minimal genus.

Theorem 1.8. *There is n_f such that for all $n \geq n_f$ the following holds: every minimal genus Heegaard splitting of N_{f^n} is isotopic to the standard one.*

Theorem 1.8 is also due to Scharlemann-Tomova [ST05] but their methods are completely different from the ones used in the present paper.

Theorem 1.1 and Theorem 1.2 are the key to all the topological results mentioned above. While Theorem 1.1 would be a consequence of a positive answer to Thurston's geometrization conjecture, the authors would like to remark that assuming the mere existence of negatively curved metric, or even hyperbolic, on N_{f^n}, they were not able to obtain any of the topological applications presented here. It is the control on the geometry of the manifolds N_{f^n} provided by Theorem 1.2 that opens the door to all the subsequent results. In fact, it was the intention of the authors to demonstrate in a concrete situation how geometric methods can be successfully applied to study topological properties of 3-manifolds *once* the geometry of the manifolds in question is well-understood. This is why we have chosen to restrict ourselves, as far as possible, to the case that M^+ and M^- are handlebodies. However, due to the nature of proofs, some theorems are proved in greater generality.

We should also remark that in [Na05] Theorem 1.1, Theorem 1.2 and therefore appropriate versions of other theorems mentioned here are generalized to a more general class of Heegaard splittings. However the method of construction of the negatively curved metric is different from here and involves a more elaborate study of the deformation space of hyperbolic structures on a handlebody similar to work of Minsky in [Min94, Min01] in the case of hyperbolic structures on an interval bundle over a compact surface. Moreover, in [Na05] it is shown, using a theorem of Tian [Ti90], that manifolds with these Heegaard splittings are actually hyperbolic and the hyperbolic metric is C^2-close to the constructed metric. Here though we do not use Tian's theorem and all the corollaries and applications are proved by having the metric with pinched negative curvature.

After some preliminaries in sections 2 and 3 we describe in section 4 the asymptotic geometry of the end of two particular hyperbolic metrics on M^+ and M^-. In section 5 we merge these two metrics to obtain the desired metric on N_{f^n} and we prove a more general version of Theorem 1.1 and Theorem 1.2. In section 6 we start obtaining topological consequences of Theorem 1.1 and Theorem 1.2 proving Theorem 6.1, a more general version of Theorem 1.3. Theorem 6.1 generalizes results of Lackenby [Lac02] and Easson [Eas04] which were proved using different methods. In section 7 we combine the geometric control obtained earlier on with the ideas used in [Sou05] to prove Theorem 1.4 and Theorem 1.5. The next two results, Theorem 1.6 and Theorem 1.7 are proved in section 8, and, finally, we show Theorem 1.8 in section 9.

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2. Some topology

Recall that an orientable 3-manifold is *irreducible* if every sphere bounds a ball and that it is *atoroidal* if every incompressible embedded torus is parallel to the boundary. It is well-known that every compact 3-manifold whose interior admits a complete metric with negative curvature is irreducible and atoroidal.

Suppose M is a compact orientable 3-manifold and S is a component of the boundary ∂M. A *meridian* on S is an essential simple closed curve on S which is homotopically trivial in M. We consider the set $\mathcal{M}$ of meridians on S as a subset of $\mathcal{PML}(S)$, the *space of projectivized measured laminations* on S; let $\mathcal{M}'$ be the closure of $\mathcal{M}$ in $\mathcal{PML}$. Recall that the space $\mathcal{ML}$ of measured lamination is endowed with the intersection form $i(\cdot,\cdot)$ and that it is unambiguous to say that $i(\lambda,\mu)=0$ for $\lambda,\mu\in\mathcal{PML}$.

Definition 2.1. *A projective measured lamination $\lambda\in\mathcal{PML}$ is* filling *if $i(\lambda,\gamma)>0$ for every essential simple closed curve γ; a filling lamination λ is* generic *if $i(\lambda,\mu)>0$ for every $\mu\in\mathcal{M}'$.*
A pseudo-Anosov mapping class $f\in\mathcal{MCG}(S)$ is said to be generic *if its stable lamination λ^+ is generic.*

Almost by definition, the stable lamination of a pseudo-Anosov mapping class is filling. On the other hand, the set of those laminations which appear as the stable lamination of some pseudo-Anosov mapping class are dense in $\mathcal{PML}$. In order to explain why the term *generic pseudo-Anosov* is appropriate it remains to recall that Kerckhoff [Ker90] proved that the set of all laminations λ with $i(\lambda,\mu)>0$ for all $\mu\in\mathcal{M}'$ is an open set of full-measure with respect to the canonical measure class of $\mathcal{PML}$.

Before going further we observe the following crucial fact:

Lemma 2.1. *Let $f\in\mathcal{MCG}(S)$ be a generic pseudo-Anosov mapping class. Then, for every $\alpha\in\pi_1(S)$ there is n_α with $f^n(\alpha)\notin\mathrm{Ker}(\pi_1(S)\to\pi_1(M))$ for all $n\geq n_\alpha$.*

Proof. Assume first that α is a simple closed curve. The sequence $(f^n(\alpha))_n$ converges in $\mathcal{PML}$ to the stable lamination of f and this implies that $f^n(\alpha)$ can only be a meridian for finitely many n. If α is not simple let $\gamma_1, \dots, \gamma_k$ be the finite collection of all possible essential simple closed curves which can be obtained by surgery at α. If $f^n(\alpha)$ is in the kernel of the homomorphism $\pi_1(S) \to \pi_1(M)$, then it follows from the Loop theorem that there is some $i_n \in \{1, \dots, k\}$ with $f^n(\gamma_{i_n}) \in \mathrm{Ker}(\pi_1(S) \to \pi_1(M))$. Now, the claim follows as above. $\square$

A compact, orientable, irreducible and atoroidal 3-manifold C is called a *compression body* when it has a boundary component $\partial_e C$ called the *exterior boundary* such that $\pi_1(\partial_e C)$ surjects onto $\pi_1(C)$. The union $\partial_{int} C$ of all the boundary components of C different from the exterior boundary is said to be the *interior boundary*; observe that every component of the interior boundary is incompressible. Here we assume no boundary component of a compression body is a torus. A compression body is a *handlebody* if it has empty interior boundary and it is said to be *trivial* if the exterior boundary is incompressible. A trivial compression body is homeomorphic to a trivial interval bundle over a closed orientable surface.

Compression bodies arise in a natural way when studying more general compact, orientable, irreducible and atoroidal 3-manifolds M. More precisely, if S is a component of ∂M then a compression body C_S can be associated to S as follows. Let $\{m_1, \dots, m_k\}$ be a maximal collection of disjoint non-parallel meridians in S. By Dehn's lemma, there is a collection of properly embedded disjoint disks $\{D_1, \dots, D_k\}$ with $\partial D_i = m_i$. Let C_S be the co-dimension 0 submanifold of M obtained by taking a regular neighborhood of $S \cup D_1 \cup \dots \cup D_k$ and capping off boundary spheres. It is due to Bonahon [Bon83] that the compression body C_S is, up to isotopy, independent of the collection of meridians we started with. Moreover, it is characterized by being the unique compression body in M with exterior boundary S whose fundamental group $\pi_1(C_S)$ injects into $\pi_1(M)$; C_S is said to be the *relative compression body* associated to S.

We conclude this section with the following well-known observation which will be of use below.

Lemma 2.2. *Let C be a compression body with exterior boundary of genus g and let $\Sigma \subset C$ be a closed embedded surface of genus $g' \leq g$. If $\pi_1(\Sigma)$ surjects onto $\pi_1(C)$ then $g' = g$. If moreover,* $\mathrm{rank}(\pi_1(C)) = 2g$, *i.e. if C is a trivial interval bundle, then Σ is incompressible and hence isotopic to a component of ∂C.* $\square$

3. HYPERBOLIC STRUCTURES ON 3-MANIFOLDS

A *hyperbolic structure* on a compact 3-manifold M is the conjugacy class of a discrete and faithful representation $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$, such that $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ is homeomorphic to the interior of M by a homeomorphism inducing ρ. If S is a boundary component of M, then the *S-end* of N_ρ is the end corresponding to S under this homeomorphism. The hyperbolic structure N_ρ is *convex-cocompact* if there is a convex $\rho(\pi_1(M))$-invariant subset $K \subset \mathbb{H}^3$ with $K/\rho(\pi_1(M))$ compact. Equivalently, the manifold N_ρ contains a compact convex submanifold C such that $N_\rho \setminus C$ is homeomorphic to $\partial C \times \mathbb{R}$. The end corresponding to a component S of ∂M is *convex-cocompact* if N_ρ contains a convex-submanifold C such that $N_\rho \setminus C$ is a neighborhood of the S-end of N_ρ homeomorphic to $S \times \mathbb{R}$. Before going further we remind the reader of the following characterization of the convex-cocompact structures:

Lemma 3.1. *A hyperbolic structure N_ρ is convex-cocompact if and only if for some choice, and hence for all, of $p_{\mathbb{H}^3} \in \mathbb{H}^3$ the map*

$$\pi_1(M) \to \mathbb{H}^3, \quad \gamma \mapsto (\rho(\gamma))(p_{\mathbb{H}^3})$$

is a quasi-isometric embedding with respect to the left-invariant word-metric on $\pi_1(M)$ corresponding to some, and hence for all, finite generating set of $\pi_1(M)$. □

Recall that a map $\phi : X_1 \to X_2$ between two metric spaces is an (L, A)-quasi-isometric embedding if

$$\frac{1}{L} d_{X_1}(x, y) - A \leq d_{X_2}(\phi(x), \phi(y)) \leq L d_{X_1}(x, y) + A$$

for all $x, y \in X_1$. An (L, A)-quasi-isometric embedding $\phi : \mathbb{R} \to X$ is said to be a quasi-geodesic.

The geometry of convex-cocompact ends of N_ρ is well-understood using Ahlfors-Bers theory. Building on the work of Thurston, Canary [Can96] described a different sort of ends. An end $\mathcal{E}$ of N_ρ is *simply degenerate* if there is a sequence of surfaces $(S_i) \subset N_\rho$ with the following properties:

- Every neighborhood of $\mathcal{E}$ contains all but finitely many of the surfaces S_i.
- With respect to the induced path distance, the surface S_i is $CAT(-1)$ for all i.
- If $C_1 \subset N_\rho$ is a compact submanifold with $N_\rho \setminus C_1$ homeomorphic to a trivial interval bundle, then S_i is homotopic to ∂C_1 within $N_\rho \setminus C_1$.

Thurston and Canary's covering theorem [Can96] asserts that manifolds with simply degenerate ends cannot cover other manifolds in complicated ways:

Covering theorem. *Let M and N be infinite volume hyperbolic 3-manifolds with finitely generated fundamental group and $\pi : M \to N$ be a Riemannian covering. Every simply degenerate end $\mathcal{E}$ of M has a neighborhood homeomorphic to $E = S \times [0, \infty)$ such that $\pi(E) = R \times [0, \infty)$ and $\pi|_E : E \to \pi(E)$ is a finite-to-one covering.*

We will often use the Covering Theorem to show that subgroups of the fundamental group of a hyperbolic manifold are convex-cocompact. More precisely, the following is a consequence of the Covering Theorem (cf. Canary [Can96, Cor. C]).

Proposition 3.1. *Let M be a compact 3-manifold with boundary and let $\rho : \pi_1(M) \to \mathrm{PSL}_2\,\mathbb{C}$ be a hyperbolic structure on M such that $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ has positive injectivity radius. Let $\mathcal{E}_1, \dots, \mathcal{E}_k$ be the simply degenerate ends of M and for $i = 1, \dots, k$ let $E_i = S_i \times [0, \infty)$ be a neighborhood of $\mathcal{E}_i$. Denote also by $[\pi_1(E_i)]$ the image of $\pi_1(E_i)$ in $\pi_1(M)$.*
If $\Gamma \subset \pi_1(M)$ is a finitely generated subgroup such that $\gamma\Gamma\gamma^{-1} \cap [\pi_1(E_i)]$ has infinite index in $[\pi_1(E_i)]$ for every $\gamma \in \pi_1(M)$ and $i = 1, \dots, k$ then the manifold $\mathbb{H}^3/\rho(\Gamma)$ is convex-cocompact.

