THE SPINE WHICH WAS NO SPINE

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Abstract. Let $\mathcal{T}_n$ be the Teichmüller space of flat metrics on the n-dimensional torus $\mathbb{T}^n$ and identify $\mathrm{SL}_n\,\mathbb{Z}$ with the corresponding mapping class group. We prove that the subset $\mathcal{Y}$ consisting of those points at which the systoles generate $\pi_1(\mathbb{T}^n)$ is, for $n \geq 5$, not contractible. In particular, $\mathcal{Y}$ is not a $\mathrm{SL}_n\,\mathbb{Z}$-equivariant deformation retract of $\mathcal{T}_n$.

For $n \geq 2$ let $\mathcal{T}_n$ be the Teichmüller space of flat metrics with unit volume on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. To be more precise, $\mathcal{T}_n$ is the set of equivalence classes of unit volume flat metrics on $\mathbb{T}^n$ where two metrics ρ and ρ' are equivalent if there is an orientation preserving diffeomorphism $\phi \in \mathrm{Diff}_+(\mathbb{T}^n)$ homotopic to the identity with $\rho' = \phi^*\rho$. We consider on the Teichmüller space $\mathcal{T}_n$ the topology with respect to which the classes of two flat metrics ρ and ρ' are close if there is a diffeomorphism $\phi \in \mathrm{Diff}_+(\mathbb{T}^n)$ homotopic to the identity such that ρ' and $\phi^*\rho$ are close as tensors.

Every element $A \in \mathrm{SL}_n\,\mathbb{Z}$ induces an orientation preserving diffeomorphism $A \in \mathrm{Diff}_+(\mathbb{T}^n)$ which is said to be *linear*. We obtain thus a right action of $\mathrm{SL}_n\,\mathbb{Z}$ on $\mathcal{T}_n$:

$$\mathcal{T}_n \times \mathrm{SL}_n\,\mathbb{Z} \to \mathcal{T}_n,\ (\rho, A) \mapsto A^*\rho$$

which is properly discontinuous. There exists a finite index subgroup Γ of $\mathrm{SL}_n\,\mathbb{Z}$ which acts freely; in particular, the contractibility of $\mathcal{T}_n$ implies that for any such subgroup Γ, the quotient $\mathcal{T}_n/\Gamma$ is an Eilenberg-MacLane space for Γ.

The systole $\mathrm{syst}(\rho)$ of a point $\rho \in \mathcal{T}_n$ is the length of the shortest homotopically essential geodesic in the flat torus $(\mathbb{T}^n, \rho)$. Let $\mathcal{S}(\rho)$ be the set of homotopy classes of geodesics in $(\mathbb{T}^n, \rho)$ with length $\mathrm{syst}(\rho)$; the elements in $\mathcal{S}(\rho)$ are known as the *systoles* of $(\mathbb{T}^n, \rho)$. Ash [1] proved that the systole function

$$\mathcal{T}_n \to (0, \infty), \quad \rho \mapsto \mathrm{syst}(\rho)$$

is a $\mathrm{SL}_n\,\mathbb{Z}$-equivariant topological Morse function, and so it is not surprising that it can be used to construct a particularly nice $\mathrm{SL}_n\,\mathbb{Z}$-equivariant spine, i.e., deformation retract, of $\mathcal{T}_n$. More precisely, the following result was proved in a different language and much greater generality by Ash [2]:

Theorem 1 (Ash)**.** *The subset $\mathcal{X}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates a finite index subgroup of $\pi_1(\mathbb{T}^n)$ is an $\mathrm{SL}_n\mathbb{Z}$-equivariant spine of $\mathcal{T}_n$.*

From a geometric point of view, that the systoles generate a finite index subgroup of $\pi_1(\mathbb{T}^n)$ seems to be a peculiar condition. This led the authors to wonder whether the subset $\mathcal{Y}$ of $\mathcal{T}_n$ consisting of those points $\rho \in \mathcal{T}_n$ with the property that the systoles generate the full group $\pi_1(\mathbb{T}^n)$ could be a $\mathrm{SL}_n\mathbb{Z}$-equivariant deformation retract as well. For $n = 2$, 3 and 4, this is known, as for these cases the sets $\mathcal{X}$ and $\mathcal{Y}$ coincide [8, 9]. The goal of this note is to show that this fails to be true for $n \geq 5$, although the complex $\mathcal{Y}$ is always a CW-complex of dimension $\frac{n(n-1)}{2}$.

