

H_∞ ORIENTATIONS ON BP

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ABSTRACT. We show, at the primes 2 and 3, that no map from MU to BP defining a universal p -typical formal group law on BP is H_∞ . In particular, no such map is E_∞ .

This builds on McClure's work on determining if Quillen's orientation on BP is an H_∞^2 map. By direct computation, we show that the necessary condition he derives for Quillen's orientation to be H_∞^2 fails at the primes 2 and 3. We go on to show that this implies the more general result above.

We also provide a reinterpretation of McClure's conditions in the language of formal group laws.

1. INTRODUCTION

This paper arose out of the authors' attempts to answer the long-standing open question:

Does the Brown-Peterson spectrum admit an E_∞ or H_∞ ring structure?

We offer a partial answer to an intimately related question:

Does the Brown-Peterson spectrum admit an H_∞ ring structure compatible with the H_∞ ring structure on MU ?

At the primes 2 and 3, the answer is no. At these primes, this implies that BP does not admit the structure of a commutative MU -algebra.

When making computations in homotopy theory, it is convenient to work p -locally. Quillen constructed a p -local, idempotent map

$$\varepsilon : MU_{(p)} \rightarrow MU_{(p)}$$

that splits $MU_{(p)}$ into a wedge of suspensions of the Brown-Peterson spectrum BP . More specifically, we have the following:

Theorem 1.1 ([Qui69]). *There is a map $r : MU_{(p)} \rightarrow BP$ classifying a universal p -typical formal group law. This map admits a section s , and the composite sr is the map ε above.*

As a consequence, for many calculations there is no loss of information when replacing $MU_{(p)}$ with BP . This is often desirable given that the Brown-Peterson spectrum has a much smaller coefficient ring and is more amenable to computations. For these reasons one usually makes computations using BP rather than $MU_{(p)}$.

The category of E_∞ ring spectra is equivalent to the subcategory of D -algebras in the category of spectra for a suitable monad D , while the category of H_∞ ring spectra is equivalent to the subcategory of D -algebras in the *homotopy* category of spectra.¹

It follows that every E_∞ ring spectrum becomes an H_∞ ring spectrum after passing to the homotopy category. Elmendorf-Kriz-Mandell-May [EKMM97] show that E_∞ ring spectra are weakly equivalent to commutative S -algebras (*i.e.*, commutative monoids in the category of S -modules). They go on to show that many constructions in algebra can be mimicked in the categories of modules over an S -algebra. This correspondence requires a deep point-set level theory of spectra, but has dramatically improved our understanding of the stable homotopy category.

Although the notion of an H_∞ ring structure is too weak to perform such constructions, Bruner-May-McClure-Steinberger [BMMS86] have shown that a spectrum with an H_∞ ring structure is equivalent to a cohomology theory with a well-behaved theory of power operations in degree 0 and

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¹ This is not the homotopy category of D -algebras, whose underlying objects are E_∞ ring spectra.

this is enough for many applications. They also develop the theory of H_∞^d ring spectra, which have a larger family of power operations.

For the spectrum MU , the underlying geometry of cobordism endows it with a standard E_∞ ring structure [May77]. The associated H_∞ ring structure comes from the H_∞^2 structure constructed by tom Dieck [tD68]. These structures descend to E_∞ and H_∞^2 structures on $MU_{(p)}$.

A map of H_∞ ring spectra is a natural transformation on cohomology functors that respects power operations. Since computations in BP -cohomology rely heavily on the retraction of $MU_{(p)}$ onto BP , a useful H_∞ ring structure should appear in this way.

Since the construction of BP involves maps into and out of MU , it is natural to ask if either map can be made into a map of H_∞ ring spectra. The first case has already been considered by Hu-Kriz-May:

Theorem 1.2 (See [HKM01, 2.11] and [BM04, App. B]). *There are no H_∞ ring maps from BP to $MU_{(p)}$.*²

Our work concerns maps out of MU into BP . Taking [BMMS86] as a starting point, we directly address the question of H_∞^2 ring structure for BP under MU and deduce the non-existence of H_∞ ring structure under MU as a consequence.

Our main result (Theorem 2.2) is that, at the primes 2 and 3, there are no H_∞ ring maps from MU to BP which endow BP with a universal p -typical formal group law. The precise statement and proof are given in Section 2.

In addition to some of our own reductions, this theorem relies on some results of McClure [BMMS86, VIII] and his formula for possible obstructions to H_∞^2 structure. We outline the reductions in Section 2 and give an alternate formula for McClure's obstruction in Section 5. Section 3 reviews E_∞ , H_∞ , and H_∞^2 ring spectra. In Section 4 we give an equivalent formulation of the problem in the language of formal group laws.

Using our formula for McClure's obstructions, we partially compute the obstructions at the primes 2 and 3 in Section 6 and show that Quillen's map is not a map of H_∞ ring spectra. Using this calculation we deduce that there are no maps of H_∞ ring spectra from $MU_{(p)}$ to BP when p equals 2 or 3.

Throughout this paper we will refer to a map of ring spectra $MU \rightarrow E$ as a (complex) orientation on E . For convenience, we will assume all spectra are localized at a prime p . We will also use the usual shorthand $E^* \equiv E^*(*)$ for the E cohomology of a point.

2. MAIN THEOREMS

Definition 2.1. *An orientation $f : MU \rightarrow E$ is called p -universal if for any orientation*

$$g : MU \rightarrow F$$

that defines a p -typical formal group law on F , there is a unique map $h : E \rightarrow F$ up to homotopy such that $g = hf$.

Definition 2.1 implies that all spectra admitting a p -universal orientation are canonically homotopy equivalent under MU . The map r in Theorem 1.1 is the standard example of a p -universal orientation on BP , so any E admitting a p -universal orientation can be thought of as an alternative form of the Brown-Peterson spectrum. The main result is as follows:

Theorem 2.2. *At the primes 2 and 3, there are no p -universal orientations of H_∞ ring spectra from MU to BP .*

Corollary 2.3. *At the primes 2 and 3, there are no p -universal orientations of BP that make it a commutative MU -algebra.*

We also have a reinterpretation of this result in the language of formal group laws.

²Although the cited theorem is stated in more restrictive terms, their proof directly applies to this more general result.

Corollary 2.4 (Theorem 4.18). *At the primes 2 and 3, the formal group law \mathcal{VP} (Proposition 4.17) over $BP^{BC_p}[\chi^{-1}]$ is not p -typical.*

We do not see anything special about the primes 2 and 3 here, and we conjecture that the results above hold for all primes. We organize the proof of Theorem 2.2 into the following steps. The relevant definitions and notation are given in Sections 3 and 5.

Step 1 (Observation 3.8). *If there is an H_∞ p -universal orientation on any spectrum E , then there is an H_∞ ring structure on BP such that Quillen's map r is a map of H_∞ ring spectra.*

Step 2 (Theorem 3.17). *Quillen's orientation on BP is H_∞ if and only if it is H_∞^2 ; that is, if and only if for all n and each $\pi \leq \Sigma_n$, there are compatible power operations $\mathcal{P}_{\pi, BP}$ such that the diagram in Figure 1 commutes naturally in X .*

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{\mathcal{P}_{\pi, MU}} & MU^{2n*}(D_\pi X) \\ r_* \downarrow & & \downarrow r_* \\ BP^{2*}(X) & \xrightarrow{\mathcal{P}_{\pi, BP}} & BP^{2n*}(D_\pi X) \end{array}$$

Figure 1: Naturality for an H_∞^2 Orientation on BP

For the sake of the reader, we outline the remaining sequence of reductions due to McClure [BMMS86, VIII]:

Step 3 ([BMMS86, VIII, p. 287]). *For all n and each $\pi \leq \Sigma_n$ there is at most one map $\mathcal{P}_{\pi, BP}$ such that the diagram in Figure 1 commutes.*

Moreover, if all such diagrams commute then this system of maps defines an H_∞^2 ring structure on BP and r is a map of H_∞^2 ring spectra.