The condition on Γ in Proposition 3.1 is perhaps a little bit confusing but it becomes clear when considered in a simple situation where $[\pi_1(E_i)] = \pi_1(M)$.

Corollary 3.1. *Let M be trivial interval bundle or a handlebody and let $\rho : \pi_1(M) \to \mathrm{PSL}_2\,\mathbb{C}$ be a hyperbolic structure on M such that $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ has positive injectivity radius. If $\Gamma \subset \pi_1(M)$ is a finitely generated subgroup of infinite index then $\mathbb{H}^3/\rho(\Gamma)$ is convex-cocompact.* $\square$

Pleated surfaces are one of the many ways to construct $CAT(-1)$-surfaces in hyperbolic 3-manifolds. Given a boundary component S of M and Σ a point in the Teichmüller space $\mathcal{T}(S)$ of S, a map $p : \Sigma \to N_\rho$ in the correct homotopy class is said to be a pleated surface if it preserves the lengths of paths and if every point in Σ is contained in a geodesic segment which is isometrically embedded by p. We say that a lamination λ in S is *realized* in N_ρ, if there exists a pleated surface $p : \Sigma \to N_\rho$ which maps every leaf of λ to a geodesic in N_ρ.

Below we will need the following result which is essentially due to Thurston (see Brock [Bro00] for proofs and for very extensive treatment of similar results).

Proposition 3.2. *Let λ be a generic measured lamination in S.*

- *If λ is realized in N_ρ then there is a constant L with*
$$\liminf_i \frac{l_\rho(\gamma_i)}{l_S(\gamma_i)} \geq L$$
for every sequence (γ_i) of simple closed curves in S converging to λ in $\mathcal{PML}(S)$.
- *If λ is not realized then for every sequence (γ_i) of curves converging to λ in $\mathcal{PML}(S)$*
$$\lim_i \frac{l_\rho(\gamma_i)}{l_S(\gamma_i)} = 0$$

Here l_S is the length in some arbitrarily fixed metric on S.

Proof. To begin with identify S with a surface in N_ρ. Assume that the generic measured lamination λ is realized and that (γ_i) is a sequence of simple closed curves in S converging to λ. Up to passing to a subsequence we may assume that the sequence (γ_i) converges in the Hausdorff topology to a lamination $\lambda_{\mathcal{H}} \subset S$ which contains the support of λ. It is a result of Otal [Ota94] that for all ϵ there is a train-track[1] $\tau \subset S$ carrying $\lambda_{\mathcal{H}}$ and homotopic in N_ρ to a graph τ^* consisting of geodesic segments of at least length $\frac{1}{\epsilon}$ such that the angle at the switches is at most ϵ. For sufficiently large i, the curve γ_i is carried by τ and hence is homotopic in N_ρ to a curve, contained in τ^*, consisting of long geodesic segments and small angles. It is well-known that each one of these curves is an L_ϵ-bi-Lipschitz embedding (if ϵ is smaller than some universal constant) where L_ϵ depends only on ϵ. In particular, the length of the geodesic homotopic to γ_i in N_ρ is comparable to the length of the representative in $\tau*$, hence comparable to the length in τ, and hence to the length in S. This proves the first claim.

In order to prove the second assertion we claim that λ is realized if there is some sequence (γ_i) of simple closed curves converging to λ and such that the ratios $\frac{l_\rho(\gamma_i)}{l_S(\gamma_i)}$ are bounded from below. For all i let r_i be the minimal distance between the curve $\gamma_i \subset S$ and its geodesic representative γ_i^* in N_ρ. It is well-known that
$$l_\rho(\gamma_i) = l_{N_\rho}(\gamma_i^*) \leq \frac{1}{\cosh(r_i)} l_S(\gamma_i)$$
This implies that the sequence r_i is bounded and hence that there is some compact subset $K \subset N_\rho$ with $\gamma_i^* \cap K \neq \emptyset$ for all sufficiently large i. Each one of the curves γ_i is realized by some pleated surface in

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[1] See Bonahon [Bon86] for a discussion of train-tracks.

N_ρ [KS03] and it follows from what we just said that all these pleated surfaces intersect K. Since λ is generic, Otal's compactness theorem for pleated surfaces applies (see [KS03]) and we obtain that, up to passing to a subsequence, the pleated surfaces realizing γ_i converge to a pleated surface realizing λ. This proves the second claim. $\square$

Thurston proved that every compact, orientable and irreducible 3-manifold M with $\partial M \neq \emptyset$ and which does not contain free abelian groups of rank 2 admits a convex-cocompact hyperbolic structure. In fact, the space of these structures is parametrized by the Teichmüller space of ∂M.

Suppose S is a component of ∂M and $\lambda \in \mathcal{PML}(S)$ is a generic lamination. Take a sequence of convex cocompact structures on M where the conformal structures associated to S'-ends are fixed for every $S' \neq S$ and the conformal structures associated to the S-end converge to λ, i.e. there exist appropriate lifts of these structures to the Teichmüller space of S which converge to λ in the Thurston's compactification.

Results of Kleineidam-Souto in [KS02, KS03] show that a subsequence of the above sequence is algebraically convergent. Their work together with the proof by Agol [Ago04] and Calegary-Gabai [CG04] of the Tameness Conjecture shows that the limit is a hyperbolic structure on M (see also [KS03] for the case that M is not a handlebody). Finally by using proposition 3.2, one can see that λ is not realized in the obtained hyperbolic structure on M (see [KS03]). We have the following:

Existence of degenerated hyperbolic structures. *Let M be a compact, irreducible, orientable 3-manifold whose fundamental group does not contain $\mathbb{Z}^2$. Moreover, let S be a component of ∂M and let $\lambda \in \mathcal{PML}(S)$ be a generic lamination. Then there is a hyperbolic structure N_ρ on M whose S'-ends are convex-cocompact for all $S' \neq S$ and where λ is not realized.* $\square$

In the next section we will study the geometry of the hyperbolic structures N_ρ provided by this theorem in the case that λ is the stable lamination of a generic pseudo-Anosov map.

See Canary-Epstein-Green [CEG87] and Otal [Ota88] for more about pleated surfaces and [MT98] for more on the geometry of hyperbolic 3-manifolds.

4. Asymptotic geometry of ends

In this section let M be a compact, irreducible, orientable 3-manifold whose fundamental group does not contain $\mathbb{Z}^2$, let S be a component of ∂M and let $\lambda^+ \in \mathcal{PML}(S)$ be the stable lamination of a generic pseudo-Anosov mapping class $f \in \mathcal{MCG}(S)$. Moreover let N_ρ be a hyperbolic structure on M where λ^+ is not realized as provided by the existence theorem of degenerated hyperbolic structures in section 3. In this section we show, generalizing a result of McMullen [McM96], that the S-end of N_ρ is asymptotically isometric to the positive end of the infinite cyclic cover M_f of the mapping torus T_f of f.

Recall that it is a theorem of Thurston [Thu98] that the mapping torus

$$T_f = S \times [0,1]/(x,1) \sim (f(x),0)$$

admits a hyperbolic metric. We denote by M_f the infinite cyclic cover of T_f corresponding to the fundamental group $\pi_1(S \times \{0\})$ of the fiber. The manifold M_f is homeomorphic to $S \times \mathbb{R}$. Moreover, choosing once and for ever a lift $\iota : S \to M_f$ of the inclusion of $S \hookrightarrow T_f$ we obtain a marking of M_f. Identifying the universal covering of M_f with hyperbolic space we obtain a representation

$$\rho_f : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{C}$$

with $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$; any two such representations differ by conjugacy in $\mathrm{PSL}_2\,\mathbb{C}$.

It is well-known that only two projective classes of measured laminations on S are not realized in M_f, namely the stable and the unstable laminations λ^+ and λ^- of f. Essentially the argument used in the proof of Proposition 3.2 shows that if $(\gamma_i^\pm)_i$ are sequences of simple closed curves converging in $\mathcal{PML}$ to $\lambda^\pm$ then the corresponding sequences of geodesics leave every compact set. More precisely, the geodesics corresponding to (γ_i^+) converge to an end $\mathcal{E}^+$ and the geodesics corresponding to (γ_i^-) to the other end $\mathcal{E}^-$. The laminations λ^+ and λ^- are the so-called *ending laminations* of M_f [Bon86]. We will refer to the end $\mathcal{E}^+$ as the *positive end* of M_f.

Notice that the parametrization of the S-end as an interval bundle over S gives an isotopy class of embeddings of S to N_ρ; fix such an embedding $\iota_\rho : S \hookrightarrow N_\rho$. By an *asymptotic isometry* between the S-end of N_ρ and $\mathcal{E}^+$ we mean a proper embedding $h^+ : E^+ \to N_\rho$ of a neighborhood E^+ of $\mathcal{E}^+$ onto a neighborhood of the S-end of N_ρ such that:

- if $\iota' : S \to M_f$ is homotopic to ι and $\iota'(S) \subset E^+$ then $h^+ \circ \iota$ is isotopic to ι_ρ, and
- for any k, r and $\epsilon > 0$, there is a compact set $K \subset M_f$ such that h^+ is ϵ-close to an isometry in the C^k topology when restricted to any embedded r-ball contained in $E^+ \setminus K$.

The following is the main result of this section and the key point in the proof of Theorem 1.1 and Theorem 1.2 in the next section:

Proposition 4.1. *Let S be a component of the boundary ∂M of a compact, irreducible, orientable 3-manifold M whose fundamental group does not contain $\mathbb{Z}^2$. Moreover, let $\lambda^+ \in \mathcal{PML}(S)$ be the stable lamination of a generic pseudo-Anosov mapping class $f \in \mathcal{MCG}(S)$ and let N_ρ be a hyperbolic structure on M whose S'-ends are convex-cocompact for all $S' \neq S$ and where λ^+ is not realized.*
Then the S-end of N_ρ is asymptotically isometric to the positive end $\mathcal{E}^+$ of M_f.

If S is incompressible then Proposition 4.1 is due to McMullen [McM96]. For the sake of concreteness we assume from now that S is compressible; the incompressible case is similar.

The idea of the proof of Proposition 4.1 is to consider the sequence (ρ_n) of representations of $\pi_1(S)$ into $\mathrm{PSL}_2\,\mathbb{C}$ obtained by precomposing ρ with the surjective map $\pi_1(S) \to \pi_1(M)$ induced by the inclusion $S \hookrightarrow M$ and with the map f_*^n:

$$\rho_n = \rho \circ (\pi_1(S) \to \pi_1(M)) \circ f_*^n$$

We will show that the sequence (ρ_n) converges algebraically and geometrically to the faithful and discrete representation ρ_f with $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$; once this is settled, then the proof of Proposition 4.1 is concluded as in McMullen [McM96].
As a first step we show that, up to conjugacy, the sequence (ρ_n) is pre-compact in $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2\,\mathbb{C})$, i.e. in the algebraic topology.

4.1. **Precompactness of the sequence (ρ_n).** We obtain the precompactness of the sequence (ρ_n) using a similar strategy as in Otal's proof of Thurston's Double Limit Theorem [Ota96]. Using Morgan-Shalen's compactification of the space of conjugacy classes of representations in $\mathrm{PSL}_2(\mathbb{C})$ we have that every subsequence of (ρ_n) contains a subsequence (ρ_{n_i}) that either

- converges, up to conjugation, to a representation of $\pi_1(S)$ in $\mathrm{PSL}_2(\mathbb{C})$, or
- converges, in the *sense of Morgan-Shalen*, to a nontrivial minimal action of $\pi_1(S) \curvearrowright T$ on a real tree T. Recall that this means that there are base points $p_{n_i} \in \mathbb{H}^3$ and $p \in T$ and positive $\epsilon_{n_i} \to 0$ with

$$\epsilon_{n_i} d_{\mathbb{H}^3}(\rho_{n_i}(\alpha)(p_{n_i}), p_{n_i}) \to d_T(\alpha(p), p), \quad \forall \alpha \in \pi_1(S).$$

See Bestvina [Bes88] and Paulin [Pau88] for more on the convergence of sequences of representations to actions on trees.
In particular, in order to show that the sequence (ρ_n) is pre-compact, it suffices to show that no subsequence converges to an action on a tree.