Theorem 2. *For $n \geq 5$, the subset $\mathcal{Y}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates $\pi_1(\mathbb{T}^n)$ is not contractible and hence it is not a $\mathrm{SL}_n\mathbb{Z}$-equivariant spine.*

Observe that Ash's spine $\mathcal{X}$, known as *the well-rounded retract*, is homeomorphic to a CW-complex with the same dimension as the virtual cohomological dimension $\mathrm{vcdim}(\mathrm{SL}_n\mathbb{Z}) = \frac{n(n-1)}{2}$ of $\mathrm{SL}_n\mathbb{Z}$. The complex $\mathcal{Y}$ is also a CW-complex of the correct dimension.

In order to prove Theorem 2, we make use of the well-known identification between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n\mathbb{R}$. We discuss this identification in Section 1. For the convenience of the reader, we also sketch briefly the proof of Theorem 1 in Section 2. Now let Γ be a torsion free finite index subgroup of $\mathrm{SL}_n\mathbb{Z}$. The action of Γ on S_n is free and hence the quotient $M_\Gamma = S_n/\Gamma$ is a manifold. Borel and Serre [5] constructed a compact manifold $\bar{M}_\Gamma$ with boundary $\partial\bar{M}_\Gamma$ whose interior is homeomorphic to M_Γ. In section 3 we briefly describe how to construct non-trivial homology classes in $H_{\frac{n(n-1)}{2}}(M_\Gamma)$ and $H_{n-1}(\bar{M}_\Gamma, \partial\bar{M}_\Gamma)$. These classes are then used in Section 4 to show that whenever Γ is as above and is contained in the kernel of the standard homomorphism $\mathrm{SL}_n\mathbb{Z} \to \mathrm{SL}_n\mathbb{Z}/2\mathbb{Z}$, the inclusion $\mathcal{Y}/\Gamma \to M_\Gamma$ is not surjective on the $\frac{n(n-1)}{2}$-homology; Theorem 2 follows.

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1. Generalities

We begin by fixing some notation that will be used in the sequel. We denote by $\{e_1, \ldots, e_n\}$ and $\langle \cdot, \cdot \rangle$ the standard basis and scalar product on $\mathbb{R}^n$. If v or A are vectors or matrices we let ${}^t v$ and ${}^t A$ denote their transposes. Using this notation $|v| = \sqrt{{}^t v v}$ is the standard euclidean norm on $\mathbb{R}^n$. If $\mathcal{S}$ is a subset of a group then we denote by $\langle \mathcal{S} \rangle$ the subgroup generated by $\mathcal{S}$; for example, $\mathbb{Z}^n = \langle \{e_1, \ldots, e_n\} \rangle$. If $\mathcal{S}$ is a subset of a euclidean vector space, we denote by $\langle \mathcal{S} \rangle_{\mathbb{R}}$ the $\mathbb{R}$-linear subspace generated by $\mathcal{S}$ and by $\langle \mathcal{S} \rangle_{\mathbb{R}}^{\perp}$ its orthogonal complement. We will sometimes use the same symbol to denote both an equivalence class and a representative of the equivalence class. For example, we may use the same notation for an element in $\mathrm{SL}_n \mathbb{R}$, and the corresponding element in the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R}$ or in the even smaller quotient $S_n / \mathrm{SL}_n \mathbb{Z}$. When we do want to distinguish the class of A, we denote it by $[A]$, and we will consistently denote the homology class corresponding to a cycle β by $[\beta]$. All the homology groups considered below will have coefficients in the field $\mathbb{Z}/2\mathbb{Z}$ of two elements.

These platitudes out of the way, we recall briefly the identification between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R}$. If ρ is a flat metric on $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ with unit volume $\mathrm{vol}(\mathbb{T}^n, \rho) = 1$, the universal cover $\mathbb{R}^n$ is a complete flat manifold with respect to the induced metric $\tilde{\rho}$. In particular, there is an orientation preserving isometry

$$\phi : (\mathbb{R}^n, \tilde{\rho}) \to (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$$