Step 4 ([BMMS86, VIII.7.2]). *The diagram in Figure 1 commutes naturally in X for all n and $\pi \leq \Sigma_n$ if and only if it commutes when $\pi = \Sigma_p$.*

Step 5 ([BMMS86, VIII.7.6]). *The diagram in Figure 1 commutes naturally in X for all n and $\pi \leq \Sigma_n$ if and only if it commutes when $\pi = \Sigma_p$ and $X = \mathbb{C}P^\infty$.*

Step 6 ([BMMS86, VIII.7.3]). *The diagram in Figure 1 commutes when $X = \mathbb{C}P^\infty$ and $\pi = \Sigma_p$ if and only if it commutes when the diagram in Figure 2 commutes.*

$$\begin{array}{ccc} MU^{2*}(\mathbb{C}P^\infty) & \xrightarrow{P_{C_p, MU}} & MU^{2p*}(BC_p \times \mathbb{C}P^\infty) \\ r_* \downarrow & & \downarrow r_* \\ BP^{2*}(\mathbb{C}P^\infty) & \xrightarrow{P_{C_p, BP}} & BP^{2p*}(BC_p \times \mathbb{C}P^\infty) \end{array}$$

Figure 2: Reduction to P_{C_p}

Notation 2.5. *For a formal group law F the p -series $[p]_F$ and the reduced p -series $\langle p \rangle_F$ are defined by the following equation.*

$$\overbrace{x +_F \cdots +_F x}^{p \text{ times}} = [p]_F x = x \cdot \langle p \rangle_F x$$

When clear from the context, we will drop the subscript F .

For a complex oriented cohomology theory E , let q_* denote the projection:

$$(2.6) \quad q_* : E^* BC_p \cong E^* [\xi]/[p]\xi \rightarrow E^* [\xi]/\langle p \rangle \xi.$$

Step 7 ([BMMS86, VIII.7.7]). *The diagram in Figure 2 commutes if and only if the diagram in Figure 3 commutes.*

$$\begin{array}{ccc} MU^{2*}(\ast) & \xrightarrow{q^{P_{C_p, MU}}} & MU^{2p*}(BC_p)/\langle p \rangle \xi \\ \downarrow r_* & & \downarrow r_* \\ BP^{2*}(\ast) & \xrightarrow{q^{P_{C_p, BP}}} & BP^{2p*}(BC_p)/\langle p \rangle \xi \end{array}$$

Figure 3: Reduction to a point

Since all of the rings in Figure 3 are torsion-free, it suffices to check commutativity of this diagram on the rational polynomial generators $[CP^n] \in MU^{-2n}(\ast)$ (see Proposition 4.11.) After noting that a particular Euler class

$$\chi \in BP^{2(p-1)}[\xi]/\langle p \rangle \xi$$

is not a zero divisor, we obtain our final reduction:

Step 8 ([BMMS86, VIII.7.8], Theorem 5.19). *The diagram in Figure 3 commutes if and only if the elements*

$$MC_n(\xi) = r_* q_* \chi^{2n} P_{C_p, MU}[CP^n] \in BP^{-2n(p-2)}[\xi]/\langle p \rangle \xi$$

are 0 when $n \neq p^i - 1$ for some i .

In Theorem 4.18 we give an alternate statement of this result in the language of formal group laws.

A formula for the MC_n was first given by McClure in [BMMS86, VIII.7.8]. In Sections 5.1 and 5.3 we provide the relevant notation and summarize the derivation of this formula and then go on to prove the following formula:

$$MC_n(\xi) = \chi^{2n+1} \sum_{k=0}^n r_* [CP^{n-k}] \cdot \left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} [z^k].$$

Using this formula we give computations in Section 6 that yield the following result which, by the reductions above, proves Theorem 2.2.

Theorem 2.7.

- (1) When $p = 2$, $MC_2(\xi) \neq 0$.
- (2) When $p = 3$, $MC_4(\xi) \neq 0$.

3. E_∞ AND H_∞ RING SPECTRA

Let \mathcal{S} denote the Lewis-May-Steinberger category of coordinate-free spectra and $\mathcal{h}\mathcal{S}$ the stable homotopy category.

A spectrum in this category is indexed by finite dimensional subspaces of some countable inner product space \mathcal{U} . Let π be a subgroup of Σ_n the symmetric group on n letters. The space of linear isometries $\mathcal{L}(\mathcal{U}^n, \mathcal{U})$ is a free contractible Σ_n -space and by restriction a free contractible π -space which we will denote $E\pi$.

For each subgroup π of Σ_n we have an extended power functor on unbased spaces, based spaces, and spectra. For an unbased space Z , a based space W , and a spectrum X , the definitions are

$$\begin{aligned} D_\pi Z &= E\pi \times_\pi Z^{\times n} \\ D_\pi W &= E\pi_+ \wedge_\pi W^{\wedge n} \\ D_\pi X &= E\pi \times_\pi X^{\wedge n}. \end{aligned}$$

where \times is the twisted half-smash product of [LMS86]. The functor from unbased to based spaces given by adjoining a disjoint basepoint relates the extended cartesian power on unbased spaces and the extended smash power on based spaces. For an unbased space Z , we have a homeomorphism of based spaces,

$$D_\pi(Z_+) \cong (D_\pi Z)_+.$$

In this paper we study power operations on unreduced cohomology theories, and therefore focus on unbased rather than based spaces. The extended cartesian power on unbased spaces is related to the extended smash power on spectra by the following: For an unbased space Z

$$(3.1) \quad D_\pi \Sigma_+^\infty(Z) = D_\pi \Sigma^\infty(Z_+) \cong \Sigma^\infty D_\pi(Z_+) \cong \Sigma^\infty(D_\pi Z)_+ = \Sigma_+^\infty D_\pi Z.$$

With Equation 3.1 in mind, we may implicitly apply the functor Σ_+^∞ and will use the notation $D_\pi Z$ to denote either an unbased space or a spectrum, as determined by context.

The extended power construction, D_π is multiplicative, but not additive. For spectra X and Y , $D_{\Sigma_n}(X \vee Y)$ is given by the following formula [BMMS86, II.1.1]:

$$(3.2) \quad D_{\Sigma_n}(X \vee Y) = \bigvee_{i+j=n} D_{\Sigma_i} X \wedge D_{\Sigma_j} Y.$$

Definition 3.3. *Let D be the functor on \mathcal{S} such that*

$$DX = \bigvee_{n \geq 0} D_{\Sigma_n} X.$$

The following result is standard (see [Rez98] for example).

Proposition 3.4. *There are natural transformations*

$$\begin{aligned} \mu : D^2 &\rightarrow D \\ \eta : Id &\rightarrow D \end{aligned}$$

that make D a monad on \mathcal{S} .

Definition 3.5. *The category of E_∞ ring spectra is the category of D -algebras in \mathcal{S} .*

Proposition 3.6. *The monad D on \mathcal{S} descends to a monad \tilde{D} on the stable homotopy category $\mathfrak{h}\mathcal{S}$.*

Proof. In [LMS86] it is shown that this functor preserves homotopy equivalences between cell spectra and takes cellular spectra to cellular spectra. It follows that D has a well-defined functor on the stable homotopy category, modeled by cellular spectra with homotopy classes of maps and that the previous structure maps pass to the stable category. \square

Definition 3.7. *The category of H_∞ ring spectra is the category of \tilde{D} -algebras in $\mathfrak{h}\mathcal{S}$.*

Step 1 now follows from:

Observation 3.8. *Consider the following diagram:*

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ Y & \xrightarrow[\sim]{h} & Z \end{array}$$

If g is a map of \tilde{D} -algebras for some monad \tilde{D} and h is an isomorphism, then h induces a \tilde{D} algebra structure on Y such that f and h are maps of \tilde{D} -algebras.

Proposition 3.9. Let $\Gamma : \mathcal{S} \rightarrow \mathfrak{h}\mathcal{S}$ denote the canonical functor. If X is an E_∞ ring spectrum, then ΓX is an H_∞ ring spectrum.

Remark 3.10. Nearly all known H_∞ ring spectra are in the image of Γ . In [Noe09] the second author shows that the counterexample to the transfer conjecture constructed by Kraines-Lada [KL79] can be used to construct an example of a spectrum with an H_∞ ring structure which does not come from an E_∞ ring structure.

Definition 3.11. Suppose E is an H_∞ ring spectrum, X is a spectrum, and

$$f : X \rightarrow E$$

is a map representing a cohomology class in $E^0(X)$.

Define the π^{th} external cohomology operation

$$\mathcal{P}_{\pi,E} : E^0(X) \rightarrow E^0(D_\pi X)$$

by

$$(X \xrightarrow{f} E) \mapsto (D_\pi X \xrightarrow{D_\pi f} D_\pi E \rightarrow D_{\Sigma_n} E \hookrightarrow DE \xrightarrow{\mu} E).$$

If Y is a space, $Y^{\times n}$ is equipped with the π action induced by the inclusion $\pi \rightarrow \Sigma_n$. If we regard Y as a trivial π -space, then the diagonal map

$$\Delta : Y \rightarrow Y^{\times n}$$

is π -equivariant.

Definition 3.12. Suppose E is an H_∞ ring spectrum, Y is a space. Define the map $\delta : B\pi \times Y \rightarrow D_\pi Y$ to be the following composite:

$$\delta : (B\pi \times Y) \simeq E\pi \times_\pi Y \xrightarrow{E\pi \times \Delta} E\pi \times_\pi Y^n \cong D_\pi Y.$$

Define the π^{th} internal cohomology operation $P_{\pi,E} : E^0(Y) \rightarrow E^0(B\pi \times Y)$ as the composite

$$E^0(Y) \xrightarrow{\mathcal{P}_{\pi,E}} E^0(D_\pi Y) \xrightarrow{\delta^*} E^0(B\pi \times Y).$$

Notation 3.13. We will drop the subscript E from the power operations $\mathcal{P}_{\pi,E}$ and $P_{\pi,E}$, when it is clear from the context.

Definition 3.14 ([BMMS86, I.4.3]). An H_∞^d ring structure on a spectrum E is a compatible family of maps

$$D_{\Sigma_n} \Sigma^{di} E \rightarrow \Sigma^{din} E$$

for all $i \in \mathbb{Z}$.

When $i = 0$, these maps define an H_∞ structure on E . The compatibility conditions are graded analogs of those for an H_∞ ring spectrum, but are more easily understood in terms of the larger class of power operations on such a spectrum.

Let X be a spectrum and Y a space. For each $\pi \leq \Sigma_n$ and for each integer i , an H_∞^d spectrum E is equipped with the following power operations

$$\begin{aligned} \mathcal{P}_{\pi,E} &: E^{di}(X) \rightarrow E^{din}(D_\pi X) \\ P_{\pi,E} &: E^{di}(X) \rightarrow E^{din}(B\pi \times X). \end{aligned}$$

The compatibility conditions for power operations on an H_∞^d ring spectrum are merely the appropriately graded analogues of those for an H_∞ ring spectrum. In particular, when $i = 0$, these are simply the power operations for the underlying H_∞ ring structure on E .

Remark 3.15. Note that the alternate definition of H_∞^d structure given in [BMMS86, II.1.3] is flawed. The argument that an H_∞^d ring structure on E determines an H_∞ ring structure on

$$\bigvee_{i \in \mathbb{Z}} \Sigma^{di} E$$

is correct, but the given argument for the converse is not. We suspect the converse is false.

Maps of H_∞^d ring spectra are those which commute with the structure maps in Definition 3.14. It follows that the category of H_∞^d ring spectra is a subcategory of the category of H_∞ ring spectra.

3.1. The Thom isomorphism and H_∞^2 orientations. Let V_k denote the standard representation of Σ_k on \mathbb{C}^k and recall (e.g. [LMS86, Ch. X]) that

$$(3.16) \quad D_{\Sigma_k} S^{2i} \cong B\Sigma_k^{V_k \otimes \mathbb{C}^i},$$

the Thom spectrum of the tensor product $V_k \otimes \mathbb{C}^i$ of complex vector bundles over $B\Sigma_k$. Since $V_k \otimes \mathbb{C}^i$ is a complex vector bundle, for any complex oriented cohomology theory E , we have a Thom isomorphism

$$E^*(\Sigma^{2ki} B\Sigma_k) \cong E^*(B\Sigma_k^{V_k \otimes \mathbb{C}^i}).$$

Taking $\mu_{i,k}$ to be a map representing the Thom class under this isomorphism we obtain the following diagram:

$$\begin{array}{ccc} S^{2ki} & \xrightarrow{i} & D_{\Sigma_k} S^{2i} \\ & \searrow \Sigma^{2ki} e & \swarrow \mu_{i,k} \\ & & \Sigma^{2ki} E. \end{array}$$

Note that although the Thom classes $\mu_{i,k}$ clearly depend on the cohomology theory E , we will abuse notation and use the same symbol regardless of the cohomology theory.

When $E = MU$, McClure shows [BMMS86, VII] that the $\mu_{i,k}$ combine with the H_∞ structure maps

$$\mu_k : D_{\Sigma_k} MU \rightarrow DMU \xrightarrow{\mu} MU$$

to define an H_∞^2 structure for MU : The structure maps are those given by the top horizontal composite in Figure 4.

$$\begin{array}{ccccccc} D_{\Sigma_k}(\Sigma^{2i} MU) & \longrightarrow & D_{\Sigma_k} S^{2i} \wedge D_{\Sigma_k} MU & \xrightarrow{\mu_{i,k} \wedge \mu_k} & \Sigma^{2ki} MU \wedge MU & \longrightarrow & \Sigma^{2ki} MU \\ D_{\Sigma_k}(f) \downarrow & & D_{\Sigma_k} S^{2i} \wedge f \downarrow & & \downarrow \Sigma^{2ki} f \wedge f & & \downarrow \Sigma^{2ki} f \\ D_{\Sigma_k}(\Sigma^{2i} E) & \longrightarrow & D_{\Sigma_k} S^{2i} \wedge D_{\Sigma_k} E & \xrightarrow{\mu_{i,k} \wedge \mu_k} & \Sigma^{2ki} E \wedge E & \longrightarrow & \Sigma^{2ki} E. \end{array}$$

Figure 4: H_∞^2 orientations

Now consider an H_∞ complex orientation $f : MU \rightarrow E$. Figure 4 is induced by this structure and the left and right squares in this diagram commute for any orientation on E . The center square is the smash product of the following two squares:

$$\begin{array}{ccc} D_{\Sigma_k} S^{2ik} \xrightarrow{\mu_{i,k}} \Sigma^{2ki} MU & & D_{\Sigma_k} MU \xrightarrow{\mu_k} MU \\ \parallel & \searrow \Sigma^{2ki} f & \downarrow f \\ D_{\Sigma_k} S^{2ik} \xrightarrow{\mu_{i,k}} \Sigma^{2ki} E & & D_{\Sigma_k} E \xrightarrow{\mu_k} E \end{array}$$

The left square commutes since f sends MU -Thom classes to E -theory Thom classes. The right square commutes since f is an H_∞ ring map.

It follows that the center square and therefore the entire diagram commutes in Figure 4. Another elementary diagram chase, using the H_∞^2 structure of MU , shows that the bottom horizontal composite defines an H_∞^2 structure on E .

This gives the equivalence of H_∞ orientations and H_∞^2 orientations, since by neglect of structure every H_∞^2 orientation is H_∞ .

Theorem 3.17. *An orientation $MU \rightarrow E$ is H_∞ if and only if it is H_∞^2 .*

4. THE FORMAL GROUP LAW PERSPECTIVE

4.1. Formal group laws. We recall some well-known facts about complex-oriented cohomology theories and formal group laws (see [Ada95, Part II] or [Rav00] for example).

Definition 4.1. A (commutative, 1-dimensional) formal group law F over a commutative ring k is a connected bicommutative, associative, topological Hopf algebra \mathcal{A} with a specified isomorphism $\mathcal{A} \cong k[[x]]$.

By forgetting the grading, a graded Hopf algebra of the above form is a formal group law. For such Hopf algebras the completed tensor product provides the following isomorphism:

$$\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}} \cong k[[x_1, x_2]].$$

Notation 4.2. We will frequently identify a formal group law F with the formal power series:

$$x_1 +_F x_2 = \Delta(x) \in k[[x_1, x_2]].$$

Definition 4.3. Given a ring map $f : k \rightarrow k'$ and a formal group law \mathcal{A} over k , the push-forward of \mathcal{A} along f is the formal group law $\mathcal{A} \otimes_k^f k'$ over k' .

One can formally define a ring L and a formal group law \mathcal{U} over L such that

$$(4.4) \quad \mathcal{R}ing(L, k) \cong \text{Formal group laws over } k$$

$$(4.5) \quad f \rightarrow \mathcal{U} \otimes_L^f k$$

Theorem 4.6 ([Laz55]). The ring L is isomorphic to a polynomial algebra over \mathbb{Z} on infinitely many generators.

Definition 4.7. Given a commutative ring k we formally adjoin q th roots of unity ζ . A formal group law F over k is p -typical, if for all primes $q \neq p$, the formal sum over the q th roots of unity

$$\sum_{\zeta^q=1}^F \zeta x$$

is trivial.

4.2. Connection to complex orientations. Recall that if X is a space and E is a spectrum, the function spectrum

$$E^X = F(\Sigma_+^\infty X, E)$$

defines a cohomology theory satisfying

$$(4.8) \quad E^{X,*}(Y) \cong E^*(X \times Y).$$

Moreover, if E admits the structure of a ring spectrum (or an H_∞ ring spectrum or commutative S algebra respectively) then so does E^X .

Proposition 4.9 ([Lan76, 3.1]). The spectra MU^{BC_p} and BP^{BC_p} are ring spectra satisfying the following natural isomorphisms:

$$MU^{BC_p,*} X \cong MU^*(BC_p) \otimes_{MU_*} MU^*(X)$$

$$BP^{BC_p,*} X \cong BP^*(BC_p) \otimes_{BP_*} BP^*(X).$$

In complex cobordism there is a tautological element x giving an isomorphism

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[[x]],$$

and we fix an element ξ such that

$$MU^*(BC_p) \cong MU^*[[\xi]]/[p]\xi.$$

Hence we have

$$MU^{BC_p,*}(\mathbb{C}P^\infty) \cong MU^*[[\xi, x]]/[p]\xi.$$

An orientation $f : MU \rightarrow E$ fixes generators x and ξ in E -cohomology that define analogous isomorphisms.

The above tautological isomorphism in complex cobordism combined with the multiplication on $\mathbb{C}P^\infty$ classifying a tensor product of line bundles defines a formal group law over MU^* . An orientation $MU \rightarrow E$, induces a map $MU^* \rightarrow E^*$ which defines a formal group law structure (also denoted by E) on $E^*(\mathbb{C}P^\infty)$ by pushing forward the formal group law on MU , or equivalently [Ada95, II.4.6], by fixing the generator $x \in E^*(\mathbb{C}P^\infty)$ above.

Theorem 4.10 ([Qui69]). *The map*

$$L \cong \mathbb{Z}[U_1, U_2, \dots] \rightarrow MU^*$$

classifying the tautological formal group law over MU^ is an isomorphism.*

Rationally, we can describe this isomorphism explicitly in terms of the cobordism classes

$$[\mathbb{C}P^n] \in MU^{-2n}.$$

Proposition 4.11. *There is an algebra isomorphism*

$$MU^* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots].$$

With these choices, the power operation $P_{C_p, MU}$ is given by the following formula [Qui71]:

$$(4.12) \quad P_{C_p, MU}(x) = \prod_{i=0}^{p-1} ([i]\xi +_{MU} x).$$

Of course, after applying an orientation $f : MU \rightarrow E$ we obtain

$$(4.13) \quad f_* P_{C_p, MU}(x) = \prod_{i=0}^{p-1} ([i]\xi +_E x).$$

Considering Equation 4.13 as a power series in x whose coefficients are power series in ξ , we define

$$a_i \equiv a_i(\xi) \in E^{2(p-i-1)}(BC_p) \cong E^{2(p-i-1)}[[\xi]/[p]\xi], \quad \text{for } i \geq 0$$

by the following expansion:

$$(4.14) \quad f_* P_{C_p, MU}(x) = a_0 x + a_1 x^2 + a_2 x^3 + \dots$$

By pulling back along the inclusion

$$S^2 \cong \mathbb{C}P^1 \rightarrow \mathbb{C}P^\infty,$$

and applying the C_p analogue of Equation 3.16 we see that $a_0 x$ is the Euler class of the regular representation of C_p and

$$(4.15) \quad a_0 = \chi,$$

is the Euler class of the *reduced* regular representation of C_p . Equation 5.12 shows that we have ring maps after inverting χ :

Proposition 4.16. *Let X be a topological space and define*

$$\overline{P_{C_p}} : MU^{2*}(X) \rightarrow MU^{BC_p, 2*}(X)[\chi^{-1}]$$

as the map which in degree $2n$ is P_{C_p}/χ^n , then $\overline{P_{C_p}}$ and $r_ \overline{P_{C_p}}$ are maps of graded rings.*

Proof. By the above discussion the result holds when X is a point. Since MU^{BC_p} and BP^{BC_p} are Landweber exact and χ is a non-zero divisor, the proposition follows from the following natural isomorphisms:

$$\begin{aligned} MU^{BC_p, *}[\chi^{-1}](X) &\cong MU^*(X) \otimes_{MU^*} MU^{BC_p, *}[\chi^{-1}] \\ BP^{BC_p, *}[\chi^{-1}](X) &\cong BP^*(X) \otimes_{BP^*} BP^{BC_p, *}[\chi^{-1}]. \end{aligned}$$

□

$$\begin{array}{ccc}
MU^{2*}(\mathbb{C}P^\infty) & \xrightarrow{\overline{P_{C_p, MU}}} & MU^{BC_p, 2*}[\chi^{-1}](\mathbb{C}P^\infty) \\
r_* \downarrow & & \downarrow r_* \\
BP^{2*}(\mathbb{C}P^\infty) & \xrightarrow{\overline{P_{C_p, BP}}} & BP^{BC_p, 2*}[\chi^{-1}](\mathbb{C}P^\infty)
\end{array}$$

Figure 5: A formal group theoretic condition

We now obtain formal group laws on $MU^{BC_p}[\chi^{-1}]$ and $BP^{BC_p}[\chi^{-1}]$ using the previous result and the discussion preceding Theorem 4.10:

Proposition 4.17. *The maps $\overline{P_{C_p, MU}}$ and $r \circ \overline{P_{C_p, MU}}$ in Figure 5 define formal group laws $\mathcal{U}\mathcal{P}$ and $\mathcal{V}\mathcal{P}$ over $MU^{BC_p}[\chi^{-1}]$ and $BP^{BC_p}[\chi^{-1}]$ respectively.*

Theorem 4.18. *The map $r : MU \rightarrow BP$ is a map of H_∞ ring spectra if and only if $\mathcal{V}\mathcal{P}$ is p -typical.*

Proof. Since the map r is a p -universal orientation of BP , there exists a map

$$P : BP \rightarrow BP^{BC_p}[\chi^{-1}].$$

that makes Figure 5 commute if and only if $\mathcal{V}\mathcal{P}$ is p -typical. This happens if and only if the polynomial generators in MU^{-2n} map to zero under $\overline{P_{C_p, MU}}$ when $n \neq p^i - 1$. Since the cobordism classes $[\mathbb{C}P^n]$ are rationally polynomial generators and all rings in sight are torsion-free, we see that $\mathcal{V}\mathcal{P}$ is p -typical if and only if the elements MC_n described in Step 6 map to 0. \square

5. COMPUTING THE OBSTRUCTIONS

Before proving Proposition 5.21 we will need some notation.

5.1. Notation. Throughout this paper, the symbol

$$(5.1) \quad \alpha = (\alpha_0, \alpha_1, \dots)$$

with $\alpha_n = 0$ for $n \gg 0$, will be a multi-index beginning with α_0 .

As the reader will see, it will also be convenient to have notation for multi-indices starting with α_1 , so we let

$$(5.2) \quad \bar{\alpha} = (\alpha_1, \alpha_2, \dots).$$

Given an infinite list of variables a_0, a_1, a_2, \dots , we set

$$(5.3) \quad a^\alpha = a_0^{\alpha_0} a_1^{\alpha_1} \dots \quad \text{and} \quad a^{\bar{\alpha}} = a_1^{\alpha_1} a_2^{\alpha_2} \dots$$

For any integer n we define the modified multinomial coefficient $\mu(n; \bar{\alpha})$ by the formal power series expansion:

$$(5.4) \quad (1 + b_1 + b_2 \dots)^n = \sum_{\bar{\alpha}} \mu(n; \bar{\alpha}) b^{\bar{\alpha}}.$$

We also set:

$$(5.5) \quad |\alpha| = \sum_{i \geq 0} \alpha_i$$

$$(5.6) \quad |\alpha|' = \sum_{i \geq 0} i \alpha_i = |\bar{\alpha}|'.$$

Given a formal power series $S(z)$, let

$$(5.7) \quad S(z)[z^k] = \text{coefficient of } z^k \text{ in } S(z).$$

5.2. Additive and multiplicative operations. Recall that the Landweber-Novikov algebra is the subalgebra of MU^*MU whose elements define additive cohomology operations. This algebra is a free $\mathbb{Z}_{(p)}$ -module on elements

$$(5.8) \quad s_{\alpha_1, \alpha_2, \dots} = s_{\bar{\alpha}}$$

dual to the standard basis

$$(5.9) \quad t_1^{\alpha_1} t_2^{\alpha_2} \cdots = t^{\bar{\alpha}} \in MU_{2|\bar{\alpha}|} MU \cong MU_{2|\bar{\alpha}|} BU.$$

To simplify our formulas we extend the indexing to multi-indices starting with α_0 by setting

$$(5.10) \quad s_{\alpha} \equiv s_{\bar{\alpha}} \in MU^{2|\alpha|} MU.$$

Theorem 5.11 ([Qui71]). *If $x \in MU^{-2q}(X)$ and $m \gg 0$ then*

$$(5.12) \quad \chi^{m+q} P_{C_p} x = \sum_{|\alpha|=m} a^{\alpha} s_{\alpha}(x).$$

Since the right hand side of Equation 5.12 is additive in x and P_{C_p} is always multiplicative, we obtain Proposition 4.16 by inverting χ .

For any complex oriented cohomology theory E ,

$$[i]\xi +_E x \equiv i\xi \pmod{x},$$

which implies

$$(5.13) \quad \chi = a_0 \equiv (p-1)! \xi^{p-1} \pmod{\xi^p}.$$

It follows that inverting χ factors through inverting ξ , so when E is MU or BP , we have:

$$E^{BC_p, *}(X)[\chi^{-1}] \cong E^*(X)[[\xi][\chi^{-1}]/[p]\xi] \cong E^*(X)[[\xi][\chi^{-1}]/\langle p \rangle \xi].$$

Since

$$\langle p \rangle \xi = [p]\xi/\xi \equiv p \pmod{\xi}$$

and $(p-1)!$ is not divisible by p , $q_*\chi$ is not a zero-divisor in $MU^*[[\xi]]/\langle p \rangle \xi$ and similarly for $r_*q_*\chi$. It follows that, when $E = MU$ or BP , the localization map

$$E^*(X)[[\xi]]/\langle p \rangle \xi \rightarrow E^*(X)[[\xi][\chi^{-1}]/\langle p \rangle \xi]$$

is an injection. Applying Proposition 4.16 proves the following:

Proposition 5.14. *The composites*

$$\begin{aligned} q_* P_{C_p} : MU^*(\mathbb{C}P^{\infty}) &\rightarrow MU^{BC_p, *}(\mathbb{C}P^{\infty})/\langle p \rangle \xi \\ r_* q_* P_{C_p} : MU^*(\mathbb{C}P^{\infty}) &\rightarrow BP^{BC_p, *}(\mathbb{C}P^{\infty})/\langle p \rangle \xi \end{aligned}$$

are ring maps.

5.3. Derivation of MC_n . The following result shows that if $x = [\mathbb{C}P^n]$ in Equation 5.12 we can take $m = n$:

Lemma 5.15.

$$(5.16) \quad \chi^{2n} P_{C_p} [\mathbb{C}P^n] = \sum_{|\alpha|=n} a^{\alpha} s_{\alpha} [\mathbb{C}P^n].$$

Proof. By Equation 5.12, for $k \gg 0$ we have:

$$\begin{aligned} \chi^{2n+k} P_{C_p} [\mathbb{C}P^n] &= \sum_{|\alpha|=n+k} a^{\alpha} s_{\alpha} [\mathbb{C}P^n] \\ &= \sum_{\alpha_0=0}^{n+k} \sum_{|\bar{\alpha}|=n+k-\alpha_0} a_0^{\alpha_0} a^{\bar{\alpha}} s_{\bar{\alpha}} [\mathbb{C}P^n] \\ &= \sum_{\alpha_0=0}^{k-1} a_0^{\alpha_0} \sum_{|\bar{\alpha}|=n+k-\alpha_0} a^{\bar{\alpha}} s_{\bar{\alpha}} [\mathbb{C}P^n] + \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\bar{\alpha}|=n+k-\alpha_0} a^{\bar{\alpha}} s_{\bar{\alpha}} [\mathbb{C}P^n] \end{aligned}$$

Since MU^* is concentrated in non-positive degrees,

$$s_{\bar{\alpha}}([\mathbb{C}P^n]) \in MU^{2|\bar{\alpha}'|-2n} = 0$$

when $|\bar{\alpha}'| > n$.

In the first sum of the last equation, $|\bar{\alpha}| > n$. Since

$$|\bar{\alpha}'| = \sum_{i \geq 1} i\alpha_i \geq \sum_{i \geq 1} \alpha_i = |\bar{\alpha}|,$$

all terms in the first sum are trivial.

We are left with

$$\begin{aligned} \chi^{2n+k} P_{C_p}[\mathbb{C}P^n] &= \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\bar{\alpha}|=n+k-\alpha_0} a^{\bar{\alpha}} s_{\bar{\alpha}}[\mathbb{C}P^n] \\ &= a_0^k \sum_{\alpha_0=0}^n a_0^{\alpha_0} \sum_{|\bar{\alpha}|=n-\alpha_0} a^{\bar{\alpha}} s_{\bar{\alpha}}[\mathbb{C}P^n] \\ &= a_0^k \sum_{|\alpha|=n} a^\alpha s_\alpha[\mathbb{C}P^n] \end{aligned}$$

Since $a_0 = \chi$ is not a zero-divisor the lemma follows. \square

Theorem 5.17 ([Ada95, I.8.1]).

$$(5.18) \quad s_\alpha[\mathbb{C}P^n] = \mu(-(n+1); \bar{\alpha})[\mathbb{C}P^{n-|\alpha'|}]$$

We combine Equations 5.16 and 5.18 and obtain:

Theorem 5.19.

$$MC_n(\xi) \equiv r_* q_* \chi^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} \mu(-(n+1); \bar{\alpha}) r_*[\mathbb{C}P^{n-|\alpha'|}] a^\alpha.$$

Remark 5.20. After correcting a couple of typographical errors, this is a simplified version of the formula given in [BMMS86, VIII.7.8].

For $n \neq p^k - 1$, the power series $MC_n(\xi)$ are McClure's obstructions to the existence of H_∞ structure on Quillen's map $r : MU \rightarrow BP$. Note that, if $i+1$ is not a power of p then $r_*[\mathbb{C}P^i] = 0$, so many of the summands on MC_n are zero. For our calculations, we make use of the following alternate expression:

Proposition 5.21. *McClure's formula is equivalent to*

$$MC_n(\xi) = \chi^{2n+1} \sum_{k=0}^n r_*[\mathbb{C}P^{n-k}] \cdot \left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} [z^k].$$

Proof. We rearrange the sum by summing over $|\alpha'| = k$. Now the condition $|\alpha| = n$ is simply a constraint on α_0 .

$$\begin{aligned} MC_n(\xi) &= \sum_{k=0}^n \sum_{\substack{|\alpha'|=k \\ |\alpha|=n}} \mu(-(n+1); \bar{\alpha}) r_*[\mathbb{C}P^{n-|\alpha'|}] a^\alpha \\ &= \sum_{k=0}^n r_*[\mathbb{C}P^{n-k}] \sum_{\substack{|\alpha'|=k \\ |\alpha|=n}} \mu(-(n+1); \bar{\alpha}) a^\alpha. \end{aligned}$$

To simplify the inner sum, we consider the following formal series and use the definition of the modified multinomial coefficients given in Equation 5.4:

$$\begin{aligned}
a_0^{2n+1} \left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} &= a_0^n \left(1 + \frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \dots \right)^{-(n+1)} \\
&= a_0^n \sum_{\bar{\alpha}} \mu(-(n+1); \bar{\alpha}) \left(\frac{a_1}{a_0} z \right)^{\alpha_1} \left(\frac{a_2}{a_0} z^2 \right)^{\alpha_2} \dots \\
&= \sum_{\bar{\alpha}} \mu(-(n+1); \bar{\alpha}) \frac{a_0^n a_1^{\alpha_1} a_2^{\alpha_2} \dots}{a_0^{\alpha_1 + \alpha_2 + \dots}} z^{\alpha_1 + 2\alpha_2 + \dots} \\
&= \sum_{k \geq 0} z^k \left(\sum_{|\bar{\alpha}'| = k} \mu(-(n+1); \bar{\alpha}') \frac{a_0^n a_1^{\alpha_1} a_2^{\alpha_2} \dots}{a_0^{\alpha_1 + \alpha_2 + \dots}} \right)
\end{aligned}$$

Now we consider the coefficients of z^k . For $k \leq n$, the restriction $|\bar{\alpha}'| = k$ implies $|\bar{\alpha}| \leq n$. Hence we may extend to a sum over multi-indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ with $\alpha_0 = n - |\bar{\alpha}|$ which forces $|\alpha| = n$. Thus we have, for $0 \leq k \leq n$,

$$a_0^{2n+1} \left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} [z^k] = \sum_{\substack{|\alpha'| = k \\ |\alpha| = n}} \mu(-(n+1); \bar{\alpha}) a^\alpha.$$

□

5.4. Sparseness. In this section we prove that, at odd primes, many of the MC_n do in fact vanish. We also give a sparseness result for the a_i .

Proposition 5.22. *If $n \not\equiv 0 \pmod{p-1}$ then $MC_n = 0$.*

Proof. The statement is vacuously true at the prime 2, so we will assume p is odd. The summands of the equation in Theorem 5.19 are constant multiplies of $r_*[\mathbb{C}P^i]$ and a^α . The first term is nonzero only in degrees divisible by $2(p-1)$ and it follows from the lemma below that the nonzero a^α are also concentrated in degrees divisible by $2(p-1)$.

Now the left side of the equation in Theorem 5.19 is in degree $2n(p-2)$ and the right hand side is concentrated in degrees divisible by $2(p-1)$. Since 2 and $(p-2)$ are units mod p we see that MC_n can only be non-zero when n is divisible by $p-1$. □

Lemma 5.23. *The elements $a_i \in BP^*(BC_p)$ defined in Equation 4.14 are zero if $i \not\equiv 0 \pmod{p-1}$.*

Proof. Since the lemma is vacuously true for $p=2$, we will assume p is odd.

The action of C_p^\times on C_p induces an action of C_p^\times on BC_p . In $BP^*(BC_p)$, an element $v \in C_p^\times$ acts on $[i]\xi$ by

$$[i]\xi \mapsto [vi]\xi.$$

Since the product

$$\prod_{i=1}^{p-1} ([i]\xi +_{BP} x)$$

is invariant under this action, we see that $a_i \in BP^{2(p-i-1)}(BC_p)^{C_p^\times}$.

The Atiyah-Hirzebruch spectral sequence computing $BP^*(BC_p)$ collapses at the E_2 page, which is of the form $H^*(BC_p, BP^*)$. The group action above induces a group action on this page. Since the edge homomorphism $BP^*(BC_p) \rightarrow H^*(BC_p)$, is an equivariant surjection that restricts to an isomorphism along the 0th row, the associated graded of $BP^*(BC_p)^{C_p^\times}$ is isomorphic to $H^*(BC_p)^{C_p^\times} \otimes BP^* \cong \mathbb{Z}/p[\xi^{p-1}] \otimes BP^*$.

Since this last group is concentrated in degrees divisible by $2(p-1)$, if $a_i \neq 0$ then

$$a_i \in BP^{2(p-1)^*}(BC_p).$$

The congruence

$$\frac{|a_i|}{2} = (p-1-i) \equiv i \equiv 0 \pmod{p-1}$$

implies i is divisible by $p-1$. □

As a result, it is of interest to check $MC_{2(p-1)}$. In this case, one can give the formula more explicitly:

$$(5.24) \quad \begin{aligned} MC_{2(p-1)}(\xi) &= a_0^{2p-4} r_* [\mathbb{C}P^{(p-1)}] \left(-(2p-1)a_0 a_{(p-1)} \right) \\ &\quad + a_0^{2p-4} r_* [\mathbb{C}P^0] \left(-(2p-1)a_0 a_{2(p-1)} + p(2p-1)a_{(p-1)}^2 \right) \end{aligned}$$

Making the simplifications $[\mathbb{C}P^0] = 1$ and $r_*[\mathbb{C}P^{p-1}] = v_1$, we have

$$MC_{2(p-1)}(\xi) = (2p-1)a_0^{2p-4} \left(-v_1 a_0 a_{(p-1)} - a_0 a_{2(p-1)} + p a_{(p-1)}^2 \right)$$

6. CALCULATIONS

In this section, we outline the computation of the MC_n , work through an example at the prime 2, and display results at the primes 2, 3, and 5. We have developed a Mathematica package [JN09a] to automate the calculations, together with a user's guide [JN09b].

6.1. Description of calculation. We are working in $BP^*[[\xi]]/\langle p \rangle \xi$, and we emphasize reduction modulo $\langle p \rangle \xi$ by writing $\equiv \pmod{\langle p \rangle \xi}$ instead of equality. Our calculations have three parameters: the prime, p , the value of n , and a truncation number, k . All of our computations are modulo $(\xi, x)^{k+1}$. If power series $f(\xi)$ and $g(\xi)$ are equal modulo the ideal $(\xi)^{k+1}$, we write

$$f(\xi) = g(\xi) + O(\xi)^{k+1}.$$

It is important to note, because of this choice, that the range of accurate coefficients for the $a_i(\xi)$ decreases as i grows. Each a_i is accurate modulo ξ^{k-i+1} . Using the formula above, and the fact that $a_0 = (p-1)! \cdot \xi^{p-1} + \dots$, we see that $MC_{2(p-1)}$ is accurate modulo ξ^{k-p+2} .

Our package represents power series as coefficient lists, with the length of the list determined by the truncation number, k . Our implementation of multiplication and composition of series as operations on these lists preserves this accuracy, without computing unnecessary terms. We have made other efforts to streamline the computation, but our results are limited by the difficulty of formal group law calculations. Determining the series \exp_{BP} is already a task whose computation time grows quickly with the length of the input. Calculating the a_i is also a high-complexity task, and as a result we do not expect direct computation to be a feasible approach for large primes. We have not been able to work in a large enough range to detect non-zero values of MC_n for primes greater than 3.

6.2. Example calculation, $p = 2$. To give the reader a sense of how these calculations are implemented, we work through the calculation of $MC_2(\xi)$ with the minimum range of coefficients necessary to see that it is non-zero. For this, it is necessary to work modulo $(x, \xi)^8$. The formula for MC_2 is given in Proposition 5.21:

$$MC_2(\xi) = a_0^5 \sum_{k=0}^2 r_* [\mathbb{C}P^{n-k}] \cdot \left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} [z^k].$$

Now one can easily check the formal computation

$$\begin{aligned} \left(\sum_{i \geq 0} a_i z^i \right)^{-1} &= a_0^{-1} - a_1 a_0^{-2} z + (-a_2 a_0^{-2} + a_1^2 a_0^{-3}) z^2 \\ &\quad + O(z)^3 \end{aligned}$$

and hence

$$\begin{aligned} \left(\sum_{i \geq 0} a_i z^i \right)^{-3} &= a_0^{-3} - 3a_1 a_0^{-4} z + (-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5}) z^2 \\ &\quad + O(z)^3. \end{aligned}$$

The image of $[\mathbb{C}P^i] \in MU^{-2i}$ under r_* is given by

$$r_*[\mathbb{C}P^i] = \begin{cases} 0 & \text{if } i \neq p^k - 1 \\ [\mathbb{C}P^i] = p^k \ell_k & \text{if } i = p^k - 1 \end{cases}$$

The elements ℓ_k are rational generators for BP , but it is convenient to work with integral generators. For this example we choose the Hazewinkel generators v_k , but the result is independent of this choice. It will be necessary only to use $v_1 = 2\ell_1$, so we work modulo the ideal $I = (v_2, v_3, \dots)$. Modulo I we have $4\ell_2 = v_1^3$, and this will be the only additional substitution we need to use.

Returning to the calculation, we have

$$[\mathbb{C}P^0] = 1, \quad r_*[\mathbb{C}P^1] = 2\ell_1 = v_1, \quad \text{and } r_*[\mathbb{C}P^2] = 0$$

and so

$$\begin{aligned} MC_2(\xi) &= a_0^5 (-3v_1 a_0^{-4} a_1 + (-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5})) \\ &= 6a_1^2 - 3a_0 a_2 - 3v_1 a_0 a_1. \end{aligned}$$

To continue, we determine $a_0(\xi)$, $a_1(\xi)$, and $a_2(\xi)$. These are defined by the following (4.13 4.14):

$$\begin{aligned} P_{C_p, BP}(x) &= r_* P_{C_p, MU}(x) = \prod_{i=0}^1 ([i]\xi +_{BP} x) = x \cdot \exp(\log(\xi) + \log(x)) \\ &= x \cdot [a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 \\ &\quad + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \\ &\quad + O(x, \xi)^8]. \end{aligned}$$

The logarithm is

$$\log_{BP}(\xi) = \xi + \ell_1 \xi^2 + \ell_2 \xi^4 + O(\xi)^8$$

and hence the exponential is

$$\begin{aligned} \exp_{BP}(\xi) &= \xi - \ell_1 \xi^2 + 2\ell_1^2 \xi^3 + (-5\ell_1^3 - \ell_2) \xi^4 \\ &\quad + (14\ell_1^4 + 6\ell_1 \ell_2) \xi^5 \\ &\quad + (-42\ell_1^5 - 28\ell_1^2 \ell_2) \xi^6 \\ &\quad + (132\ell_1^6 + 120\ell_1^3 \ell_2 + 4\ell_2^2) \xi^7 \\ &\quad + O(\xi)^8. \end{aligned}$$

Using the logarithm and exponential, we give the reduced 2-series:

$$\begin{aligned} \langle 2 \rangle \xi &= \frac{1}{\xi} \exp(2 \log(\xi)) = 2 - 2\ell_1 \xi + 8\ell_1^2 \xi^2 \\ &\quad + (-36\ell_1^3 - 14\ell_2) \xi^3 \\ &\quad + (176\ell_1^4 + 120\ell_1 \ell_2) \xi^4 \\ &\quad + (-912\ell_1^5 - 888\ell_1^2 \ell_2) \xi^5 \\ &\quad + (4928\ell_1^6 + 6240\ell_1^3 \ell_2 + 448\ell_2^2) \xi^6 \\ &\quad + O(\xi)^7 \end{aligned}$$

Substituting the Hazewinkel generators, and working modulo v_2 ,

$$\begin{aligned}\langle 2 \rangle \xi &= 2 - v_1 \xi + 2v_1^2 \xi^2 \\ &\quad - 8v_1^3 \xi^3 \\ &\quad + 26v_1^4 \xi^4 \\ &\quad - 84v_1^5 \xi^5 \\ &\quad + 300v_1^6 \xi^6 \\ &\quad + O(\xi)^7\end{aligned}$$

and

$$\begin{aligned}P_{C_p, BP} &= x \cdot [(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)) \\ &\quad - \ell_1(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^2 \\ &\quad + 2\ell_1^2(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^3 \\ &\quad + (-5\ell_1^3 - \ell_2)(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^4 \\ &\quad + (14\ell_1^4 + 6\ell_1\ell_2)(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^5 \\ &\quad + (-42\ell_1^5 - 28\ell_1^2\ell_2) \cdot \\ &\quad \quad (\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^6 \\ &\quad + (132\ell_1^6 + 120\ell_1^3\ell_2 + 4\ell_2^2) \cdot \\ &\quad \quad (\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8))^7 \\ &\quad + O(x, \xi)^8].\end{aligned}$$

Expanding, and substituting the Hazewinkel generators, we have

$$\begin{aligned}a_0 &= \xi + O(\xi)^8 \\ a_1 &= 1 - v_1 \xi + v_1^2 \xi^2 - 2v_1^3 \xi^3 \\ &\quad + 3v_1^4 \xi^4 - 4v_1^5 \xi^5 \\ &\quad + v_1^6 \xi^6 + O(\xi)^7 \\ &\equiv 1 + v_1 \xi + v_1^4 \xi^4 + v_1^5 \xi^5 + v_1^6 \xi^6 + O(\xi)^7 \pmod{\langle 2 \rangle \xi} \\ a_2 &= v_1^2 \xi - 4v_1^3 \xi^2 + 10v_1^4 \xi^3 - 21v_1^5 \xi^4 \\ &\quad + 43v_1^6 \xi^5 + O(\xi)^6 \\ &\equiv v_1^2 \xi + v_1^5 \xi^4 + O(\xi)^6 \pmod{\langle 2 \rangle \xi}\end{aligned}$$

Substituting into the formula for MC_2 , we have (modulo v_2)

$$\begin{aligned}MC_2(\xi) &= 6a_1^2 - a_0 a_2 - 3v_1 a_0 a_1 \\ &\equiv 6(1 + v_1 \xi + v_1^4 \xi^4 + v_1^5 \xi^5 + v_1^6 \xi^6 + O(\xi)^7)^2 \\ &\quad - 3(\xi + O(\xi)^8)(v_1^2 \xi + v_1^5 \xi^4 + O(\xi)^6) \\ &\quad - 3v_1(\xi + O(\xi)^8)(1 + v_1 \xi + v_1^4 \xi^4 + v_1^5 \xi^5 + v_1^6 \xi^6 + O(\xi)^7) \\ &\quad \pmod{\langle 2 \rangle \xi} \\ &= 6 + 9v_1 \xi + 12v_1^4 \xi^4 + 18v_1^5 \xi^5 + 21v_1^6 \xi^6 + O(\xi)^7 \pmod{\langle 2 \rangle \xi}.\end{aligned}$$

Note that, although a_2 is accurate only modulo ξ^6 , the product a_0a_2 is accurate modulo ξ^7 and hence MC_2 is accurate modulo ξ^7 . Since the lowest-order term is $3 \cdot 2$, we subtract $3 \cdot \langle 2 \rangle \xi$ to give

$$MC_2(\xi) \equiv 12v_1\xi - 6v_1^2\xi^2 + v_1^3\xi^3 - 66v_1^4\xi^4 + 270v_1^5\xi^5 - 879v_1^6\xi^6 + O(\xi)^7 \pmod{\langle 2 \rangle \xi}.$$

Continuing to reduce in this way gives the following:

$$MC_2(\xi) \equiv v_1^6\xi^6 + O(\xi)^7 \pmod{\langle 2 \rangle \xi}.$$

Since the lowest-order term of the right-hand side is non-zero mod 2, the entire expression is non-zero in $BP^*[[\xi]]/\langle 2 \rangle \xi$.

DRAFT

6.3. Results at $p = 2$.

$$\begin{aligned}
\langle 2 \rangle \xi &= 2 - \xi v_1 + 2\xi^2 v_1^2 + \xi^3 (-8v_1^3 - 7v_2) + \xi^4 (26v_1^4 + 30v_1 v_2) \\
&+ \xi^5 (-84v_1^5 - 111v_1^2 v_2) + \xi^6 (300v_1^6 + 502v_1^3 v_2 + 112v_2^2) \\
&+ \xi^7 (-1140v_1^7 - 2299v_1^4 v_2 - 960v_1 v_2^2 - 127v_3) \\
&+ \xi^8 (4334v_1^8 + 9958v_1^5 v_2 + 5414v_1^2 v_2^2 + 766v_1 v_3) \\
&+ \xi^9 (-16692v_1^9 - 43118v_1^6 v_2 - 29579v_1^3 v_2^2 - 2380v_2^3 - 3579v_1^2 v_3) \\
&+ \xi^{10} (65744v_1^{10} + 189976v_1^7 v_2 + 161034v_1^4 v_2^2 + 31012v_1 v_2^3 + 17770v_1^3 v_3 + 5616v_2 v_3) \\
&+ \xi^{11} (-262400v_1^{11} - 837637v_1^8 v_2 - 838452v_1^5 v_2^2 - 240631v_1^2 v_2^3 - 86487v_1^4 v_3 \\
&\quad - 55329v_1 v_2 v_3) \\
&+ \xi^{12} (1056540v_1^{12} + 3685550v_1^9 v_2 + 4232750v_1^6 v_2^2 + 1600786v_1^3 v_2^3 + 58268v_2^4 \\
&\quad + 404198v_1^5 v_3 + 363210v_1^2 v_2 v_3) \\
&+ \xi^{13} (-4292816v_1^{13} - 16254540v_1^{10} v_2 - 21110372v_1^7 v_2^2 - 10071369v_1^4 v_2^3 - 1022466v_1 v_2^4 \\
&\quad - 1864478v_1^6 v_3 - 2193009v_1^3 v_2 v_3 - 212440v_2^2 v_3) \\
&+ O(\xi)^{14}
\end{aligned}$$

$$\begin{aligned}
MC_1(\xi) &\equiv \xi^2 v_1^2 + \xi^3 v_2 + \xi^4 (v_1^4 + v_1 v_2) + \xi^7 (v_1^7 + v_3) + \xi^8 (v_1^8 + v_1 v_3) \\
&+ \xi^9 (v_1^9 + v_1^6 v_2 + v_1^3 v_2^2 + v_2^3 + v_1^2 v_3) + \xi^{10} (v_1^{10} + v_1 v_2^3 + v_1^3 v_3) + \xi^{11} (v_1^5 v_2^2 + v_1 v_2 v_3) \\
&+ \xi^{12} (v_1^{12} + v_1^9 v_2 + v_1^6 v_2^2 + v_1^3 v_2^3 + v_2^4 + v_1^5 v_3) + \xi^{13} v_1^4 v_2^3 \\
&+ O(\xi)^{14} \pmod{\langle 2 \rangle \xi}
\end{aligned}$$

$$\begin{aligned}
MC_2(\xi) &\equiv \xi^6 (v_1^6 + v_2^2) + \xi^7 (v_1^7 + v_3) + \xi^8 (v_1^5 v_2 + v_1 v_3) + \xi^9 v_2^3 + \xi^{10} (v_1^4 v_2^2 + v_1 v_2^3) \\
&+ \xi^{11} (v_1^5 v_2^2 + v_1^2 v_2^3 + v_1^4 v_3) + \xi^{12} (v_1^9 v_2 + v_1^5 v_3) + \xi^{13} (v_1^{13} + v_1^{10} v_2 + v_1^3 v_2 v_3) \\
&+ O(\xi)^{14} \pmod{\langle 2 \rangle \xi}
\end{aligned}$$

$$\begin{aligned}
MC_3(\xi) &\equiv \xi^6 v_1^6 + \xi^7 (v_1^4 v_2 + v_1 v_2^2) + \xi^8 (v_1^8 + v_1^5 v_2 + v_1 v_3) + \xi^{10} (v_1^{10} + v_1^7 v_2 + v_1^4 v_2^2 + v_1^3 v_3 + v_2 v_3) \\
&+ \xi^{11} (v_1^{11} + v_1^8 v_2 + v_1^4 v_3 + v_1 v_2 v_3) + \xi^{12} v_1^3 v_2^3 + \xi^{13} (v_1^{13} + v_1^3 v_2 v_3 + v_2^2 v_3) \\
&+ O(\xi)^{14} \pmod{\langle 2 \rangle \xi}
\end{aligned}$$

$$\begin{aligned}
MC_4(\xi) &\equiv \xi^{10} v_1^4 v_2^2 + \xi^{11} (v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2 + v_1^4 v_3) \\
&+ \xi^{12} (v_1^9 v_2 + v_1^3 v_2^3 + v_2^4) + \xi^{13} (v_1^{10} v_2 + v_1^4 v_2^3 + v_1^6 v_3 + v_1^3 v_2 v_3 + v_2^2 v_3) \\
&+ O(\xi)^{14} \pmod{\langle 2 \rangle \xi}
\end{aligned}$$

$$MC_5(\xi) \equiv 0 + O(\xi)^{14} \pmod{\langle 2 \rangle \xi}$$

For $n > 5$, we have not found any non-zero $MC_n \pmod{(\xi^{13}, \langle 2 \rangle \xi)}$.

6.4. Results at $p = 3$.

$$\begin{aligned}
\langle 3 \rangle \xi &= 3 - 8\xi^2 v_1 + 72\xi^4 v_1^2 - 840\xi^6 v_1^3 \\
&\quad + \xi^8 (9000v_1^4 - 6560v_2) + \xi^{10} (-88992v_1^5 + 216504v_1 v_2) \\
&\quad + \xi^{12} (658776v_1^6 - 5360208v_1^2 v_2) + \xi^{14} (1199088v_1^7 + 119105576v_1^3 v_2) \\
&\quad + \xi^{16} (-199267992v_1^8 - 2424100032v_1^4 v_2 + 129120480v_2^2) \\
&\quad + \xi^{18} (5896183992v_1^9 + 45824243688v_1^5 v_2 - 8307203592v_1 v_2^2) \\
&\quad + \xi^{20} (-133449348816v_1^{10} - 807801733088v_1^6 v_2 + 336744805688v_1^2 v_2^2) \\
&\quad + \xi^{22} (2658275605728v_1^{11} + 13162584394728v_1^7 v_2 - 11021856839856v_1^3 v_2^2) \\
&\quad + \xi^{24} (-48579725371464v_1^{12} - 193206868503840v_1^8 v_2 + 314960186505360v_1^4 v_2^2 \\
&\quad \quad - 3670852206240v_2^3) \\
&\quad + O(\xi)^{26} \\
MC_2(\xi) &\equiv v_1^3 \xi^8 + 2v_2 \xi^{10} + (v_1^5 + v_2 v_1) \xi^{12} + 2v_1^2 v_2 \xi^{14} + 2v_1^7 \xi^{16} + (2v_1^8 + v_2^2) \xi^{18} \\
&\quad + (v_2 v_1^5 + v_2^2 v_1) \xi^{20} + (2v_1^{10} + 2v_2 v_1^6 + v_2^2 v_1^2) \xi^{22} + (v_1^{11} + v_2 v_1^7) \xi^{24} \\
&\quad + O(\xi)^{26} \quad \text{mod } \langle 3 \rangle \xi \\
MC_4(\xi) &\equiv 2v_1^9 \xi^{22} + 2v_1^{10} \xi^{24} + O(\xi)^{26} \quad \text{mod } \langle 3 \rangle \xi
\end{aligned}$$

For $n > 4$, we have not found any non-zero $MC_n \text{ mod } (\xi^{26}, \langle 3 \rangle \xi)$.

6.5. Results at $p = 5$.

$$\begin{aligned}
\langle 5 \rangle \xi &= 5 - 624\xi^4 v_1 + 390000\xi^8 v_1^2 - 341094000\xi^{12} v_1^3 + 347012281200\xi^{16} v_1^4 \\
&\quad - 384865568096880\xi^{20} v_1^5 + 439473937694610000\xi^{24} v_1^6 \\
&\quad - 505939291320703500000\xi^{28} v_1^7 + 580757413801495562502000\xi^{32} v_1^8 \\
&\quad + O(\xi)^{40} \\
MC_4(\xi) &\equiv \xi^{32} v_1^5 + 4\xi^{36} v_2 + O(\xi)^{40} \quad \text{mod } \langle 5 \rangle \xi
\end{aligned}$$

For $n > 4$, we have not found any non-zero $MC_n \text{ mod } (\xi^{40}, \langle 5 \rangle \xi)$.

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