Seeking a contradiction, suppose that $(\rho_{n_i})_i$ converges to a non-trivial and minimal action $\pi_1(S) \curvearrowright T$. As a first step we want to prove that the action $\pi_1(S) \curvearrowright T$ is *geometric*. Recall that dual to a measured lamination μ on S, there exists a real tree T_μ with a natural minimal and small action of $\pi_1(S)$ on T_μ. The action $\pi_1(S) \curvearrowright T$ is *geometric* if T is (equivariantly) isometric to the dual tree T_μ of some $\mu \in \mathcal{ML}(S)$. See for example Otal [Ota96] or Skora [Sko90] for a description of the tree dual to a measured lamination. Every geometric action is *small*, i.e. the stabilizer of every nondegenerate segment is abelian. On the other hand Skora proved:

Theorem (Skora [Sko90, Sko96])**.** *Every small minimal action of the fundamental group of a closed surface on a real tree is geometric.*

We prove now that $\pi_1(S) \curvearrowright T$ is small.

Lemma 4.1. *The action $\pi_1(S) \curvearrowright T$ is small, i.e. the stabilizer of every nondegenerate segment is an abelian subgroup of $\pi_1(S)$.*

Sketch of proof. The proof is exactly the same as the proof in the case where we have a sequence of discrete and faithful representations (cf. [Ota96, Thm. 2.2.9] and [Pau88]). Suppose a and $b \in \pi_1(S)$ stabilize a non-degenerate segment κ of T and suppose p and p_{n_i} denote the base points of T and $\mathbb{H}^3$ for the actions of $\pi_1(S)$. Since $\pi_1(S) \curvearrowright T$ minimally, we can assume (by possibly taking a subarc) that κ is contained in a segment $[p, h(p)]$ of T for some $h \in \pi_1(S)$. Let γ_i be the geodesic that connects p_{n_i} and $\rho_{n_i}(h)(p_{n_i})$ in $\mathbb{H}^3$. Then one can see that since a and b stabilize κ for i sufficiently large, $\rho_{n_i}(a)$ and $\rho_{n_i}(b)$ are ϵ-close to a translation on a subsegment κ_i of γ_i and with translation distances that are small compare to the length of κ_i. Hence for i sufficiently large, there exists a point in $\mathbb{H}^3$ that is moved by a very small distance by both $\rho_{n_i}([a,b])$ and $\rho_{n_i}([a,b^2])$, where $[a,b]$ denotes the commutator of a and b. (We can take this point to be the midpoint of κ_i.) The Margulis Lemma implies that the group $\langle \rho_n([a,b]), \rho_n([a,b^2]) \rangle$ is virtually abelian.

Recall that ρ_n was obtained by precomposing a faithful representation ρ of $\pi_1(M)$ with $(\pi_1(S) \to \pi_1(M)) \circ f_*^n$. Because every virtually abelian subgroup of $\pi_1(M)$ is in fact abelian, we can see that

$$f_*^{n_i}([[a,b],[a,b^2]]) \in \operatorname{Ker}(\pi_1(S) \to \pi_1(M))$$

for all sufficiently large i. Lemma 2.1 implies now that $[a,b]$ and $[a,b^2]$ must commute in $\pi_1(S)$. Since any two non-commuting elements in $\pi_1(S)$ generate a free group, we deduce that $\langle a,b \rangle$ is abelian. $\square$

By Lemma 4.1, Skora's theorem applies in the present situation. Hence, we can assume that the sequence (ρ_{n_i}) converges to the action $\pi_1(S) \curvearrowright T_\mu$ for some measured lamination $\mu \in \mathcal{ML}(S)$.

From the construction of the dual tree T_μ one deduces that every curve γ in S has translation length $i(\gamma, \mu)$ in T_μ; hence one has that the translation length $l_{\rho_{n_i}}(\gamma)$ of $\rho_{n_i}(\gamma)$ on $\mathbb{H}^3$ tends to ∞ if $i(\gamma, \mu) > 0$. More generally, in [Ota94, Ota96] Otal proves that if (γ_i) is a sequence of simple closed curves in S converging in $\mathcal{PML}(S)$ to a filling measured lamination λ with $i(\lambda, \mu) > 0$ then one has:

$$\frac{l_{\rho_{n_i}}(\gamma_i)}{l_S(\gamma_i)} \to \infty$$

As mentioned above, the stable and unstable laminations λ^+ and λ^- of the pseudo-Anosov map f are both filling and intersect each other. Hence at least one of them intersects μ. In particular, the desired contradiction follows once we show:

Lemma 4.2. *There are sequences (γ_i^+) and (γ_i^-) of simple closed curves on S converging to λ^+ and λ^- respectively such that*

$$\frac{l_{\rho_{n_i}}(\gamma_i^+)}{l_S(\gamma_i^+)}, \frac{l_{\rho_{n_i}}(\gamma_i^-)}{l_S(\gamma_i^-)} \to 0$$

Proof. To begin with the proof observe that $l_{\rho_{n_i}}(\gamma) = l_\rho(f^{n_i}(\gamma))$. In particular, if m is a meridian on S, then $\gamma_i^- = f^{-n_i}(m)$ is in the kernel of ρ_{n_i} for all i, and hence $l_{\rho_{n_i}}(\gamma_i^-) = 0$. Since the sequence (γ_i^-) converges in $\mathcal{PML}(S)$ to λ^- we have proved the claim for λ^-.

By construction λ^+ is not realized in N_ρ. In particular, it follows from Proposition 3.2 that there is a sequence $(\eta_k)_k$ of simple closed curves on S converging to λ^+ with

$$\frac{l_\rho(\eta_k)}{l_S(\eta_k)} \to 0$$

If (η_k) is a such a sequence, then the sequence $(f^{-n}(\eta_k))_k$ converges to λ^+ as well since λ^+ is a fixed point of f. Let L be a bi-Lipschitz constant for f with respect to the metric l_S and choose a subsequence (η_{n_i}) such that $(f^{-i}(\eta_{n_i}))_i$ converges to λ^+, and such that

$$\lim_i L^i \frac{l_\rho(\eta_{n_i})}{l_S(\eta_{n_i})} \to 0$$

Setting $\gamma_i^+ = f^{-i}(\eta_{n_i})$ we have

$$\frac{l_{\rho_i}(\gamma_i^+)}{l_S(\gamma_i^+)} = \frac{l_\rho(f^i(\gamma_i^+))}{l_S(\gamma_i^+)} = \frac{l_\rho(\eta_{n_i})}{l_S(f^{-i}(\eta_{n_i}))} \le L^i \frac{l_\rho(\eta_{n_i})}{l_S(\eta_{n_i})} \to 0$$

This concludes the proof of the Lemma 4.2. □

As remarked above this yields the desired contradiction on the assumption that the sequence (ρ_{n_i}) converges to an action on a tree. We have proved that the sequence (ρ_n) is pre-compact in the algebraic topology. Our next goal is to show that the sequence (ρ_n) converges, up to conjugacy, to the representation ρ_f with $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$. The difference with what we just proved is that we only showed that this is the case *up to choice of a subsequence.*

4.2. **Algebraic convergence of the sequence** (ρ_n)**.** Assume that, up to conjugacy, a subsequence $(\rho_{n_i})_i$ converges to a representation ρ_∞. The image of ρ_{n_i} is a discrete non-elementary subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ and this implies that the image of ρ_∞ is discrete as well (see for example Thurston [Thu86]). Moreover, the Margulis lemma and Lemma 2.1 imply that the representation ρ_∞ is faithful. In particular, it follows from Bonahon [Bon86] that $N_\infty = \mathbb{H}^3/\rho(\pi_1(S))$ is homeomorphic to $S \times \mathbb{R}$.

We claim that the laminations λ^+ and λ^- are not realized in N_∞. Seeking a contradiction assume that λ^+ is realized. Then, a small variation of the argument used in the proof of Proposition 3.2 (cf. Brock [Bro00]) shows that for every sequence (γ_i) of simple closed curves converging to λ^+ we have

$$\frac{l_{\rho_{n_i}}(\gamma_i^+)}{l_S(\gamma_i^+)} \geq L \tag{4.1}$$

for some positive L and all sufficiently large i. This contradicts lemma 4.2 showing that λ^+ is not realized. The same argument applies to λ^-.

Lemma 4.3. *The laminations λ^+ and λ^- are not realized in N_∞.* □

This implies, by Thurston and Bonahon's work, that λ^+ and λ^- are the ending laminations of $\mathbb{H}^3/\rho_\infty(\pi_1(S))$. On the other hand, λ^+ and λ^- are also the ending laminations of the infinite cyclic cover M_f of the mapping torus of f. Minsky [Min01] proved that two Kleinian surface groups with the same ending laminations are conjugate in $\mathrm{PSL}_2(\mathbb{C})$, provided that one of them has non-vanishing injectivity radius. In our case, M_f clearly has positive injectivity radius because it covers the closed manifold T_f. In particular, we deduce that $\mathbb{H}^3/\rho_\infty(\pi_1(S)) = M_f$ and hence, up to conjugacy, $\rho_\infty = \rho_f$.

We have proved that the limit of any, up to conjugacy, convergent subsequence of (ρ_n) is conjugate to the representation ρ_f with $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$. Since we proved in the previous section that (ρ_n) is pre-compact up to conjugacy we obtain:

Lemma 4.4. *The sequence* (ρ_n) *converges, up to conjugation, to the representation* $\rho_f : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ *with* $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$. □

4.3. **Geometric convergence of the sequence** (ρ_n)**.** We prove now that, after choosing appropriate base points p_n, the sequence of pointed manifolds (N_ρ, p_n) converges geometrically to M_f.

By Lemma 4.4 we can conjugate the representations ρ_n so that they converge to ρ_f; we assume from now on that this is the case. Choosing once and for ever a point $p_{\mathbb{H}^3}$ in $\mathbb{H}^3$ let q_n and p_f be the projections of $p_{\mathbb{H}^3}$ to $\mathbb{H}^3/\rho_n(\pi_1(S))$ and $\mathbb{H}^3/\rho_f(\pi_1(S)) = M_f$. Let also p_n be the projection of the base point q_n to $\mathbb{H}^3/\rho(\pi_1(M)) = N_\rho$.

The Margulis Lemma implies that the injectivity radius of the manifolds $\mathbb{H}^3/\rho_n(\pi_1(S))$ and N_ρ at the base points q_n and p_n is uniformly bounded from below. In particular, Gromov's Compactness Theorem implies that every subsequence of $(\mathbb{H}^3/\rho_n(\pi_1(S)), q_n)$ (resp. (N_ρ, p_n)) has a geometrically convergent subsequence $(\mathbb{H}^3/\rho_{n_i}(\pi_1(S)), q_{n_i})$ (resp. (N_ρ, p_{n_i})) with limit (N'_G, q_G) (resp. (N_G, p_G)); by construction the coverings

$$(\mathbb{H}^3/\rho_{n_i}(\pi_1(S)), q_{n_i}) \to (N_\rho, p_{n_i})$$

converge to a covering

$$(N'_G, q_G) \to (N_G, p_G)$$

Before going further observe that the manifold N_G is non-compact.

The relation between geometric convergence of hyperbolic manifolds and convergence in the Chabauty topology of discrete groups of $\mathrm{PSL}_2\,\mathbb{C}$ [BP92] implies that $N_G = \mathbb{H}^3/\Gamma_G$ where Γ_G is the group formed by all elements $\gamma \in \mathrm{PSL}_2\,\mathbb{C}$ such that there is a sequence (γ_i) in $\pi_1(S)$ with $\gamma = \lim \rho_{n_i}(\gamma_i)$. In particular $\rho_f(\pi_1(S)) \subset \Gamma_G$ and hence the manifold M_f is a Riemannian cover of N'_G. In other words, we have the following tower of coverings

$$M_f \to N'_G \to N_G$$

We claim that all these coverings are isometries. In order to prove that this is the case it suffices to show that $M_f \to N_G$ has degree 1.