The action by deck-transformations of the fundamental group $\pi_1(\mathbb{T}^n)$ on $(\mathbb{R}^n, \tilde{\rho})$ is isometric. Conjugating this action by ϕ we obtain an action of $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$ on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, also by isometries. It follows from a classical result of Bieberbach [10] that the group $\phi \pi_1(\mathbb{T}^n) \phi^{-1}$ is a group of translations of $\mathbb{R}^n$. In other words, the isometry ϕ induces a homomorphism

$$\mathbb{Z}^n \to \mathbb{R}^n, \quad \gamma \mapsto \{x \mapsto (\phi \circ \gamma \circ \phi^{-1})(x)\}$$

with discrete and co-compact image. Any such homomorphism is the restriction to $\mathbb{Z}^n$ of an element in $\mathrm{SL}_n \mathbb{R}$. Different choices for the isometry ϕ yield homomorphisms which differ by post-composition with an orthogonal transformation of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, and hence elements in $\mathrm{SL}_n \mathbb{R}$ which differ by left-multiplication with an element in SO_n. Thus, to every flat metric on $\mathbb{T}^n$ we can associate a well-defined point in the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R}$. Moreover, equivalent flat metrics on $\mathbb{T}^n$ induce the same point in S_n. We have thus a well-defined map

$$\mathcal{T}_n \to S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R} \tag{1.1}$$

The map (1.1) is a homeomorphism. Observe that under the identification (1.1), the action of $\mathrm{SL}_n \mathbb{Z}$ on $\mathcal{T}_n$ corresponds to the action on S_n by right multiplication.

As defined in the introduction, the systole $\mathrm{syst}(\rho)$ of a point $\rho \in \mathcal{T}_n$ is the length of the shortest non-trivial geodesic in $(\mathbb{T}^n, \rho)$ and $\mathcal{S}(\rho)$ is the set of shortest non-trivial geodesics. Under the identification (1.1), for $A \in \mathrm{SL}_n \mathbb{R}$ we have

$$\mathrm{syst}(A) = \min_{v \in \mathbb{Z}^n, v \neq 0} |Av|$$

and

$$\mathcal{S}(A) = \{v \in \mathbb{Z}^n, |Av| = \mathrm{syst}(A)\}$$

In particular, Ash's well rounded spine $\mathcal{X}$ and the complex $\mathcal{Y}$ considered in Theorem 2 are given by:

$$\begin{aligned} \mathcal{X} &= \{\rho \in \mathcal{T}_n | \langle \mathcal{S}(\rho) \rangle \text{ has finite index in } \pi_1(\mathbb{T}^n)\} \\ &= \{A \in S_n | \langle \mathcal{S}(A) \rangle \text{ has finite index in } \mathbb{Z}^n\} \\ \mathcal{Y} &= \{\rho \in \mathcal{T}_n | \langle \mathcal{S}(\rho) \rangle = \pi_1(\mathbb{T}^n)\} \\ &= \{A \in S_n | \langle \mathcal{S}(A) \rangle = \mathbb{Z}^n\} \end{aligned}$$

As was also mentioned in the introduction, Ash [1] proved that the systole function

$$\mathcal{T}_n \to (0, \infty), \quad \rho \mapsto \mathrm{syst}(\rho)$$

is an $\mathrm{SL}_n \mathbb{Z}$-equivariant topological Morse function. Here we will only use that the systole function is proper when considered as a function on $S_n / \mathrm{SL}_n \mathbb{Z}$.

Mahler's compactness theorem. *For every $\epsilon > 0$, the set of those $A \in S_n / \mathrm{SL}_n \mathbb{Z}$ with $\mathrm{syst}(A) \geq \epsilon$ is compact.*

Computations are simpler with matrices than with flat metrics, and so in the sequel we will mainly work in the symmetric space S_n.

2. The well-rounded retract

In this section we discuss briefly the proof of Theorem 1. See [2] for a complete proof of a more general version of this theorem.

Theorem 1 (Ash)**.** *The subset $\mathcal{X}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates a finite index subgroup of $\pi_1(\mathbb{T}^n)$ is an $\mathrm{SL}_n\mathbb{Z}$-equivariant spine of $\mathcal{T}_n$.*

Recall that given $\rho \in \mathcal{T}_n$ we denote by $\langle\mathcal{S}(\rho)\rangle$ the subgroup $\pi_1(\mathbb{T}^n)$ generated by the shortest non-trivial geodesics in $(\mathbb{T}^n, \rho)$. Identifying $\pi_1(\mathbb{T}^n)$ with $\mathbb{Z}^n$ we see that the subgroup $\langle\mathcal{S}(\rho)\rangle$ is a free abelian group with rank in $\{1, \dots, n\}$. Moreover, $\mathrm{rank}\langle\mathcal{S}(\rho)\rangle = n$ if and only if $\langle\mathcal{S}(\rho)\rangle$ has finite index in $\pi_1(\mathbb{T}^n)$. For $k = 1, \dots, n$ consider the set $\mathcal{X}_k$ of those points $\rho \in \mathcal{T}_n$ for which we have $\mathrm{rank}\langle\mathcal{S}(\rho)\rangle \geq k$. We have thus the following chain of nested $\mathrm{SL}_n\mathbb{Z}$-invariant subspaces:

$$\mathcal{X} = \mathcal{X}_n \subset \mathcal{X}_{n-1} \subset \cdots \subset \mathcal{X}_1 = \mathcal{T}_n$$

In order to prove Theorem 1 it suffices to show that for $k = 1, \dots, n-1$ the space $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n\mathbb{Z}$-equivariant spine of $\mathcal{X}_k$. In order to see that this is the case we use freely the identification (1.1) discussed above between the Teichmüller space $\mathcal{T}_n$ and the symmetric space $S_n = \mathrm{SO}_n \setminus \mathrm{SL}_n\mathbb{R}$.

Under this identification, a point $A \in S_n$ belongs to $\mathcal{X}_k \setminus \mathcal{X}_{k+1}$ if and only if the set $\mathcal{S}(A)$ generates a rank k subgroup of $\mathbb{Z}^n$. Equivalently, $\mathcal{S}(A)$ generates a k-dimensional $\mathbb{R}$-linear subspace $\langle\mathcal{S}(A)\rangle_\mathbb{R}$ of $\mathbb{R}^n$. Given $A \in \mathcal{X}_k$ and $\lambda \in \mathbb{R}$, consider the one-parameter family of linear maps

$$T_A^\lambda \in \mathrm{SL}_n\mathbb{R}, \quad T_A^\lambda(v) = \begin{cases} e^{(n-k)\lambda}v & \text{for } v \in A\langle\mathcal{S}(A)\rangle_\mathbb{R} \\ e^{-k\lambda}v & \text{for } v \in (A\langle\mathcal{S}(A)\rangle_\mathbb{R})^\perp \end{cases} \tag{2.1}$$

where $(A\langle\mathcal{S}(A)\rangle_\mathbb{R})^\perp$ is the orthogonal complement in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ of the image under A of $\langle\mathcal{S}(A)\rangle_\mathbb{R}$.

Now $T_A^0 A = A$, and if $A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1}$, there is some λ positive with $T_A^\lambda A \in \mathcal{X}_{k+1}$. For $A \in \mathcal{X}_k$ let $\tau(A) \geq 0$ be maximal such that

$$T_A^\lambda A \in \mathcal{X}_k \setminus \mathcal{X}_{k+1} \quad \text{for all } \lambda \in (0, \tau(A))$$

By definition $\tau(A) = 0$ for $A \in \mathcal{X}_{k+1}$. The function $A \mapsto \tau(A)$ is continuous on $\mathcal{X}_k$, which implies that

$$[0,1] \times \mathcal{X}_k \to \mathcal{X}_k, \quad (t, A) \mapsto T_A^{t\tau(A)}A \tag{2.2}$$

is continuous as well. By definition, this homotopy is $\mathrm{SL}_n\mathbb{Z}$-equivariant, starts with the identity, and ends with a projection of $\mathcal{X}_k$ to $\mathcal{X}_{k+1}$. This proves that $\mathcal{X}_{k+1}$ is an $\mathrm{SL}_n\mathbb{Z}$-equivariant spine of $\mathcal{X}_k$ for $k = 1, \dots, n-1$, concluding the sketch of the proof of Theorem 1.

Remark. Something must be done to verify the continuity of (2.2) as the map

$$\mathbb{R} \times \mathcal{X}_k \to \mathrm{SL}_n\mathbb{R}, \quad (\lambda, A) \mapsto T_A^\lambda A$$

itself is not continuous. The key point is that this map is continuous on $\mathbb{R} \times (\mathcal{X}_k \setminus \mathcal{X}_{k+1})$, and by definition $\tau(A) = 0$ for $A \in \mathcal{X}_{k+1}$.

We conclude this section with a couple of additional remarks about the structure of the well-rounded retract $\mathcal{X}$ and a computation of the virtual cohomological dimension of $\mathrm{SL}_n \mathbb{Z}$.