The Covering Theorem implies that the cover $M_f \to N_G$ is finitely sheeted, that N_G is homeomorphic to $R \times \mathbb{R}$ where R is a compact surface and that this covering is actually induced by a finite sheeted covering between surfaces $S \to R$. Observe that this implies that N_G is not homeomorphic to N_ρ and hence the sequence of base points $(p_{n_i})_i$ cannot stay in a bounded region of N_ρ. We claim now that this sequence leaves N_ρ in the direction of the S-end. In fact, since the geometric limit N_G contains some closed geodesic one obtains from geometric convergence that there is some uniform constant L such that for all

i there is some closed geodesic in N_ρ which intersects the radius L neighborhood around p_{n_i}. By construction, all ends of the manifold N_ρ but the S-end are convex-cocompact and hence have a neighborhood which is disjoint of the set of closed geodesics. This implies that p_{n_i} tends towards the S-end.

We claim that the cover $S \to R$ has degree 1. This suffices in order to prove that $M_f \to N_G$ is an isometry. Seeking a contradiction assume that this is not the case. Identify the surface R with the image of $R \times \{0\}$ under the homeomorphism $R \times \mathbb{R} \simeq N_G$ and let $\kappa_{n_i} : R \to N_\rho$ be, for sufficiently large i, the almost isometric embeddings provided by geometric convergence. Choose a neighborhood U homeomorphic to $S \times (0, \infty)$ of the S-end of N_ρ. From the above we obtain that $\kappa_i(R) \subset U$ for all sufficiently large i. In particular we obtain that the group $(\kappa_i)_*(\pi_1(R))$ is contained, up to conjugacy, for all but finitely many i in the image $[\pi_1(S)]$ of $\pi_1(S)$ in $\pi_1(N_\rho)$. On the other hand, the fact that N_G is covered by $M_f = \mathbb{H}^3/\rho_f(\pi_1(S))$ implies that $(\kappa_i)_*(\pi_1(R))$ contains, again up to conjugacy, the group $[\pi_1(S)]$. In other words we have $(\kappa_i)_*(\pi_1(R)) = [\pi_1(S)]$. Let C_S be the relative compression body of $N_\rho \setminus U$ corresponding to ∂U and $C'_S = C_S \cup U$. From the above we obtain that for all sufficiently large i the surface $\kappa_i(R)$ is an embedded surface in C'_S such that the homomorphism $\pi_1(\kappa_i(R)) \to \pi_1(C'_S)$ is surjective. It follows from lemma 2.2 that R has at least the same genus as S. This contradicts the assumption that the cover $S \to R$ had degree more than 2. We have proved:

Lemma 4.5. *The pointed manifolds $(N_\rho, p_n)_n$ converge geometrically to the manifold (M_f, p_f).* □

In the language of Kleinian groups, Lemma 4.5 means that the sequence (ρ_n) converges *strongly*, i.e. algebraically and geometrically, to ρ_f.

4.4. **Proof of Proposition 4.1.**

Because of the geometric convergence of the sequence (N_ρ, p_n) to (M_f, p_f) there exists a sequence of maps

$$\kappa_n : (M_f, p_f) \to (N_\rho, p_n)$$

such that on every compact set $K \subset M_f$ the sequence κ_n tends to an isometry in the C^∞ topology on K.

We can now prove Proposition 4.1 with essentially the same argument as in McMullen [McM96]:

Proof of Prop. 4.1. Recall that $\iota : S \to M_f$ is an embedding which, at the level of fundamental groups, induces ρ_f and set $\Sigma_0 = \iota(S)$. The manifold M_f admits an isometry

$$\Psi : M_f \to M_f$$

which acts like f on the level of the fundamental groups. We chose ι such that $\Sigma_0 \cap \Psi^k(\Sigma_0) = \emptyset$ for all k. We denote $\Psi^n(\Sigma_0)$ by Σ_n for $n \in \mathbb{Z}$ and by $[\Sigma_n, \Sigma_m]$, we mean the subset of M_f which is enclosed between Σ_n and Σ_m. Let $E^+ \subset M_f$ be the neighborhood of the positive end $\mathcal{E}^+$ given by

$$E^+ = [\Sigma_0, \infty) = \bigcup_{n>0} [\Sigma_{n-1}, \Sigma_n].$$

Let $K = [\Sigma_{-3}, \Sigma_3]$. Also let $\iota^+ : S \hookrightarrow N_\rho$ be an embedding in the same homotopy class as the embedding $S \hookrightarrow M$.

Because of Lemma 4.4 and Lemma 4.5 for all n sufficiently large, κ_n restricted to K is a smooth embedding which is C^∞-close to an isometry in the correct homotopy class, i.e. $\kappa_n \circ \iota$ is homotopic to $\iota^+ \circ f^n$. We are going to construct the desired embedding h^+ essentially by gluing together the maps

$$h_n = \kappa_n \circ \Psi^{-n} : [\Sigma_{n-3}, \Sigma_{n+3}] \to N_\rho.$$

The maps h_n and h_{n+1} are both defined on $[\Sigma_{n-2}, \Sigma_{n+3}]$ but they do not coincide on this set. However, both are for sufficiently large n, close in the C^∞ topology and therefore they can be, for n sufficiently large, slightly modified so that they coincide. More precisely, let $\theta : M_f \to [0,1]$ be a smooth bump function supported on $[\Sigma_{-1}, \Sigma_1]$, such that $\sum_{-\infty}^{\infty} \theta(\Psi^k(x)) = 1$ for all x. Following McMullen [McM96] define

$$h^+(x) = \sum_k \theta(\Psi^{-k}(x)) h_k(x),$$

for all $x \in [\Sigma_{-1}, \infty]$, where the sum is interpreted as the hyperbolic barycenter of the weighted points. Note that for any x at most two terms, say $h_n(x)$ and $h_{n+1}(x)$ have nonzero weights. Both maps h_n and h_{n+1} provide almost isometric embeddings of a definite neighborhood of Σ_n into N^+. Moreover they are both in the same homotopy class which is compatible with the markings ι and ι^+ of M_f and N^+ and therefore they are close in the C^∞ topology. Thus, for all sufficiently large n, the map $h_+|_{[\Sigma_n, \infty)}$ is a well-defined almost isometric embedding in the correct homotopy class. This concludes the proof of Proposition 4.1. $\square$

5. Gluing

In this section, we prove Theorem 1.1 and Theorem 1.2.

Let M^+ and M^- be compact, irreducible, orientable 3-manifolds whose fundamental groups do not contain $\mathbb{Z}^2$ and let $\partial_0 M^+$ and $\partial_0 M^-$ be boundary components which are identified with an oriented surface S of genus $g > 1$ in such a way that the orientation of S agrees with the

induced orientation of $\partial_0 M^+$ and does not agree with the induced orientation of $\partial_0 M^-$. Let also $f \in \mathcal{MCG}(S)$ be a pseudo-Anosov mapping class which is generic with respect to M^+ and M^-, i.e. the stable (resp. unstable) lamination of f is not a limit of meridians of $\partial_0 M^+$ (resp. $\partial_0 M^-$). Abusing terminology we refer from now to such a pseudo-Anosov mapping class as being *generic*. Observe that this convention is consistent with the use of the word generic in the introduction.
Let N^+ (resp. N^-) be a hyperbolic structure on M^+ (resp. M^-) as provided by the Existence Theorem for Degenerated Structures in section 3 such that the S'-end is convex-cocompact for every boundary component $S' \neq \partial_0 M^\pm$ of $M^\pm$ and $\lambda^\pm$ is not realized in $N^\pm$. We also assume that $\iota^\pm : S \hookrightarrow N^\pm$ are embeddings of S that represent the isotopy class of $\partial_0 M^\pm \hookrightarrow M^\pm$.
In this section we will merge the metrics of N^+ and N^- to construct negatively curved metrics on $N_{f^n} = M^+ \cup_{f^n} M^-$ for all sufficiently large n.

Theorem 5.1. *Let M^+ and M^- be compact, irreducible, orientable 3-manifold whose fundamental groups do not contain $\mathbb{Z}^2$ and let $\partial_0 M^+$ and $\partial_0 M^-$ be boundary components which are identified with an oriented surface S of genus $g > 1$ in such a way that the orientation of S agrees with the induced orientation of $\partial_0 M^+$ and does not agree with the induced orientation of $\partial_0 M^-$. Let also $f \in \mathcal{MCG}(S)$ be a generic pseudo-Anosov mapping class. For an arbitrary $\epsilon > 0$ there is n_ϵ, such that for all $n \geq n_\epsilon$, the manifold*

$$N_{f^n} = M^+ \cup_{f^n} M^-$$

admits a Riemannian metric $\rho_{n,\epsilon}$ with all sectional curvatures pinched by $-1-\epsilon$ and $-1+\epsilon$. Moreover, the injectivity radius of the metric $\rho_{n,\epsilon}$ is bounded from below independently of n and ϵ.

Proof. Recall from section 4 that M_f is the infinite cyclic cover of the mapping torus of f. Let $\Psi : M_f \to M_f, \iota, \Sigma_0, \Sigma_n, [\Sigma_m, \Sigma_n] \subset M_f$ be as in the proof of Proposition 4.1 and set

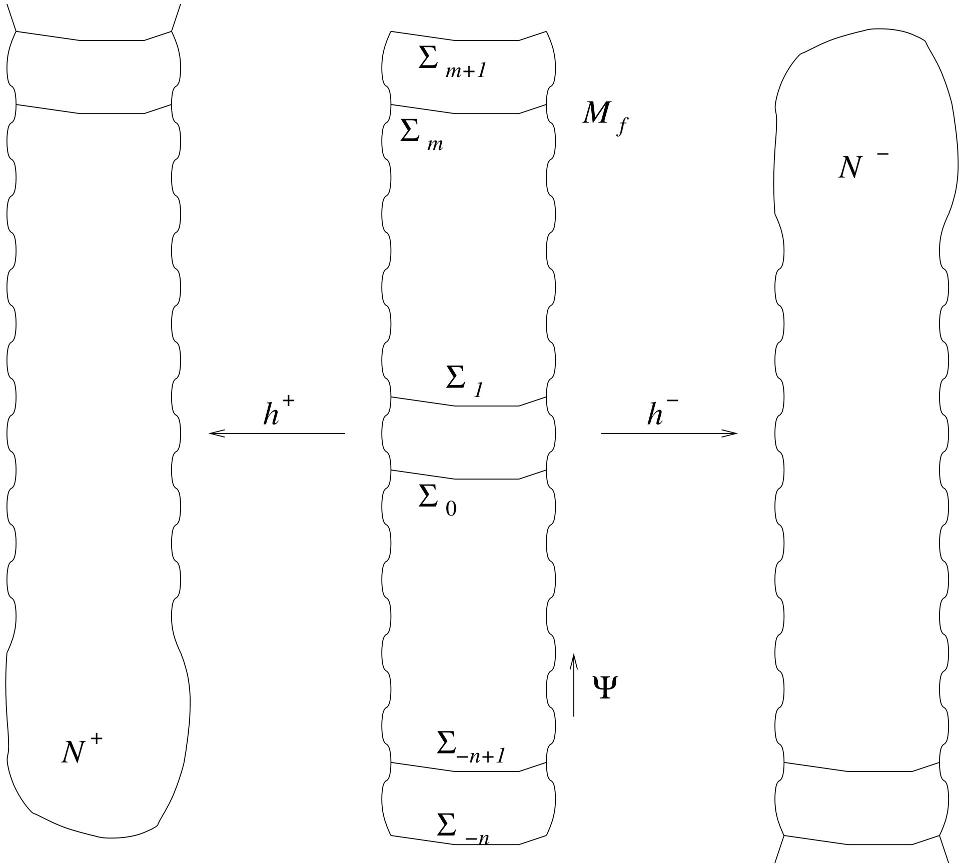
$$(\Sigma_n, \infty) = [\Sigma_n, \infty) \setminus \Sigma_n, \quad (-\infty, \Sigma_n) = (-\infty, \Sigma_n] \setminus \Sigma_n.$$

We also observe that the manifolds M_f and $M_{f^{-1}}$ can be identified by an orientation reversing isometry. From Proposition 4.1 we obtain embeddings

$$h^+ : E^+ \to N^+, \quad h^- : E^- \to N^-,$$

where $E^+ = [\Sigma_0, \infty)$ and $E^- = (-\infty, \Sigma_0]$, such that:

(1) For any k, r and $\epsilon > 0$, there is a compact set $K \subset E^{\pm}$ such that $h^{\pm}$ is ϵ-close to an isometry in the C^k topology when restricted to any embedded r-ball contained in $E^{\pm} \setminus K$.
(2) The embedding $h^{\pm} \circ \iota$ is homotopic to $\iota^{\pm}$.