It is not difficult to prove that $\mathcal{X}_k$ is a co-dimension $k-1$ semi-algebraic set given by a locally finite collection of inequalities and quadratic algebraic equations. Hence $\mathcal{X}$ is homeomorphic to a CW-complex of dimension

$$\dim(\mathcal{X}) = \dim S_n - (n-1) = \frac{n(n-1)}{2}$$

It is also easy to see that the well-rounded retract $\mathcal{X}$ is cocompact, although $\mathcal{X}_k$ is not cocompact for $k < n$.

The symmetric space S_n is contractible, hence so is $\mathcal{X}$. In particular, if Γ is a subgroup of $\mathrm{SL}_n \mathbb{Z}$ which acts freely on S_n, then $\mathcal{X}/\Gamma$ is an Eilenberg-MacLane space for Γ, giving us the following upper bound on its cohomological dimension:

$$\mathrm{cdim}(\Gamma) \leq \dim(X) = \frac{n(n-1)}{2}$$

The group $\mathrm{SL}_n \mathbb{Z}$ contains subgroups Γ of finite index which are torsion free and thus act freely on S_n. This yields the upper bound

$$\mathrm{vcdim}(\mathrm{SL}_n \mathbb{Z}) \leq \frac{n(n-1)}{2}$$

for the virtual cohomological dimension of $\mathrm{SL}_n \mathbb{Z}$. One can see the upper bound is sharp as follows: Let N be the $\frac{n(n-1)}{2}$-dimensional subgroup of $\mathrm{SL}_n \mathbb{R}$ consisting of upper triangular matrices with units in the diagonal. The intersection $N \cap \mathrm{SL}_n \mathbb{Z}$ is a cocompact subgroup of N; hence for Γ as above $N/(N \cap \Gamma)$ is a closed manifold of dimension $\frac{n(n-1)}{2}$. The group N is contractible, hence $N/(N \cap \Gamma)$ is an Eilenberg-MacLane space for $N \cap \Gamma$. Thus we have

$$\mathrm{cdim}(\Gamma) \geq \mathrm{cdim}(N \cap \Gamma) = \dim(N/(N \cap \Gamma)) = \frac{n(n-1)}{2}$$

This implies that $\mathrm{vcdim}(\mathrm{SL}_n \mathbb{Z}) = \frac{n(n-1)}{2}$.

In the next section we will give an elementary argument to prove that the homology class $[N/(N \cap \Gamma)] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ is non-trivial.

3. Some topology

As mentioned some lines above, $\mathrm{SL}_n\,\mathbb{Z}$ contains a torsion free subgroup of finite index, and any such subgroup acts not only discretely, but also freely on S_n; hence the quotient $M_\Gamma = S_n/\Gamma$ is a manifold. Borel and Serre [5] proved that M_Γ is homeomorphic to the interior of a compact manifold $\bar{M}_\Gamma$ with boundary $\partial\bar{M}_\Gamma$. Identifying $\bar{M}_\Gamma$ with the complement of an open regular neighborhood of $\partial\bar{M}_\Gamma$ we consider from now on the former as a submanifold of M_Γ and choose a map

$$p : M_\Gamma \to \bar{M}_\Gamma \tag{3.1}$$

whose restriction to $\bar{M}_\Gamma$ is the identity.

Remark. Grayson [7] gave a construction of $\bar{M}_\Gamma$ directly as a submanifold of M_Γ, giving a new proof of some of Borel's and Serre's results. If we are only interested in constructing a compactification $\bar{M}_\Gamma$ as above, we can do the following: For $A \in \mathrm{SL}_n\,\mathbb{R}$ the series $\sum_{v\in\mathbb{Z}^n} e^{-|Av|}$ converges, and its value depends only on the class of A in S_n. In particular, the function

$$F : S_n \to \mathbb{R}, \quad F(A) = \sum_{v\in\mathbb{Z}^n} e^{-|Av|}$$

is well-defined, smooth, and descends to a function $f : M_\Gamma \to \mathbb{R}$. The function f is proper, and there is some constant L which bounds above the critical values of f. This implies that $f^{-1}[L,\infty)$ is a product, hence we can set $\bar{M}_\Gamma = f^{-1}[0,L]$.

Borel and Serre constructed the compactification $\bar{M}_\Gamma$ to study homological properties of Γ. We will only need some basic facts, well-known probably to experts and non-experts alike, which we deduce in an elementary way.

Recall that we always consider homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. By Lefschetz duality there is a non-degenerate pairing

$$\iota : H_{\frac{n(n-1)}{2}}(M_\Gamma) \times H_{n-1}(\bar{M}_\Gamma, \partial\bar{M}_\Gamma) \to \mathbb{Z}/2\mathbb{Z}$$

which can be computed as follows. Given homology classes $[\alpha] \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ and $[\beta] \in H_{n-1}(\bar{M}_\Gamma, \partial\bar{M}_\Gamma)$, represent them by cycles α and β in general position. Then $\iota([\alpha],[\beta])$ is just the parity of the cardinality of the set $\alpha\cap\beta$.

Remark. This is the simplest version of the Alexander-Whitney product in homology, which dualizes the cup product.

In particular, in order to prove that the $\frac{n(n-1)}{2}$-cycle $\alpha = N/(N\cap\Gamma)$ represents a non-trivial homology class it suffices to find a cycle $\beta \in$

$C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ which intersects α transversally at a single point. In order to find such a cycle β we consider the subgroup Δ of $\mathrm{SL}_n \mathbb{R}$ consisting of diagonal matrices with positive entries and the map $\Delta \to M_\Gamma$ which maps every $H \in \Delta$ to its class in $M_\Gamma = \mathrm{SO}_n \setminus \mathrm{SL}_n \mathbb{R}/\Gamma$. By Mahler's compactness theorem, the systole function is proper on $S_n/\mathrm{SL}_n \mathbb{Z}$; since Γ has finite index in $\mathrm{SL}_n \mathbb{Z}$ it is also proper on M_Γ. Then the following lemma implies that the map $\Delta \to M_\Gamma$ is proper as well.

Lemma 1. *Let $H \in \Delta$ be a diagonal matrix with positive entries. Then* $\mathrm{syst}(H)$ *is the minimum of the entries in the diagonal of H. In particular* $\mathrm{syst}(H) \leq 1$*, with equality if and only if* $H = \mathrm{Id}$.

Proof. Let $a_1, \dots, a_n$ be the diagonal entries of H, and for the sake of concreteness assume that a_1 is minimal. Then for $v = {}^t(v_1, \dots, v_n) \in \mathbb{Z}^n$ with, say, $v_i \neq 0$, we have

$$|Av| = \sqrt{a_1^2 v_1^2 + \cdots + a_n^2 v_n^2} \geq |a_i v_i| \geq a_i \geq a_1$$

with equality if, for example, $v_1 = 1$ and $v_2 = \cdots = v_n = 0$. This proves the first claim of the lemma. The second claim follows from the fact that $a_1 \dots a_n = 1$ so that either some a_i is less than 1 or all of the a_i's are equal to 1. □

Composing the proper map $\Delta \to M_\Gamma$ with the projection (3.1) we obtain a cycle β in $C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$. We denote by $[\Delta] = [\beta]$ the homology class of β.

Lemma 2. *Let $A \in N$ be an upper triangular matrix with* 1 *at the diagonal. Then* $\mathrm{syst}(A) = 1$.

Proof. Given $v = {}^t(v_1, \dots, v_n) \in \mathbb{Z}^n$, let i be minimal such that $v_j = 0$ for all $j > i$. Then we have that v_i is the i-th coordinate of Av and hence $|Av| \geq |v_i| \geq 1$, with equality when, for example, $v_1 = 1$ and $v_2 = \cdots = v_n = 0$. □

The intersection points of the cycles $\alpha = N/(N \cap \Gamma)$ and β in M_Γ correspond bijectively to the set of those $H \in \Delta$ for which there is $A \in \Gamma$ with $HA \in N$. For any such H we have by Lemma 2

$$1 = \mathrm{syst}(HA) = \mathrm{syst}(H)$$

and hence $H = \mathrm{Id}$; thus α and β intersect at a single point. Moreover, their intersection is locally modeled by the intersection of the images of Δ and N in S_n and hence it is transversal; therefore $\iota([\alpha], [\beta]) = 1$. This implies that $[\alpha] = [N/(N \cap \Gamma)]$ and $[\beta] = [\Delta]$ are not homologically trivial.

Lemma 3. *If Γ is a torsion-free subgroup of $\mathrm{SL}_n\mathbb{Z}$ then the classes $[N/N\cap\Gamma]\in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ and $[\Delta]\in H_{n-1}(\bar{M}_\Gamma,\partial\bar{M}_\Gamma)$ have intersection*

$$\iota([N/N\cap\Gamma],[\Delta])=1$$

and hence are not trivial. □

4. Proof of Theorem 2

Taking into account the title of this section, it can hardly be surprising that we now prove:

Theorem 2. *For $n\geq 5$, the subset $\mathcal{Y}$ of $\mathcal{T}_n$ consisting of those points ρ with the property that $\mathcal{S}(\rho)$ generates $\pi_1(\mathbb{T}^n)$ is not contractible and hence it is not a $\mathrm{SL}_n\mathbb{Z}$-equivariant spine.*

Let all the notation be as in the previous section. As mentioned in the introduction, in order to prove Theorem 2 we will show that there is a finite index torsion free subgroup $\Gamma\subset\mathrm{SL}_n\mathbb{Z}$ for which the map

$$H_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma)\to H_{\frac{n(n-1)}{2}}(M_\Gamma) \tag{4.1}$$

is not surjective. More precisely, we will show that this is the case for those torsion-free finite index subgroups Γ contained in the kernel of the homomorphism

$$\mathrm{SL}_n\mathbb{Z}\to\mathrm{SL}_n\mathbb{Z}/2\mathbb{Z} \tag{4.2}$$

Fix such a Γ and let $A\in\mathrm{SL}_n\mathbb{R}$ be the upper triangular matrix which, up a factor, is the identity on the upper left $(n-1)\times(n-1)$ quadrant and with entries equal to $\frac{1}{2}$ in the last column

$$A=2^{-\frac{1}{n}}\begin{pmatrix}1&0&\dots&0&\frac{1}{2}\\0&1&\dots&0&\frac{1}{2}\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\dots&1&\frac{1}{2}\\0&0&\dots&0&\frac{1}{2}\end{pmatrix} \tag{4.3}$$

The assumption that Γ is contained in the kernel of (4.2) implies that every element $B\in\Gamma$ can be written as $B=\mathrm{Id}+B'$ where every entry of B' is even. In particular, we have for any such B that ABA^{-1} has integer entries and hence that

$$A\Gamma A^{-1}\subset\mathrm{SL}_n\mathbb{Z}$$

Observe that we have a diffeomorphism $\mathcal{A} : M_{A\Gamma A^{-1}} \to M_\Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} S_n & \xrightarrow{\{[B]\mapsto[BA]\}} & S_n \\ \downarrow & & \downarrow \\ M_{A\Gamma A^{-1}} & \xrightarrow{\mathcal{A}} & M_\Gamma \end{array}$$

The diffeomorphism $\mathcal{A}$ maps the non-trivial, by Lemma 3, homology classes

$$[N/(N\cap(A\Gamma A^{-1}))] \in H_{\frac{n(n-1)}{2}}(M_{A\Gamma A^{-1}}), [\Delta] \in H_{n-1}(\bar{M}_{A\Gamma A^{-1}}, \partial\bar{M}_{A\Gamma A^{-1}})$$

to, a fortiori, non-trivial classes with

$$\iota(\mathcal{A}_*[\Delta], \mathcal{A}_*([N/(N\cap(A\Gamma A^{-1}))])) = 1$$

Observe that the class $\mathcal{A}_*[\Delta] \in H_{n-1}(\bar{M}_\Gamma, \partial\bar{M}_\Gamma)$ is represented by a cycle supported in $\{HA|H \in \Delta\} \cap \bar{M}_\Gamma$. Below we will prove

Lemma 4. *Assume that $n \geq 5$, that A is the matrix given in* (4.3) *and that $H \in \Delta$ is a diagonal matrix. Then we have:*

- *$A \in \mathcal{X} \setminus \mathcal{Y}$, and*
- *$HA \in \mathcal{X}$ if and only if $H = \mathrm{Id}$.*

Lemma 4 implies that the homologically non-trivial class $\mathcal{A}_*[\Delta]$ is supported by a cycle which does not intersect $\mathcal{Y}/\Gamma$. This implies then that the class $\mathcal{A}_*([N/(N\cap(A\Gamma A^{-1}))]) \in H_{\frac{n(n-1)}{2}}(M_\Gamma)$ is not represented by any cycle in $C_{\frac{n(n-1)}{2}}(\mathcal{Y}/\Gamma)$. In particular, we deduce that as claimed (4.1) is not surjective. We can now conclude the proof of Theorem 2. If $\mathcal{Y}$ were contractible, then $\mathcal{Y}/\Gamma$ would be an Eilenberg-MacLane space for Γ and the inclusion $\mathcal{Y}/\Gamma \hookrightarrow S_n/\Gamma = M_\Gamma$ a homotopy equivalence, contradicting the lack of surjectivity of (4.1).

It just remains to prove Lemma 4:

Proof of Lemma 4. We start proving that $A \in \mathcal{X} \setminus \mathcal{Y}$. For every vector $v = {}^t(v_1, \ldots, v_n) \in \mathbb{Z}^n$ we have that

$${}^t(Av) = 2^{-\frac{1}{n}}\left(v_1 + \frac{v_n}{2}, \ldots, v_{n-1} + \frac{v_n}{2}, \frac{v_n}{2}\right)$$

If v_n is odd, then $|Av| \geq \frac{\sqrt{n}}{2}2^{-\frac{1}{n}}$. On the other hand, if v_n is even every vector has at least length $2^{-\frac{1}{n}}$ with, for example, equality for e_1. This proves that $\mathrm{syst}(A) = 2^{-\frac{1}{n}}$ and one can easily see that $\mathcal{S}(A)$ consists of the following $2n$ vectors in $\mathbb{Z}^n$

$$\pm e_1, \dots, \pm e_{n-1}, \pm(2e_n - \sum_{i=1}^{n-1} e_i)$$

This implies that $\mathcal{S}(A)$ generates the subgroup of $\mathbb{Z}^n$ consisting of vectors whose last coordinate is even. This is a proper subgroup with index 2, hence $A \notin \mathcal{Y}$ but $A \in \mathcal{X}$.

Continuing with the proof of the lemma let $H \in \Delta$ be a diagonal matrix with positive entries $a_1, \dots, a_n$. When we multiply H and A we obtain:

$$HA = 2^{-\frac{1}{n}} \begin{pmatrix} a_1 & 0 & \dots & 0 & \frac{a_1}{2} \\ 0 & a_2 & \dots & 0 & \frac{a_2}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & \frac{a_{n-1}}{2} \\ 0 & 0 & \dots & 0 & \frac{a_n}{2} \end{pmatrix} \tag{4.4}$$

For any such HA and $i = 1, \dots, n-1$ we have $|HAe_i| = 2^{-\frac{1}{n}} a_i$. We also have $|HA(2e_n - \sum_{i=1}^{n-1} e_i)| = 2^{-\frac{1}{n}} a_n$. This shows that

$$\operatorname{syst}(HA) \leq 2^{-\frac{1}{n}} \min\{a_i | i = 1, \dots, n\} \tag{4.5}$$

Assume from now on that HA belongs to the well-rounded retract $\mathcal{X}$ and recall that this means that the set $\mathcal{S}(HA)$ of those $v \in \mathbb{Z}^n$ with $|HAv| = \operatorname{syst}(HA)$ generates a finite index subgroup of $\mathbb{Z}^n$. In particular, there is a shortest vector $v = {}^t(w_1, \dots, w_n) \in \mathcal{S}(HA)$ with $w_n > 0$. For such a v one has

$$\operatorname{syst}(HA) = |HAv| \geq 2^{-\frac{1}{n}} \frac{w_n}{2} a_n$$

We deduce then from (4.5) that w_n is either 1 or 2. We claim that $w_n = 2$. Otherwise one has

$$|HAv| \geq \frac{1}{2} \sqrt{a_1^2 + \dots + a_{n-1}^2 + a_n^2} \geq 2^{-\frac{1}{n}} \frac{\sqrt{n}}{2} \min\{a_i | i = 1, \dots, n\}$$

contradicting (4.5), as $n \geq 5$. Hence there is a shortest vector with last coefficient $w_n = 2$. Among all these vectors, HAv is minimal if and only if $v = 2e_n$; thus $\operatorname{syst}(HA) = 2^{-\frac{1}{n}} a_n$. The assumption that $HA \in \mathcal{X}$ implies that for $i = 1, \dots, n-1$ there is also some vector v' with $|HAv'| = \operatorname{syst}(HA) = 2^{-\frac{1}{n}} a_n$ and whose i-th coefficient w'_i does not vanish. By the discussion above, the last coefficient of v' must vanish and hence the i-th coefficent of HAv is $2^{-\frac{1}{n}} w'_i a_i$. This implies that $a_i = a_n$. We have proved that if $HA \in \mathcal{X}$ then $H = \mathrm{Id}$. $\square$

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