For $m, n \geq 0$, $N^+ \setminus h^+((\Sigma_m, \infty))$ and $N^- \setminus h^-((-\infty, \Sigma_{-n}))$ are respectively homeomorphic to M^+ and M^-, via homeomorphisms that are compatible with the identifications of N^+ and N^- with the interiors of M^+ and M^-, and $h^+(\Sigma_m)$ and $h^-(\Sigma_{-n})$ are boundary components of these two manifolds associated to $\partial_0 M^+$ and $\partial_0 M^-$. The map

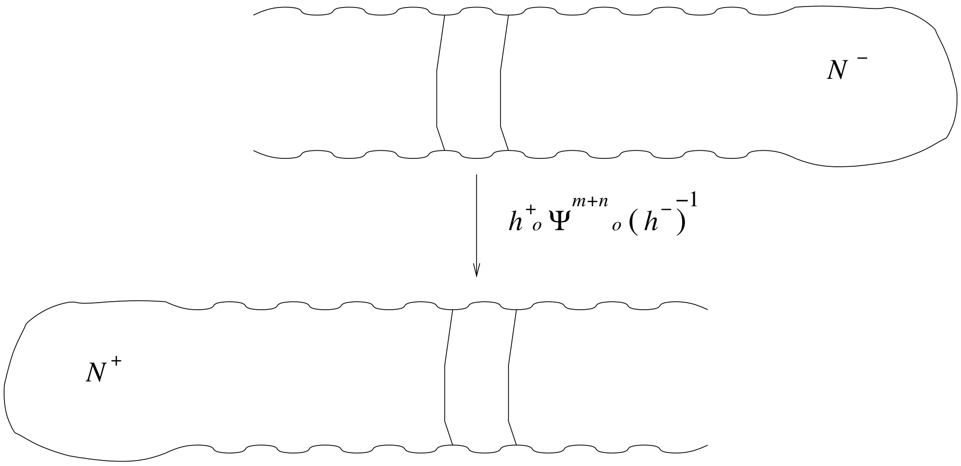
$$h^+ \circ \Psi^{m+n} \circ (h^-)^{-1}$$

provides an identification of these boundaries. If we use this map to glue these two manifolds the result is homeomorphic to $N_{f^{m+n}}$ because $\Psi^{m+n} \circ \iota$ is homotopic to $\iota \circ f^n$ and therefore $h^+ \circ \Psi^{m+n} \circ (h^-)^{-1} \circ \iota^-$ is homotopic to $\iota^+ \circ f^{m+n}$.
Observe that the interior of $N_{f^{m+n}}$ is also homeomorphic to

$$(N^+ \setminus h^+((\Sigma_m, \infty)) \cup_{h^+ \circ \Psi^{m+n} \circ (h^-)^{-1}} (N^- \setminus (h^-)((-\infty, \Sigma_{-n+1})))$$

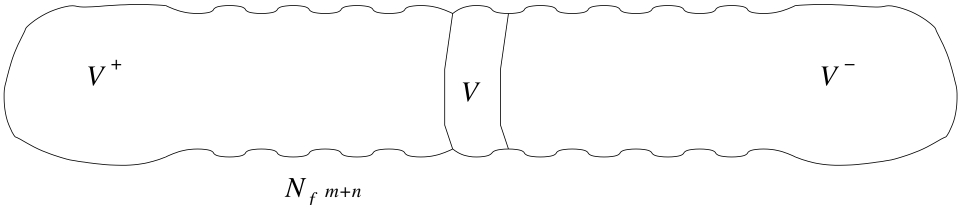
where we consider the map Ψ^{m+n} as a homeomorphism between the collars

$$[\Sigma_{-n}, \Sigma_{-n+1}] \to [\Sigma_m, \Sigma_{m+1}].$$



We denote the image of this collar in $N_{f^{m+n}}$ by V and the two components of $N_{f^{m+n}} \setminus V$ by V^+ and V^-.

The hyperbolic metric of N^+ (resp. N^-) induces a hyperbolic metric ν^+ on $N_{f^{m+n}} \setminus V^-$ (resp. ν^- on $N_{f^{m+n}} \setminus V^+$). These metrics do not coincide on V and this is why, in order to merge both, we consider a smooth bump function $\theta : [\Sigma_0, \Sigma_1] \to \mathbb{R}$ with values in $[0,1]$ and such that $\theta|_{\Sigma_1} \equiv 1$ and $\theta|_{\Sigma_0} \equiv 0$. Using Ψ^m we can identify $[\Sigma_0, \Sigma_1]$ and V and therefore we can consider θ as a bump function on V with $\theta|_{\partial V^+} \equiv 1$ and $\theta|_{\partial V^-} \equiv 0$.



We can extend θ to a map defined on $N_{f^{m+n}}$ by defining it to be constant on V^+ and V^-.

Now we define the metric ν to be:

$$\nu(x) = \theta(x) \cdot \nu^+(x) + (1 - \theta(x)) \cdot \nu^-(x),$$

for any $x \in V$. This metric is smooth and of course hyperbolic on $N_{f^{m+n}} \setminus V$. Moreover, when m, n are sufficiently large, the metrics ν^+

and ν^- are C^∞ close on V to the metric ν_f induced by the hyperbolic metric on M_f. In particular we have:

$$\nu^+ = \nu_f + \xi^+ \text{ and } \nu^- = \nu_f + \xi^-,$$

where ξ^+ and ξ^- are 2-tensors which are C^2-close to zero for $m, n \gg 0$. On V we have

$$\nu = \nu_f + \theta\, \xi^+ + (1-\theta)\, \xi^-$$

and since θ does not depend on m and n, by assuming that they are sufficiently large, we can make sure that ν and ν_f are as C^2-close as we want. The sectional curvature of ν depends only on the first and second derivatives of the metric and therefore all sectional curvatures of ν stay in the interval $[-1-\epsilon, -1+\epsilon]$ if n and m are large.
It is also obvious that the injectivity radius for ν at every point is at least half the lower bound for the injectivity radius on M_f and we have proved Theorem 5.1. □

> **Convention:** Using the above Theorem, for the rest of this article we assume N_{f^n} is equipped with a metric ρ_n constructed in the proof for a small constant ϵ_n which tends to 0 as $n \to \infty$.

Recall that $N^\pm$ is the interior of $M^\pm$ together with a complete hyperbolic metric and also that M^+ and M^- have a canonical embedding in $N_{f^n} = M^+ \cup_{f^n} M^-$. The construction that we just gave to prove Theorem 5.1 implies directly that:

Theorem 5.2.

- *Any geometric limit of a subsequence of the manifolds N_{f^n} is isometric to either N^+, N^- or M_f.*
- *If N is either isometric to N^+, N^- or M_f then there are basepoints $p_n \in N_{f^n}$ such that the sequence (N_{f^n}, p_n) converges geometrically to a Riemannian manifold isometric to N.*
- *If the geometric limit of a subsequence of N_{f^n} is isometric to either N^+ or N^- then the almost isometric embeddings provided by the geometric convergence are such that the following diagram commutes up to isotopy:*

$$\begin{tikzcd}
N^\pm \arrow[r, "\text{id.}"] \arrow[dr] & M^\pm \arrow[d, hook] \\
 & N_{f^n}
\end{tikzcd}$$

where the diagonal map is the almost isometric embedding and the other two maps are the canonical embeddings.

- *If the geometric limit of a subsequence of* N_{f^n} *is isometric to* M_f *then embedded incompressible surfaces in* M_f *are mapped by the almost isometric embeddings provided by geometric convergence of the sequence to surfaces isotopic to* $S = \partial M^+ = \partial M^-$. □

We observe that Theorem 1.1 and Theorem 1.2 are special cases of Theorem 5.1 and Theorem 5.2. The following Corollary follows immediately as well:

Corollary 5.1. *For every* n*,* N_{f^n} *can be decomposed into a union of disjoint subsets* $N_{f^n} = X^+ \cup Y \cup X^-$ *such that:*

- X^+ *and* X^- *are homeomorphic to* M^+ *and* M^- *respectively and their diameters are bounded independent of* n*.*
- Y *is homeomorphic to* $S \times \mathbb{R}$ *and the distance between its boundary components tends to infinity as* $n \to \infty$*.* □

6. Asymptotic faithfulness of the homomorphisms $\pi_1(M^+) \to \pi_1(N_{f^n})$

Let M^+, M^-, f and N_{f^n} be as in the statement of Theorem 5.1. Recall that by our convention in the last section N_{f^n} is endowed with the metric provided by Theorem 5.1 where the curvature tends to -1 as $n \to \infty$. We also fix N^+, N^-, M_f to be the geometric limits obtained in Theorem 5.2.

In this section we study properties of the homomorphism $\pi_1(M^+) \to \pi_1(N_{f^n})$ induced by the inclusion

$$M^+ \hookrightarrow M^+ \cup_{f^n} M^- = N_{f^n}$$

In some sense, the most fundamental question is if these sequence of homomorphisms is asymptotically faithful, i.e. if

$$\cap_{i=1}^{\infty} \cup_{n=i}^{\infty} \left\{\ker(\pi_1(M^+) \to \pi_1(N_{f^n}))\right\}$$

is trivial. This is clearly the case if M^+ and M^- have incompressible boundary since each of the homomorphisms $\pi_1(M^+) \to \pi_1(N_{f^n})$ is injective. If one of the manifolds M^+ or M^- has acylindrical boundary then it is due to Lackenby [Lac02] that the sequence $(\pi_1(M^+) \to \pi_1(N_{f^n}))_n$ is asymptotically faithful. However, the case that both M^+ or M^- have compressible boundary, for instance if they are handlebodies, seems to be untreatable with only topological methods. We derive that the sequence $(\pi_1(M^+) \to \pi_1(N_{f^n}))_n$ is asymptotically faithful as a consequence of the results proved in the previous section:

Lemma 6.1. *With the same assumptions as in Theorem 5.1 we have:* $\cap_{i=1}^{\infty} \cup_{n=i}^{\infty} \ker(\pi_1(M^+) \to \pi_1(N_{f^n})) = \mathrm{Id}$.

Proof. By Theorem 5.2, there are base points $p_n \in N_{f^n}$ such that the sequence (N_{f^n}, p_n) converges geometrically to N^+; moreover, the almost isometric embeddings provided by geometric convergence are such that the following diagram commutes up to isotopy:

$$\begin{tikzcd}
N^{\pm} \arrow[r, "\text{id.}"] \arrow[dr] & M^{\pm} \arrow[d, hook] \\
& N_{f^n}
\end{tikzcd}$$

By Proposition 4.1 the injectivity radius of N^+ is bounded from below. In particular, every non-trivial element $\gamma \in \pi_1(N^+)$ determines uniquely a non-trivial geodesic γ^* in N^+. For sufficiently large n, the image of γ^* under the almost isometric embeddings given by geometric convergence has very small geodesic curvature. Since the manifold N_{f^n} has, for sufficiently large n, curvature pinched by $-2 \leq \kappa_{N_{f^n}} \leq -\frac{1}{2}$ we derive from the Morse lemma that the image of γ^* is not homotopically trivial for all n sufficiently large. This shows that $\gamma \notin \ker(\pi_1(M^+) \to \pi_1(N_{f^n}))$ for all but finitely many n. $\square$

We would like to observe that Theorem 5.2 plays a central role in the proof of Lemma 6.1. In particular, the authors didn't succeed deriving Lemma 6.1 only from the existence of negatively curved, even hyperbolic, metrics on the manifolds N_{f^n}. And even less so using topological arguments.

If Γ is a cyclic subgroup of $\pi_1(M^+)$ then it follows from Lemma 6.1 that, again for sufficiently large n, the homomorphism $\Gamma \to \pi_1(N_{f^n})$ is injective. Our next goal is to extend this property to every infinite index finitely generated subgroup of $\pi_1(M^+)$.

Theorem 6.1. *Let M^+, M^-, f and N_{f^n} be as in Theorem 5.1 and let $\Gamma \subset \pi_1(M^+)$ be a finitely generated subgroup of $\pi_1(M^+)$ such that*

$$(*)\ \gamma\Gamma\gamma^{-1} \cap \pi_1(C_S) \text{ has infinite index in } \pi_1(C_S) \text{ for all } \gamma \in \pi_1(M^+)$$

where C_S is the relative compression body associated to S. Then there is some n_Γ such that for all $n \geq n_\Gamma$ the homomorphism $\Gamma \to \pi_1(N_{f^n})$ is injective.

Proof. As in the proof of the above lemma choose base points $p_n \in N_{f^n}$ such that the sequence (N_{f^n}, p_n) converges geometrically to N^+; again the almost isometric embeddings provided by geometric convergence are such that the following diagram commutes up to isotopy:

$$\begin{tikzcd}
N^{\pm} \arrow[r, "\text{id.}"] \arrow[dr] & M^{\pm} \arrow[d, hook] \\
& N_{f^n}
\end{tikzcd}$$

Suppose Γ is as in the statement of the Theorem. Proposition 3.1 implies that the Riemannian cover $N_\Gamma \to N^+$ associated to Γ is convex-cocompact. Let $K \subset \mathbb{H}^3$ be a convex Γ-invariant subset with K/Γ compact and let the maps $K \to N^+ \to N_{f_n}$ be obtained as the juxtaposition of the restriction of the covering map and the almost isometric embeddings provided by the geometric convergence. These maps lift to maps $K \to \widetilde{N_{f^n}}$ to the universal cover of N_{f^n} and compactness of K/Γ implies that there for sufficiently large n they are quasi-isometric embeddings and hence map geodesics to quasi-geodesics. In particular this applies to the axis of every non-trivial $\gamma \in \Gamma$. This proves that for all n sufficiently large the kernel of the map $\Gamma \to \pi_1(N_{f^n})$ is trivial. $\square$

Theorem 6.1 is due to Easson [Eas04] in the case that M^+ has acylindrical boundary and Γ is the fundamental group of a closed surface. Observe also that Theorem 1.3 from the introduction is a particular case of Theorem 6.1 (cf. Cor. 3.1).

Before going further we observe that a similar argument to the proof of the above Theorem shows:

Lemma 6.2. *If the geometric limit of a subsequence of N_{f^n} is homeomorphic to $S \times \mathbb{R}$ and Γ is an infinite index subgroup of $\pi_1(S)$, for n sufficiently large the homomorphism $\Gamma \to \pi_1(N_{f^n})$ induced by the almost isometric embeddings is injective.* $\square$

We conclude this section with an application of Theorem 6.1 which will be needed later on. Suppose that a sequence of 3-manifolds N_{f^n} is obtained by gluing a handlebody M^- via iterations of a generic pseudo-Anosov f onto the a boundary component S of a compact 3-manifold with incompressible boundary M^+. If $S' \subset \partial M^+$ is a different boundary component, either S and S' are isotopic and hence M^+ is a trivial interval bundle, or $\gamma\pi_1(S')\gamma^{-1} \cap \pi_1(S)$ has infinite index in $\pi_1(S)$ for all γ. Hence, we obtain from Theorem 6.1:

Corollary 6.1. *Let M^+, M^-, f and N_{f^n} be as in Theorem 5.1. Furthermore assume M^+ has incompressible boundary, M^- is a handlebody and N_{f^n} is a handlebody for infinitely many n. Then M^+ is homeomorphic to a trivial interval bundle.* $\square$

7. Rank of $\pi_1(N_{f^n})$

In this section we prove Theorem 1.4. As in the introduction from now on we assume M^+ and M^- are orientable 3-dimensional handlebodies whose boundary is identified with a surface S of genus $g > 1$. We also assume $f \in \mathcal{MCG}(S)$ is generic. Before launching the proof we need to recall some facts about the relation between Nielsen equivalence classes of generating sets of the fundamental group of a connected Riemannian manifold and carrier graphs.

Following White [Whi02] we say that a map $\phi : X \to M$ of a connected graph X into a connected manifold M is a *carrier graph* if the induced homomorphism $\phi_* : \pi_1(X) \to \pi_1(M)$ is surjective. Two carrier graphs $\phi : X \to M$ and $\psi : Y \to M$ are *equivalent* if there is a homotopy equivalence $h : X \to Y$ such that ϕ and $\psi \circ h$ are freely homotopic.
To every generating set $\mathcal{S} = (g_1, \ldots, g_r)$ of $\pi_1(M)$ one can associate an equivalence class of carrier graphs as follows: Let $\mathbb{F}_{\mathcal{S}}$ be the free non-abelian group generated by the set $\mathcal{S}$, $\phi_{\mathcal{S}} : \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ the homomorphism given by mapping the free basis $\mathcal{S} \subset \mathbb{F}_{\mathcal{S}}$ to the generating set $\mathcal{S} \subset \pi_1(M)$ and $X_{\mathcal{S}}$ a graph with $\pi_1(X_{\mathcal{S}}) = \mathbb{F}_{\mathcal{S}}$. The homomorphism $\phi_{\mathcal{S}} : \mathbb{F}_{\mathcal{S}} \to \pi_1(M)$ determines a free homotopy class of maps $\phi : X_{\mathcal{S}} \to M$, i.e. a carrier graph, and any two carrier graphs obtained in this way are equivalent. The so determined equivalence class is said to be the *equivalence class of carrier graphs associated to* $\mathcal{S}$. In fact, the following well-known lemma, which is essentially due to Nielsen, states that we have a bijection between the set of Nielsen equivalence classes of generating sets of $\pi_1(M)$ and the set of equivalence classes of carrier graphs.

Lemma 7.1. *Let $\mathcal{S}$ and $\mathcal{S}'$ be finite generating sets of $\pi_1(M)$ with the same cardinality. Then the following are equivalent:*

1. *$\mathcal{S}$ and $\mathcal{S}'$ are Nielsen equivalent.*
2. *$\mathcal{S}$ and $\mathcal{S}'$ have the same associated equivalence classes of carrier graphs.* □

From now on we will only consider generating sets of minimal cardinality. Equivalently, we consider only carrier graphs $\phi : X \to M$ with $\mathrm{rank}(\pi_1(X)) = \mathrm{rank}(\pi_1(M))$. Minimality asserts that whenever $Y \subset X$ is a connected subgraph such that $\phi_*(\pi_1(Y))$ is a free subgroup of $\pi_1(M)$ then $\phi_* : \pi_1(X) \to \pi_1(M)$ is injective when restricted to $\pi_1(Y)$ because

$$\mathrm{rank}(\phi_*(\pi_1(Y))) = \mathrm{rank}(\pi_1(Y))$$

and every surjective homomorphism between two free groups of the same rank is an isomorphism.

Assume now that M is Riemannian with, for concreteness, pinched negative curvature. Given a carrier graph $\phi : X \to M$ and a path I in X we say that its length is the length, with respect to the metric of M, of the path $\phi(I)$. Measuring the minimal length of a path joining two points in X we obtain a semi-distance $d_{\phi:X\to M}$ on X and we define the *length* $l_{\phi:X\to M}(X)$ of the carrier graph $\phi : X \to M$ as the sum of the lengths of the edges of X with respect to $d_{\phi:X\to M}$. The semi-distance $d_{\phi:X\to M}$ induced on X is not always a distance since there may be some edges of length 0 but minimality of the generating set ensures that collapsing these edges we obtain an equivalent carrier graph on which the induced semi-distance is in fact a distance. Moreover, this collapsing process does not change the length of the carrier graph. From now on we will assume without further remark that the semi-distance $d_{\phi:X\to M}$ is in fact a distance.

Definition. *A carrier graph $\phi : X \to M$ has* minimal length *if*

$$l_{\phi:X\to M}(X) \leq l_{\phi':X'\to M}(X)$$

for every equivalent carrier graph $\phi' : X' \to M$.

If M is closed then it follows from Arzela-Ascoli's Theorem that every equivalence class of carrier graphs contains a carrier graph with minimal length:

Lemma 7.2. *If M is a closed Riemannian 3-manifold (or more generally if M is a Riemannian cover of a closed manifold), then every equivalence class of carrier graphs contains a carrier graph with minimal length. Moreover, every such minimal length carrier graph is trivalent, hence it has $3(\mathrm{rank}(M)-1)$ edges, the image in M of its edges are geodesic segments and the angle between any two adjacent edges is $\frac{2\pi}{3}$.* □

See White [Whi02, Section 2] for a proof of Lemma 7.2 in the case that M is hyperbolic, the proof in the general case is word-by-word the same.

If $\phi : X \to M$ is a carrier graph in a 3-manifold M we denote by $\tilde{\phi} : \tilde{X} \to \tilde{M}$ the lift of ϕ to a map between the universal covers of X and M. We will be mainly interested in closed manifolds with negative curvature, in particular $\pi_1(M)$ is not going to be free and hence the map $\tilde{\phi}$ cannot be an embedding. However, subgraphs of X may well be quasi-isometrically embedded.

Definition. *Let M be a manifold with negative curvature and $A > 0$. A connected subgraph $Y \subset X$ of a carrier graph $\phi : X \to M$ is A-*quasi-convex *for some $A > 0$ if:*

- *The restriction* $\tilde{\phi}|_{\tilde{Y}} : \tilde{Y} \to \tilde{M}$ *of the map* $\tilde{\phi}$ *to the universal cover* $\tilde{Y}$ *of* Y *is an* (A, A)*-quasi-isometric embedding.*
- *Every point in* $\tilde{Y}$ *is at most at distance* A *of the axis of some element of* $\pi_1(Y)$*. If* $\tilde{Y}$ *is compact we assume that it has diameter at most* A*.*
- *The translation length of every element* $\phi_*(\gamma)$ *in* $\tilde{M}$ *is at least* $\frac{1}{A}$ *for every* $\gamma \in \pi_1(Y)$*.*

If Y is a graph and $g : Y \to M$ is a map whose lift $\tilde{g} : \tilde{Y} \to \mathbb{H}^3$ is a quasi-isometric embedding then the image $g_*(\pi_1(Y))$ is a free convex-cocompact subgroup. Intuitively, considering A-quasi-convex graphs amounts to considering uniformly convex-cocompact free subgroups.

Theorem 1.4. *There is* n_f *with* $\operatorname{rank}(\pi_1(N_{f^n})) = g$ *for all* $n \geq n_f$*. Moreover, every minimal generating set of* $\pi_1(N_{f^n})$ *is Nielsen equivalent to a standard generating set for all sufficiently large* n*. In particular* $\pi_1(N_{f^n})$ *has at most 2 Nielsen equivalence classes of minimal generating sets.*

Recall that $N_{f^n} = M^+ \cup_{f^n} M^-$ where M^+ and M^- are genus g handlebodies and f is a generic pseudo-Anosov mapping class.

Proof. Following our convention we consider N_{f^n} with the metric provided by Theorem 1.1. For all n let $\mathcal{S}_n$ be a generating set of $\pi_1(N_{f^n})$ with minimal cardinality and $\phi_n : X_n \to N_{f^n}$ a minimal length carrier graph in the equivalence class determined by $\mathcal{S}_n$. As remarked in the introduction

$$\operatorname{rank}(\pi_1(N_{f^n})) \leq g$$

and hence X_n has at most $3(g-1)$ edges. We denote by $X_n^{<t}$ the subgraph of X_n consisting of all the edges of X_n of length less than t.

Claim 1. *For every* D *there are* n_D *and* A_D *such that for every proper subgraph* Y_n *of* X_n *of length less than* D *one has:* Y_n *is* A_D*-quasi-convex for all* $n \geq n_D$*.*

Proof of Claim 1. To begin with observe that the injectivity radius of the manifold N_{f^n} is bounded from below by Theorem 1.1 by some uniform constant δ. In particular, the last condition in the definition of A-quasi-convex is automatically satisfied for every A with $A^{-1} \leq \delta$. Seeking a contradiction to the claim assume that for some D there are sequences $A_i, n_i \to \infty$ such that for all i there is a proper subgraph Y_{n_i} of X_{n_i} which has length less than D and fails to be A_i-quasi-convex. Choose base points in $p_n \in Y_n$. By Theorem 1.2 there is a subsequence, say the whole sequence, such that the pointed manifolds $(N_{f^{n_i}}, \phi_{n_i}(p_{n_i}))$

converge geometrically to a manifold N_G which is either homeomorphic to a genus g handlebody or to $\Sigma_g \times \mathbb{R}$. The bound on the lengths on Y_{n_i} implies that the graphs (Y_{n_i}, p_{n_i}) converge to some graph Y_G and the Arzela-Ascoli theorem implies that the maps $\phi_{n_i} : Y_{n_i} \to N_{f^{n_i}}$ converge to a map $\phi_G : Y_G \to N_G$. Geometric convergence implies that

$$\mathrm{rank}(\pi_1(Y_G)) < \mathrm{rank}(\pi_1(X_{n_i})) \leq g \leq \mathrm{rank}(\pi_1(N_G))$$

Since $\pi_1(N_G)$ is either free or it is a surface group,

$$(\phi_G)_*(\pi_1(Y_G)) < \pi_1(N_G)$$

is a free infinite index-subgroup of $\pi_1(N_G)$. Hence, by Corollary 3.1, $(\phi_G)_*(\pi_1(Y_g))$ is a free convex-cocompact subgroup of $\pi_1(N_G)$. The same argument as in the proof of Theorem 6.1 shows that there is some A such that Y_{n_i} is A-quasi-convex for all i. □

We prove now that the length of the graph X_n is uniformly bounded from above.

Claim 2. *There is t with $X_n = X_n^{<t}$ for all n.*

Proof of Claim 2. If $X_n = X_n^{<1}$ for all n then we are done; assume that this is not the case for some subsequence (n_i) and let $Y^1_{n_i,1}, \dots, Y^{k(n_i,1)}_{n_i,1}$ be the components of $X^{<1}_{n_i}$. It follows from Claim 1 that the subgraphs $Y^j_{n_i,1}$ are A_1-quasi-convex for all but finitely many i, say for all.
The following proposition is proved in Souto [Sou05] for hyperbolic manifolds; the proof in the general case remains the same.

Proposition 7.1. *For all $A, s > 0$ there is L such that whenever M is a complete Riemannian 3-manifold with sectional curvature pinched by $-2 \leq \kappa_M \leq \frac{1}{2}$, $\phi : X \to M$ is a minimal length carrying graph with s edges and $Y_1, \dots, Y_k$ are disjoint connected A-quasi-convex subgraphs of X then either*

- *$\tilde{\phi} : \tilde{X} \to \tilde{M}$ is a quasi-isometric embedding and hence $\pi_1(M)$ is free, or*
- *the graph $X \setminus \cup_i Y_i$ contains an edge of at most length L.* □

By Proposition 7.1 there is a constant t_1 such that $X^{<1}_{n_i}$ is a proper subgraph of $X^{<t_1}_{n_i}$ for all i. If again $X^{<t_1}_{n_i}$ is a proper subgraph of X_{n_i} for infinitely many i, say for all i, then we can repeat this process getting t_2 such that $X^{<t_1}_{n_i}$ is a proper subgraph of $X^{<t_2}_{n_i}$ for all i. The bound on the number of edges of X_{n_i} ensures that this enlarging process can be repeated at most $3(g-1)$ times. □

We can now conclude the proof of Theorem 1.4. As in the proof of Claim 1 we obtain that, up to passing to a subsequence, the graphs

$\phi_n : X_n \to N_{f^n}$ converge geometrically to a graph $\phi_G : X_G \to N_G$ with $\mathrm{rank}(X_G) \leq g$ and such that $(\phi_G)_*(\pi_1(X_G))$ has finite index in $\pi_1(N_G)$. In particular, N_G cannot be homeomorphic to $\Sigma_g \times \mathbb{R}$ and hence it is homeomorphic to a genus g handlebody by Theorem 1.2. Moreover, since a rank g free group does not contain proper finite index subgroups of rank $\leq g$ we obtain that $(\phi_G)_*$ is surjective. In particular,

$$g \geq \mathrm{rank}(\pi_1(X_n)) \geq \mathrm{rank}(\pi_1(X_G)) \geq \mathrm{rank}(\pi_1(N_G)) = g$$

This proves the first statement of Theorem 1.4. Also note that the above implies that $(\phi_G)_* : \pi_1(X_G) \to \pi_1(N_G)$ is an isomorphism and therefore it determines a Nielsen equivalence class of minimal generating sets of N_G. But N_G is either N^+ or N^- and we know that for the free group every two minimal generating sets are Nielsen equivalent. □

We use now the same strategy to prove Theorem 1.5.

Theorem 1.5. *There is n_f such that for every $n \geq n_f$ the following holds: every proper subgroup $\Gamma \subset \pi_1(N_{f^n})$ with $\mathrm{rank}(\Gamma) \leq 2g-2$ is free.*

Since the proof has many points in common to the proof of Theorem 1.4 we will only sketch it and explain the differences.

Given a sequence Γ_n of proper subgroups of $\pi_1(N_{f^n})$ with $\mathrm{rank}(\Gamma_n) \leq 2g-2$ we have to show that Γ_n is free for all but finitely many n. Let $\mathcal{S}_n$ be a minimal generating set of Γ_n, consider the cover $\widetilde{N_{f^n}}/\Gamma_n \to N_{f^n}$ and let, for all n, $\phi_n' : X_n \to \widetilde{N_{f^n}}/\Gamma_n$ be a minimal length carrier graph in the equivalence class corresponding to $\mathcal{S}_n$. Denote by $\phi_n : X_n \to N_{f^n}$ the composition of ϕ_n' with the covering map. The same argument used to conclude the proof of Theorem 1.4 implies that Theorem 1.5 follows once the following claim is settled.

Claim. *For every D there are n_D and A_D such that for every subgraph Y_n of X_n of length less than D one has: Y_n is A_D-quasi-convex for all $n \geq n_D$.*

The proof of the claim is similar to the Claim 1 above. As above one argues by contradiction assuming that for some D there are sequences $A_i, n_i \to \infty$ such that for all i there is a subgraph Y_{n_i} of X_{n_i} which has length less than D and fails to be A_i-quasi-convex. Passing to geometric limits one obtains that the maps $\phi_{n_i}|_{Y_{n_i}} : Y_{n_i} \to N_{f^{n_i}}$ converge geometrically to a map $\phi_G : Y_G \to N_G$ and, again as above, it suffices to show that $(\phi_G)_*(\pi_1(Y_G))$ is free and convex-cocompact. We derive from the explicit description of the possible geometric limits of the sequence (N_{f^n}) provided by Theorem 1.2 that N_G is either homeomorphic to a handlebody of genus g or to $\Sigma_g \times \mathbb{R}$. Since under the almost isometric embeddings $\pi_1(N_G)$ surjects onto $\pi_1(N_{f^{n_i}})$ we obtain that $(\phi_G)_*(\pi_1(Y_G))$ is a proper subgroup of $\pi_1(N_G)$. A computation of Euler-characteristics proves that a proper finite index subgroup of $\Sigma_g \times \mathbb{R}$ has at least rank $4g-2$ and that a proper finite index subgroup of a rank g free group has at least rank $2g-1$. Since $\text{rank}((\phi_G)_*(\pi_1(Y_G)) \leq \text{rank}(\Gamma_n) \leq 2g-2$ we deduce as desired that $(\phi_G)_*(\pi_1(Y_G))$ has infinite index in $\pi_1(N_G)$ and hence it is free and convex-cocompact. The claim follows as in the proof of Claim 1 during the proof of Theorem 1.5. □

We conclude this section constructing an example to show that the bound given in Theorem 1.5 is sharp. Let $M^+ = M^-$ be oriented 3-dimensional handlebodies and let

$$\sigma : \pi_1(M^+) = \pi_1(M^-) \to \mathbb{Z}/2\mathbb{Z}$$

be a non-trivial homomorphism. The kernel of σ is a rank $2g-1$ free group. Take a generic pseudo-Anosov $f : \partial M^\pm \to \partial M^\pm$ which preserves the kernel of the composition of $\pi_1(\partial M^\pm) \to \pi_1(M^\pm)$ with σ. This can be achieved by taking suitable powers. Let N_{f^n} be as always the manifold obtained by gluing M^+ and M^- via f^n. By construction the homomorphism σ induces a homomorphism $\sigma' : \pi_1(N_{f^n}) \to \mathbb{Z}/2\mathbb{Z}$ whose kernel is a proper subgroup of index 2 and hence it is not free. Moreover, its rank is at most $2g-1$.

8. More on free subgroups of $\pi_1(N_{f^n})$

In this section we prove Theorem 1.6 and Theorem 1.7. Notation is as in the last section.

Theorem 1.7. *For every g' there is $n_{g'}$ such that $\pi_1(N_{f^n})$ does not contain any subgroup isomorphic to the fundamental group of a closed surface of genus g' for all $n \geq n_{g'}$.*

Proof. To begin with fix g' and let n_0 be such that the manifold N_{f^n} has curvature pinched by -2 and $\frac{-1}{2}$ for all $n \geq n_0$. Recall also that the manifolds N_{f^n} have injectivity radius uniformly bounded from below by some positive constant ϵ. By Corollary 5.1, there is some $n_1 \geq n_0$ such that for all $n \geq n_1$ the manifold N_{f^n} contains two parallel Heegaard surfaces S_n, S'_n which are at least at distance $\frac{8g'}{\epsilon}+1$. We claim that for all $n \geq n_1$ the manifold N_{f^n} does not contain π_1-injective surfaces of genus g'.

Seeking a contradiction assume that for some $n \geq n_1$ there is $\Sigma \subset N_{f^n}$ π_1-injective and of genus g'. It is a result of Sacks-Uhlenbeck [SU] and Schoen-Yau [SY79] that S is homotopic to an immersed minimal surface F.

Bounded diameter lemma for minimal surfaces. [Lac, Sou] *If F is a minimal surface of genus g in a complete Riemannian manifold M with curvature $\kappa_M \leq \frac{-1}{2}$ and injectivity radius* $\mathrm{inj}(M) \geq \epsilon$ *then one has*

$$\mathrm{diam}(F) \leq \frac{8g}{\epsilon}$$

Here $\mathrm{diam}_M(F)$ *is the diameter of F in M.*

Sketch of the proof. We follow Thurston's proof of the bounded diameter lemma for pleated surfaces. The fact that F is minimal implies that F has extrinsic negative curvature and hence has intrinsic curvature $\kappa_F \leq \frac{-1}{2}$. The Gauß-Bonnet theorem implies:

$$\mathrm{vol}(F) \leq 2\left|\int_F \frac{-1}{2} d\,\mathrm{vol}\right| \leq 2\left|\int_F \kappa_F d\,\mathrm{vol}\right| \leq 4\pi(g-1) \tag{8.1}$$

On the other hand, the monotonicity formula implies that for every $p \in F$ one has

$$\mathrm{vol}(F \cap B_p^M(\epsilon)) \geq \pi\epsilon^2$$

In particular, there are at most $\frac{4}{\epsilon^2}(g-1)$ points in F whose pairwise distance in M is at least 2ϵ. This yields that any two points in F are at most at distance $2\epsilon + \frac{8}{\epsilon}(g-1) \leq \frac{8g}{\epsilon}$ as claimed. □

Continuing with the proof of Theorem 1.7 observe that the bounded diameter lemma and the fact that the surfaces S_n and S'_n are at at least distance $\frac{8g'}{\epsilon} + 1$ implies that the minimal surface F is disjoint of at least one of them. However, the minimal surface F is π_1-injective and a π_1-injective surface in a closed manifold cannot be disjoint of any Heegaard surface. This proves that for all $n \geq n_1$ the manifold N_{f^n} does not contain π_1-injective surfaces of genus g'. □

Compare the above result with the of K. Hartshorn [Ha02] where by using purely topological methods he is able to bound from below the genus of an embedded π_1-injective surface in terms of the splitting.
As mentioned in the introduction we obtain Theorem 1.6 as a consequence of Theorem 1.7:

Theorem 1.6. *For all g' there is $n_{g'}$ such that for every $n \geq n_{g'}$ the following holds: every infinite index subgroup $\Gamma \subset \pi_1(N_{f^n})$ with* $\mathrm{rank}(\Gamma) \leq g'$ *is free.*

Proof. By Kurosh theorem, it suffices to show that for sufficiently large n there is no infinite index subgroup $\Gamma_n \subset \pi_1(N_{f^n})$ of rank g' which does not split as a free product. The manifold $\widetilde{N_{f^n}}/\Gamma_n$ is, by a theorem of Scott [Sco73], homotopy equivalent to a compact irreducible 3-manifold C_n. Moreover, the manifold C_n has non-empty boundary since the index of Γ_n is infinite. Furthermore, the assumption that Γ_n does not split as a free product implies that the boundary ∂C_n is incompressible; in particular, every component of ∂C_n has at least genus 2 since C_n is irreducible and $\pi_1(N_{f^n})$ is Gromov-hyperbolic. We can estimate the Euler-characteristic of ∂C_n as follows:

$$\begin{aligned}\chi(\partial C_n) &= 2\chi(C_n) = 2(1 - b_1(C_n) + b_2(C_n)) \\ &\geq 2(1 - \operatorname{rank}(\Gamma_n)) = 2 - 2g'\end{aligned}$$

This implies that every component of ∂C_n has at most genus g'. We have proved that whenever $\pi_1(N_{f^n})$ contains a freely indecomposable infinite index subgroup of at most rank g', then $\pi_1(N_{f^n})$ also contains a subgroup isomorphic to the fundamental group of a closed surface of at most genus g'. This can only happen finitely many times by Theorem 1.7. □

9. Minimal genus Heegaard splittings

We continue with the same notation as in the preceding section: N_{f^n} is the manifold obtained by gluing the two handlebodies M^+ and M^- of genus $g \geq 2$ along the boundary by f^n where f is a generic pseudo-Anosov map. The decomposition $N_{f^n} = M^+ \cup M^-$ is a genus g Heegaard splitting, to which we will refer as being standard, and hence $g(N_{f^n}) \leq g$ for all n. Here $g(N_{f^n})$ is the Heegaard genus of N_{f^n}, i.e. the minimal possible genus of a Heegaard splitting. On the other hand, the rank of the fundamental group bounds the Heegaard genus from below. Hence, we obtain from Theorem 1.4:

Corollary 1.2. *There is n_f with $g(N_{f^n}) = g$ for all $n \geq n_f$.* □

The remaining of this section is devoted to prove Theorem 1.8.

Theorem 1.8. *There is n_f such that for all $n \geq n_f$ the following holds: every minimal genus Heegaard splitting of N_{f^n} is isotopic to the standard one.*

Recall that a Heegaard splitting $N = U \cup V$ of a closed, orientable and irreducible 3-manifold is *stabilized* if there are two properly embedded essential disks $D_U \subset U, D_V \subset V$ whose boundaries ∂D_U and ∂D_V intersect transversally in a single point. If a Heegaard splitting is stabilized then it is obtained from a splitting with smaller genus by attaching a trivial handle; in particular, a minimal genus Heegaard splitting cannot be stabilized. A Heegaard splitting $N = U \cup V$ is *reducible* if there are two properly embedded essential disks $D_U \subset U, D_V \subset V$ whose boundaries coincide: $\partial D_U = \partial D_V$. Every stabilized Heegaard splitting is reducible. On the other hand, it is theorem of Waldhausen [Wal68] that every reducible Heegaard splitting of an irreducible manifold is stabilized. Since we are interested in minimal genus Heegaard splittings of the manifolds N_{f^n} and these manifolds are (for sufficiently large n) irreducible we can restrict ourselves to consider non-reducible Heegaard splittings. A non-reducible splitting $N = U \cup V$ is *weakly reducible* if there are properly embedded essential disks $D_U \subset U$ and $D_V \subset V$ whose boundaries do not intersect at all. If a Heegaard splitting is neither reducible nor weakly reducible then it is said to be *strongly irreducible*. Casson-Gordon [CG87] observed that if an irreducible manifold admits a weakly reducible Heegaard splitting then it is Haken and in fact they showed that the manifold contains an embedded π_1-injective surface whose genus is smaller than the genus of the Heegaard surface. In particular, we derive from Theorem 1.7:

Corollary 9.1. *There is n_f such that every minimal genus Heegaard splitting of N_{f^n} is strongly irreducible for all $n \geq n_f$.* □

After these preparatory remarks we launch the proof of Theorem 1.8. Seeking a contradiction, assume that there is a subsequence $N_{f^{n_i}}$ which admits minimal genus Heegaard splittings given by surfaces S_{n_i} which are not isotopic to the privileged splitting. The first step is to show that the surfaces S_{n_i} converge to some surface in some geometric limit. The link between the geometry of the manifolds $N_{f^{n_i}}$ and the Heegaard splittings is established by the following result of Pitts-Rubinstein (see [Sou] for a proof):

Existence theorem of minimal surfaces (Pitts-Rubinstein)**.** *Let S be a strongly irreducible Heegaard surface in a Riemannian 3-manifold N. Then there is an embedded minimal surface $X \subset N$ such that S is either isotopic to X or to the boundary of a regular neighborhood $\mathcal{N}(X)$ of X with a single handle added by taking the boundary of a regular neighborhood of a fiber of the bundle $\mathcal{N}(X) \to X$.*

Let X_{n_i} be the minimal surface provided by the existence theorem for minimal surfaces when applied to the surface S_{n_i} and for all i choose a base point $p_{n_i} \in X_{n_i}$ and a geometrically convergent subsequence of $(N_{f^{n_i}}, p_{n_i})$, say the whole sequence, with limit (N_G, p_G). By Theorem 5.2, the geometric limit N_G is either homeomorphic to $S \times \mathbb{R}$ or to a genus g handlebody and in both cases the image of a boundary parallel surface in N_G under the almost isometric maps provided by geometric convergence determines the standard Heegaard splitting of $N_{f^{n_i}}$.

The bounded diameter lemma for minimal surfaces (cf. section 8) implies that there is some D with $\mathrm{diam}(X_i) \leq D$ for all i. Let $U \subset N_G$ be a compact submanifold which contains the ball of radius $2D$ around p_G and such that $N_G \setminus U$ is a product region. For all i large enough let $\kappa_i : U \to N_{f^{n_i}}$ be the almost isometric embeddings provided by the geometric convergence. Then for all but finitely many i, say for all, we have $X_{n_i} \subset \kappa_i(U)$. Moreover the same computation as in (8.1) shows that surfaces X_{n_i}, and hence the surfaces $\kappa_i^{-1}(X_{n_i})$, have area uniformly bounded from above. Moreover, the surfaces X_{n_i} are minimal and hence have vanishing mean curvature. This implies that the maximum of the mean curvature of the surfaces $\kappa_i^{-1}(X_{n_i})$ tends to 0 when i goes to ∞. All this implies that Choi and Schoen's compactness theorem for minimal surfaces applies [CH85] and the sequence $\kappa_i^{-1}(X_{n_i})$ contains a subsequence, say the whole sequence, which converges to some closed embedded minimal surface $X \subset U \subset N_G$. The convergence of the surfaces $\kappa_i^{-1}(X_{n_i})$ to X can be explicitly described (compare with [PR02]) but we will only need the following two simple facts:

- (*) The surfaces $\kappa_i^{-1}(X_{n_i})$ are contained for sufficiently large i in some regular neighborhood $\mathcal{N}(X) \subset U$ of X.
- (**) The surface X has at least genus 2 because N_G is hyperbolic and at most genus g. Moreover, X has genus g if and only if (1) X_{n_i} has genus g and (2) X is isotopic to $\kappa_i^{-1}(X_{n_i})$ within $\mathcal{N}(X)$.

We observe that, up isotopy, we also have that $S_{n_i} \subset \kappa_i(\mathcal{N}(X))$. This implies that the homomorphism

$$(\kappa_i)_* : \pi_1(X) \to \pi_1(N_{f^{n_i}}) \tag{9.1}$$

is surjective. Since $\pi_1(N_{f^{n_i}})$ is not a surface group we have that the homomorphism (9.1) cannot be injective. Let $[\pi_1(X)]$ be the image of $\pi_1(X)$ in $\pi_1(N_G)$.

Assume for the time being that N_G is homeomorphic to $S \times \mathbb{R}$; we can assume without loss of generality that $S \times \{0\} \subset \partial U$. Since the surface X has genus g we obtain that $[\pi_1(X)]$ has at most rank $2g$. On the other hand since the homomorphism (9.1) is not injective we derive from Lemma 6.2 that $[\pi_1(X)]$ has finite index in $\pi_1(N_G)$. But this is only possible if $[\pi_1(X)]$ has at least rank $2g$. This shows that X has genus g and is isotopic, within U, to $S \times \{0\}$. Since X has genus g

we obtain from (**) that X_{n_i} has genus g and $\kappa_i^{-1}(X_{n_i})$ is isotopic to X within $\mathcal{N}(X)$ and hence isotopic to $S \times \{0\}$ within U. Moreover, the fact that both the Heegaard surface S_{n_i} and the Pitts-Rubinstein minimal surface X_{n_i} have genus g implies that both are isotopic. This proves that S_{n_i} is isotopic to $\kappa_i(S \times \{0\})$. Theorem 5.2 shows that S_{n_i} is isotopic in $N_{f^{n_i}}$ to the standard Heegaard surface. This concludes the proof of Theorem 1.8 if N_G is homeomorphic to $S \times \mathbb{R}$.

Assume now that N_G is homeomorphic to a handlebody of genus g. Recall that we have chosen above a compact submanifold $U \subset N_G$ containing X in its interior and such that $N_G \setminus U$ is a product region. In this case U is also a handlebody of genus g and $N_G \setminus U$ is homeomorphic to $S \times \mathbb{R}$. A similar argument as above shows that $\pi_1(X)$ must surject onto $\pi_1(N_G)$ and hence that X has genus g and that for all sufficiently large i the surfaces $\kappa_i(X)$ and S_{n_i} are isotopic in $N_{f^{n_i}}$. Since $\pi_1(X)$ surjects into $\pi_1(N_G)$ the surface X separates the handlebody U into two components; let V be the closure of the component of $U \setminus X$ containing ∂U and $W = U \setminus V$. The surfaces $\kappa_i(X)$ and $\kappa_i(\partial U)$ are Heegaard surfaces of genus g in $N_{f^{n_i}}$ and $\kappa_i(V)$ is the region enclosed by them. This shows that X and ∂U are incompressible i V. Also since $\pi_1(X)$ surjects onto $\pi_1(N_G) = \pi_1(U)$, the manifold V is irreducible and does not contain $\mathbb{Z}^2$ in its fundamental group.

By Theorem 5.2 N_G, and hence U, is identified with M^+ or M^-, say M^+, in a way compatible with the canonical inclusion $M^+ \hookrightarrow N_{f^{n_i}}$ and with the embeddings $\kappa_i : U \longrightarrow N_{f^{n_i}}$; in particular we have canonical identifications between $\partial U \simeq \partial M^+ \simeq S \simeq \partial M^-$. This implies that the manifolds $N_{f^{n_i}} \setminus \kappa_i(W)$ and $V \cup_{f^n} M^-$ are homeomorphic. On the other hand, $N_{f^{n_i}} \setminus \kappa_i(W)$ is homeomorphic to a handlebody because it is the closure of one of the components of $N_{f^{n_i}} \setminus S_i$. In other words, for infinitely many n the manifold $V \cup_{f^n} M^-$ is a handlebody. Corollary 6.1 implies that V is a trivial interval bundle and hence we have that for all sufficiently large i the Heegaard surface S_{n_i} is isotopic to the standard Heegaard surface. This concludes the proof of Theorem 1.8 and concludes this paper.

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HOSSEIN NAMAZI

MATHEMATICS DEPARTMENT, PRINCETON UNIVERSITY

JUAN SOUTO

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO