DEGREES OF ORDERS ON TORSION-FREE ABELIAN GROUPS

ASHER M. KACH, KAREN LANGE, AND REED SOLOMON

Abstract. We construct two computable presentations of computable torsion-free abelian groups, one of isomorphism type $\mathbb{Q}^\omega$ and one of isomorphism type $\mathbb{Z}^\omega$, having computable orders but not having orders of every Turing degree.

1. Introduction

In this paper, we present two results concerning computability theoretic properties of the spaces of orderings on abelian groups. To motivate these properties, we compare the known results on computational properties of orderings on abelian groups with those for fields. We refer the reader to [6] and [8] for a more complete introduction to ordered abelian groups and to [9] for background on ordered fields.

Definition 1.1. An *ordered abelian group* consists of an abelian group $\mathcal{G} = (G; +, 0)$ and a linear order $\leq_\mathcal{G}$ on G such that $a \leq_\mathcal{G} b$ implies $a + c \leq_\mathcal{G} b + c$ for all $c \in G$. An abelian group $\mathcal{G}$ which admits such an order is *orderable*.

Definition 1.2. The *positive cone* $P(\mathcal{G}; \leq_\mathcal{G})$ of an ordered abelian group $(\mathcal{G}; \leq_\mathcal{G})$ is the set of positive elements

$$P(\mathcal{G}; \leq_\mathcal{G}) := \{g \in G \mid 0_\mathcal{G} \leq_\mathcal{G} g\}.$$

Because $a \leq_\mathcal{G} b$ if and only if $b - a \in P(\mathcal{G}; \leq_\mathcal{G})$, there is an effective one-to-one correspondence between positive cones and orderings. Furthermore, an arbitrary subset $X \subseteq G$ is the positive cone of an ordering on $\mathcal{G}$ if and only if X is a semigroup such that $X \cup X^{-1} = G$ and $X \cap X^{-1} = \{0_\mathcal{G}\}$, where $X^{-1} := \{-g \mid g \in X\}$. We let $\mathbb{X}(\mathcal{G})$ denote the space of all positive cones on $\mathcal{G}$. Notice that the conditions for being a positive cone are Π^0_1.

The definitions for ordered fields are much the same, and we let $\mathbb{X}(\mathcal{F})$ denote the space of all positive cones on the field $\mathcal{F}$. (We will not give the definitions here as the results for fields are only used as motivation.) As in the case of abelian groups, the conditions for a subset of F to be a positive cone are Π^0_1.

Classically, a field $\mathcal{F}$ is orderable if and only if it is formally real, i.e., if $-1_\mathcal{F}$ is not a sum of squares in $\mathcal{F}$; and an abelian group $\mathcal{G}$ is orderable if and only if it is torsion-free, i.e., if $g \in G$ and $g \neq 0_\mathcal{G}$ implies $ng \neq 0_\mathcal{G}$ for all $n \in \mathbb{N}$ with $n > 0$. In both cases, the effective version of the classical result is false: Rabin [12] constructed a computable formally real field that does not admit a computable order, and Downey and Kurtz [4] constructed a computable torsion-free abelian group (in fact, isomorphic to $\mathbb{Z}^\omega$) that does not admit a computable order.

Date: January 12, 2012.
2010 *Mathematics Subject Classification.* Primary: 03D45; Secondary: 06F20.
Key words and phrases. ordered abelian group, degree spectra of orders.

Despite the failure of these classifications in the effective context, we have a good measure of control over the orders on formally real fields and torsion-free abelian groups. Because the conditions specifying the positive cones in both contexts are Π^0_1, the sets $\mathbb{X}(F)$ and $\mathbb{X}(G)$ are closed subsets of 2^F and 2^G respectively, and hence under the subspace topology they form Boolean topological spaces. If $\mathcal{F}$ and $\mathcal{G}$ are computable, then the respective spaces of orders form Π^0_1 classes, and therefore computable formally real fields and computable torsion-free abelian groups admit orders of low Turing degree.

For fields, one can say considerably more. Craven [1] proved that for any Boolean topological space T, there is a formally real field $\mathcal{F}$ such that $\mathbb{X}(\mathcal{F})$ is homeomorphic to T. Translating this result into the effective setting, Metakides and Nerode [11] proved that for any nonempty Π^0_1 class $\mathcal{C}$, there is a computable formally real field $\mathcal{F}$ such that $\mathbb{X}(\mathcal{F})$ is homeomorphic to $\mathcal{C}$ via a Turing degree preserving map. Friedman, Simpson, and Smith [5] proved the corresponding result in reverse mathematics that WKL_0 is equivalent to the statement that every formally real field is orderable.

Most of the corresponding results for abelian groups fail because a countable torsion-free abelian group either has two orderings (if the group has rank one) or has continuum many orderings. Even if one only considers Π^0_1 classes of separating sets (which have size continuum except in trivial cases) and only requires that the map from $\mathbb{X}(\mathcal{F})$ into the Π^0_1 class be degree preserving, one cannot represent all such classes by spaces of orders on computable torsion-free abelian groups. (See Solomon [15] for a precise statement and proof of this result.) However, the connection to Π^0_1 classes is preserved in the context of reverse mathematics as Hatzikiriakou and Simpson [7] proved that WKL_0 is equivalent to the statement that every torsion-free abelian group is orderable.

Because torsion-free abelian groups are a generalization of vector spaces, it is not surprising that notions such as linear independence play a large role in studying these objects.

Definition 1.3. Let $\mathcal{G}$ be a torsion-free abelian group. Elements $g_0, \dots, g_n$ are *linearly independent* (or just *independent*) if for all $c_0, \dots, c_n \in \mathbb{Z}$

$$c_0 g_0 + c_1 g_1 + \cdots + c_n g_n = 0_{\mathcal{G}}$$

implies each $c_i = 0$. An infinite set of elements is *independent* if every finite subset is independent. A maximal independent set is a *basis* and the cardinality of any basis is the *rank* of $\mathcal{G}$.

It is known that if $\mathcal{G}$ is a computable torsion-free abelian group of rank at least two and B is a basis for $\mathcal{G}$, then $\mathcal{G}$ has orders of every Turing degree greater than or equal to the degree of B. (See Solomon [15] and Dabkowska, Dabkowski, Harizanov, and Tonga [2].) Therefore, the set

$$\deg(\mathbb{X}(\mathcal{G})) := \{\mathbf{d} \mid \mathbf{d} = \deg(P) \text{ for some } P \in \mathbb{X}(\mathcal{G})\}$$

contains all the Turing degrees when the rank of $\mathcal{G}$ is finite (but not one) and contains cones of degrees when the rank is infinite. Furthermore, Dobritsa [3] proved that for every computable torsion-free abelian group $\mathcal{G}$, there is a computable $\mathcal{H} \cong \mathcal{G}$ such that $\mathcal{H}$ has a computable basis. Therefore, every computable torsion-free abelian group has a computable copy that has orders of every Turing degree.

These facts bring us to the motivating question for this paper – if $\mathcal{G}$ is a computable torsion-free abelian group, is $\deg(\mathbb{X}(\mathcal{G}))$ necessarily closed upwards in the Turing degrees? We show that the answer to this question is no by constructing computable copies of $\mathbb{Q}^\omega$ and $\mathbb{Z}^\omega$ that have computable orders but not orders of every Turing degree.

Theorem 1.4. *There is a computable presentation $\mathcal{G}$ of the group $\mathbb{Q}^\omega$ and a noncomputable, computably enumerable set C such that:*

- *The group $\mathcal{G}$ has exactly two computable orders.*
- *Every C-computable order on $\mathcal{G}$ is computable.*

Thus, the set of degrees of orders on $\mathcal{G}$ is not closed upwards.

Theorem 1.5. *There is a computable presentation $\mathcal{G}$ of the group $\mathbb{Z}^\omega$ and a noncomputable, computably enumerable set C such that:*

- *The group $\mathcal{G}$ has exactly two computable orders.*
- *Every C-computable order on $\mathcal{G}$ is computable.*

Thus, the set of degrees of orders on $\mathcal{G}$ is not closed upwards.

We demonstrate Theorem 1.4 and Theorem 1.5 in Section 2 and Section 3, respectively. We finish with additional remarks and open questions in Section 4. Our notation is mostly standard. In particular we use the following convention from the study of linear orders: If $\leq_\mathcal{G}$ is a linear order on $\mathcal{G}$, then $\leq_\mathcal{G}^*$ denotes the linear order defined by $x \leq_\mathcal{G}^* y$ if and only if $y \leq_\mathcal{G} x$. Note that if $(\mathcal{G}; \leq_\mathcal{G})$ is an ordered abelian group, then $(\mathcal{G}; \leq_\mathcal{G}^*)$ is also an ordered group.

2. Proof of Theorem 1.4

We divide the proof into several steps. First, we describe our general method of building the computable presentation $\mathcal{G} = (G; +_\mathcal{G})$. Second, we describe how the computable ordering $\leq_\mathcal{G}$ on $\mathcal{G}$ is constructed. (The second computable order on $\mathcal{G}$ is $\leq_\mathcal{G}^*$.) Third, we give the construction of C and the diagonalization process to ensure the only C-computable orders on $\mathcal{G}$ are $\leq_\mathcal{G}$ and $\leq_\mathcal{G}^*$.

Part 1. General Construction of $\mathcal{G}$. The group $\mathcal{G}$ is constructed in stages, with G_s denoting the finite set of elements in G at the end of stage s. We maintain $G_s \subseteq G_{s+1}$ and let $G := \bigcup_s G_s$. We define a partial function $+_s$ on G_s giving the addition facts declared by the end of stage s. For $\mathcal{G}$ to be a computable group, we cannot change any addition fact once it is declared, so we need to maintain

$$x +_s y = z \quad \Longrightarrow \quad (\forall t \geq s)\,[x +_t y = z]$$

for all $x, y, z \in G_s$. Furthermore, for any pair of elements $x, y \in G_s$, there must eventually be a stage t and an element $z \in G_t$ such that we declare $x +_t y = z$. Of course, our declared addition facts must also satisfy the axioms for a torsion-free abelian group.

To define the addition function, we use an approximation $\{b_0^s, b_1^s, \ldots, b_s^s\} \subseteq G_s$ to an initial segment of our eventual basis for G. During the construction, each approximate basis element b_i^s will be redefined at most finitely often (in fact, at most once), so each will eventually reach a limit. We let $b_i := \lim_s b_i^s$ denote this limit. The first approximate basis element b_0^s will never be redefined, so although we often use the notation b_0^s (for uniformity), we have $b_0 = b_0^s$ for all s.

At stage 0, we begin with $G_0 := \{0, 1\}$. We let 0 denote the identity element $0_{\mathcal{G}}$ and we assign 1 the label b_0^0. We declare $0_{\mathcal{G}} +_0 0_{\mathcal{G}} = 0_{\mathcal{G}}$, $0_{\mathcal{G}} +_0 b_0^0 = b_0^0$, and $b_0^0 +_0 0_{\mathcal{G}} = b_0^0$.

More generally, at stage s, each element $g \in G_s$ is assigned a $\mathbb{Q}$-linear sum over the stage s approximate basis of the form

$$q_0^s b_0^s + \cdots + q_n^s b_n^s$$

where $n \leq s$, $q_i^s \in \mathbb{Q}$ for $i \leq n$, and $q_n^s \neq 0$. This assignment is required to be one-to-one, and the identity element $0_{\mathcal{G}}$ is always assigned the empty sum. It will often be convenient to extend such a sum by adding more approximate basis elements on the end of the sum with coefficients of zero. For example, the element $0_{\mathcal{G}}$ can be represented by any sum in which all the coefficients are zero. We trust that using such extensions will not cause confusion.

We define the partial function $+_s$ on G_s by letting $x +_s y = z$ (for $x, y, z \in G_s$) if the sums for x and y add together to form the sum for z. Thus, the function $+_s$ is commutative and associative (on the elements for which it is defined) and satisfies $x +_s 0_{\mathcal{G}} = x$ for all $x \in G_s$.

During stage $s + 1$, we do one of two things – either we leave our approximate basis unchanged or we add a dependency relation for a single b_ℓ^s (for some $\ell \leq s$). The diagonalization process dictates which happens.

(1) If we leave our approximate basis unchanged, then we define $b_i^{s+1} := b_i^s$ for all $i \leq s$. For each $g \in G_s$ (viewed as an element of G_{s+1}), we define $q_i^{s+1} := q_i^s$ and assign g the same sum with b_i^{s+1} and q_i^{s+1} in place of b_i^s and q_i^s, respectively. It follows that $x +_{s+1} y = z$ (for $x, y, z \in G_s$) if $x +_s y = z$.

We add two new elements to G_{s+1}, labeling the first by b_{s+1}^{s+1} and labeling the second by $q_0^{s+1} b_0^{s+1} + \cdots + q_n^{s+1} b_n^{s+1}$, where $\langle q_0^{s+1}, \ldots, q_n^{s+1} \rangle$ is the least tuple of rationals (under some fixed computable enumeration of all tuples of rationals) such that $n \leq s$, $q_n^{s+1} \neq 0$, and this sum is not already assigned to any element of G_{s+1}. This completes the description of G_{s+1} in this case.

(2) If we redefine an approximate basis element b_ℓ^s (for the sake of diagonalizing) by adding a new dependency relation, then we proceed as follows. We define $b_i^{s+1} := b_i^s$ for all $i \leq s$ with $i \neq \ell$. The diagonalization process will tell us either to set $b_\ell^s = q b_0^{s+1}$ for some rational q or to set $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$ for some rational q and some index $j < \ell$. (That is, we make the element b_ℓ^s dependent either on b_0^{s+1} or on b_0^{s+1} and b_j^{s+1}.) For each $g \in G_s$ (viewed as an element of G_{s+1}), we assign g the same sum except we replace each b_i^s by b_i^{s+1} (for $i \leq s$ and $i \neq \ell$) and we replace b_ℓ^s by either $q b_0^{s+1}$ or $b_j^{s+1} + q b_0^{s+1}$ (depending on how we made b_ℓ^s dependent). For example, if the diagonalization process tells us to make $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$, then the sum for $g \in G_s$ changes from

$$g = q_0^s b_0^s + \cdots q_j^s b_j^s + \cdots + q_\ell^s b_\ell^s + \cdots + q_s^s b_s^s$$

at stage s (where we have added zero coefficients if necessary) to

$$\begin{aligned} g &= q_0^s b_0^{s+1} + \cdots + q_j^s b_j^{s+1} + \cdots + q_\ell^s (b_j^{s+1} + q b_0^{s+1}) + \cdots + q_s^s b_s^{s+1} \\ &= (q_0^s + q_\ell^s q) b_0^{s+1} + \cdots + (q_j^s + q_\ell^s) b_j^{s+1} + \cdots + q_s^s b_s^{s+1} \end{aligned}$$

at stage $s+1$. Therefore, we set $q_0^{s+1} := q_0^s + q_\ell^s q$, $q_j^{s+1} := q_j^s + q_\ell^s$, and $q_\ell^{s+1} := 0$, while leaving $q_i^{s+1} := q_i^s$ for all $i \notin \{0, j, \ell\}$.

There are four points to notice about these transformations. First, the approximate basis element b_ℓ^{s+1} (which we have yet to define) does not appear in the new sum for any element of G_s viewed as an element of G_{s+1}. Second, the approximate basis elements b_0^{s+1} and b_j^{s+1} may have nonzero coefficients in the new sum for g even if b_0^s and b_j^s had zero coefficients in the old sum. However, these are the only approximate basis elements that can have coefficients that change from zero to nonzero. Third, we need to make sure that the assignment of sums to elements of G_s (viewed as elements of G_{s+1}) is one-to-one when we choose q. The diagonalization process will place some restrictions on the value of q, but as long as there are infinitely many possible choices for q, we can assume q is chosen to maintain the one-to-one assignment of sums to elements of G_{s+1}. Fourth, by the linearity of this substitution, if $x +_s y = z$, then $x +_{s+1} y = z$.

To complete stage $s+1$, we add three new elements to G_{s+1}. The first is labelled b_ℓ^{s+1}, the second is labelled b_{s+1}^{s+1}, and the third is labelled $q_0^{s+1} b_0^{s+1} + \cdots + q_n^{s+1} b_n^{s+1}$, where $\langle q_0^{s+1}, \ldots, q_n^{s+1} \rangle$ is the least tuple of rationals such that $n \leq s$, $q_n^{s+1} \neq 0$, and this sum is not assigned to any element of G_{s+1}. This completes the description of G_{s+1} in our second case.

It remains to check various properties of this construction. We work under the assumption that each approximate basis element is only redefined finitely often and hence $b_i := \lim_s b_i^s$ exists for all i.

Lemma 2.1. *For each $g \in G$ there is a stage t such that g is assigned a sum $q_0^t b_0^t + \cdots + q_n^t b_n^t$ that is not later changed in the sense that, for all stages $u \geq t$, the element g is assigned the sum $q_0^u b_0^u + \cdots + q_n^u b_n^u$ with $b_i^u = b_i^t$ and $q_i^u = q_i^t$ for all $i \leq n$.*

Proof. When g enters G, it is assigned a sum. The coefficients in this sum only change when a diagonalization occurs. In this case, some approximate basis element b_ℓ^s with nonzero coefficient in the sum for g is made dependent via a relation of the form $b_\ell^s = q b_0^{s+1}$ or $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$ with $j < \ell$. Therefore, each time the sum for g changes, some approximate basis element with nonzero coefficient is replaced by rational multiples of approximate basis elements with lower indices. This process can only occur finitely often before terminating. $\square$

We refer to the sum in Lemma 2.1 as the *limiting sum* for g and denote it by $q_0 b_0 + \cdots + q_n b_n$.

Lemma 2.2. *For each rational tuple $\langle q_0, \ldots, q_n \rangle$ with $q_n \neq 0$, there is an element $g \in G$ such that the limiting sum for g is $q_0 b_0 + \cdots + q_n b_n$.*

Proof. For a contradiction, suppose there is a rational tuple violating this lemma. Fix the least such tuple $\langle q_0, \ldots, q_n \rangle$ in our fixed computable enumeration of rational tuples. Let $s \geq n$ be a stage such that $b_0^s, \ldots, b_n^s$ have reached their limits and each tuple before $\langle q_0, \ldots, q_n \rangle$ has appeared as the limiting sum of an element in G_s. By our construction, at stage $s+1$, either there is an element that is assigned the sum $q_0 b_0^{s+1} + \cdots + q_n b_n^{s+1}$ or else we add a new element to G_{s+1} and assign it this sum. In either case, this element has the appropriate limiting tuple since $b_0^{s+1}, \ldots, b_n^{s+1}$ have reached their limits (and thus we obtain our contradiction). $\square$

Lemma 2.3. *If $x +_s y = z$, then $x +_t y = z$ for all $t \geq s$. In particular, if $x +_s y = z$, then the limiting sums for x and y add to form the limiting sum for z.*

Proof. In both cases of stage $s+1$ of our construction, we checked that $x +_s y = z$ implies $x +_{s+1} y = z$. Thus, the result follows by induction. □

Lemma 2.4. *For each pair $x, y \in G_s$, there is a stage $t \geq s$ and an element $z \in G_t$ such that $x +_t y = z$. For each $x \in G_s$, there is a stage $t \geq s$ and an element $z \in G_t$ such that $x +_t z = 0_{\mathcal{G}}$.*

Proof. For the first statement, fixing $x, y \in G_s$, let $s' \geq s$ be a stage at which x and y have been assigned their limiting sums and let $q_0 b_0^{s'} + \cdots q_n b_n^{s'}$ be the sum of these limiting sums. By Lemma 2.2, let $t \geq s'$ be a stage such that there is a $z \in G_t$ assigned to the sum $q_0 b_0^t + \cdots + q_n b_n^t$. Then $x +_t y = z$.

The proof of the second statement is similar. □

Using Lemma 2.3 and Lemma 2.4, we define the additive function $+_{\mathcal{G}}$ on $\mathcal{G}$ by putting $x + y = z$ if and only if there is a stage s such that $x +_s y = z$.

Lemma 2.5. *The group $\mathcal{G}$ is a computable copy of $\mathbb{Q}^\omega$.*

Proof. The domain and addition function on $\mathcal{G}$ are computable. By Lemma 2.4, every element of $\mathcal{G}$ has an inverse, and it is clear from the construction that the addition operation satisfies the axioms for a torsion-free abelian group.

Notice that $x + y = z$ in $\mathcal{G}$ if and only if the limiting sums for x and y add to form the limiting sum for z. Therefore, the limiting sums for each element give us a Δ^0_2 map from G into the standard computable presentation of the divisible torsion-free abelian group of countably infinite rank. □

Part 2. Defining the Computable Orders on $\mathcal{G}$. We define the computable ordering of $\mathcal{G}$ in stages by specifying a partial binary relation $\leq_s$ on G_s at each stage s. To make the ordering relation computable, it suffices to satisfy

$$x \leq_s y \implies (\forall t \geq s)\,[x \leq_t y] \tag{1}$$

for all $x, y \in G_s$. Typically, the relation $\leq_s$ will not describe the ordering between every pair of elements of G_s, but it will have the property that for every pair of elements $x, y \in G_s$, there is a stage $t \geq s$ at which we declare $x \leq_t y$ or $y \leq_t x$, and not both unless $x = y$. Since we will be considering several orderings on $\mathcal{G}$, for an ordering $\preceq$ on $\mathcal{G}$, we let $(g_1, g_2)_\preceq$ denote the set $\{g \in G \mid g_1 \prec g \prec g_2\}$. Moreover, given $a_1, a_2 \in \mathbb{R}$, we let $(a_1, a_2)_{\leq_\mathbb{R}}$ denote the interval $\{a \in \mathbb{R} \mid a_1 <_\mathbb{R} a <_\mathbb{R} a_2\}$.

To specify the computable order on $\mathcal{G}$, we build a Δ^0_2-map from G into $\mathbb{R}$. (Thus our order will be archimedean.) To describe this order, let $\{p_i\}_{i \geq 1}$ enumerate the prime numbers in increasing order. Our map will send the basis element b_0 to the real 1. For $i \geq 1$, we will assign (in the limit of our construction) a real number r_i to the basis element b_i such that r_i is a rational multiple of $\sqrt{p_i}$. We choose the r_i in this manner simply so that they are algebraically independent over $\mathbb{Q}$. In the construction of $\mathcal{G}$, each element $g \in G$ is assigned a limiting sum

$$g = q_0 b_0 + \cdots + q_n b_n.$$

We define our Δ^0_2-map into $\mathbb{R}$ to send g to the real $q_0 r_0 + \cdots + q_n r_n$ (and to send $0_{\mathcal{G}}$ to 0). Since $r_0 = 1$, the rationals will be in the image of this map.

We need to approximate this Δ^0_2-map during the construction. At each stage s, we keep a real number r_i^s as an approximation to r_i, viewing r_i^s as our current target for the image of b_i. The real r_0^s is always 1 and the real r_i^s is always a rational multiple of $\sqrt{p_i}$. Exactly which rational multiple may change during the course of the diagonalization process.

We could generate a computable order on G_s by mapping G_s into $\mathbb{R}$ using a linear extension of the map sending each b_i^s to r_i^s. However, this would restrict our ability to diagonalize. Therefore, at stage s, we assign each b_i^s (for $i \geq 1$) an interval $(a_i^s, \widehat{a}_i^s)_{\leq_{\mathbb{R}}}$ where $a_i^s, \widehat{a}_i^s \in \mathbb{Q}$, $r_i^s \in (a_i^s, \widehat{a}_i^s)_{\leq_{\mathbb{R}}}$, and $\widehat{a}_i^s - a_i^s \leq 1/2^s$. Because b_0^s is always assigned 1 and $b_0 = b_0^s$ for all s, these intervals give us a range of ordering relations between b_i^s and b_0 in G. That is, we think of them as declaring that in $\mathcal{G}$, the element b_i^s is contained in the interval $(a_i^s b_0, \widehat{a}_i^s b_0)_{\leq_s}$.

Because each $x \in G_s$ is assigned a sum describing its relationship to the current approximate basis, we can generate an interval approximating the image of x in $\mathbb{R}$ under the Δ_2^0-map. That is, suppose x is assigned the sum

$$x = q_0^s b_0^s + \cdots + q_n^s b_n^s$$

at stage s. The interval constraints on the image of each b_i^s in $\mathbb{R}$ translate into a rational interval constraint on the image of x in $\mathbb{R}$. (The endpoints of this constraint can be calculated using the coefficients of the sum for x and the rationals a_i^s and $\widehat{a}_i^s$. The exact form depends on the signs of these numbers.) If $n = 0$, then the image of x in $\mathbb{R}$ is q_0 since $b_0^s = b_0 = 1$.

To define $\leq_s$ on G_s at stage s, we look at the interval constraints for each pair of distinct elements $x, y \in G_s$. If the interval constraint for x is disjoint from the interval constraint for y, then we declare $x \leq_s y$ or $y \leq_s x$ (depending on which inequality is forced by the constraints). If the interval constraints are not disjoint, then we do not declare any ordering relation between x and y at stage s. Of course, we also declare $x \leq_s x$ for each $x \in G_s$.

It remains to see how these constraints interact with the construction of $\mathcal{G}$ and to verify that we obtain the appropriate properties during the construction. In particular, to maintain the implication in (1), we need to check that $x \leq_s y$ implies $x \leq_{s+1} y$. It suffices to ensure that for each $x \in G_s$, the interval constraint for x at stage $s+1$ is contained within the interval constraint for x at stage s.

Recall that at stage $s+1$, there are two possible cases in the construction of G_{s+1}.

(1) Suppose that we leave the approximate basis unchanged. We set $r_0^{s+1} := 1$. For each $1 \leq i \leq s$, we define $r_i^{s+1} := r_i^s$ (so we maintain our guess at the target rational multiple of $\sqrt{p_i}$ for b_i) and define a_i^{s+1} and $\widehat{a}_i^{s+1}$ so that

$$(a_i^{s+1}, \widehat{a}_i^{s+1})_{\leq_{\mathbb{R}}} \subseteq (a_i^s, \widehat{a}_i^s)_{\leq_{\mathbb{R}}}, \quad r_i^{s+1} \in (a_i^{s+1}, \widehat{a}_i^{s+1})_{\leq_{\mathbb{R}}}, \quad \text{and} \quad \widehat{a}_i^{s+1} - a_i^{s+1} < 1/2^{s+1}.$$

Because we nest the constraint intervals for these approximate basis elements, the interval constraints for each $x \in G_s$ at stage $s+1$ are nested within the constraints at stage s. For the approximate basis element b_{s+1}^{s+1} introduced at this stage, we set r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ (as directed by the diagonalization process) and define $a_{s+1}^{s+1}, \widehat{a}_{s+1}^{s+1} \in \mathbb{Q}$ so that $r_{s+1}^{s+1} \in (a_{s+1}^{s+1}, \widehat{a}_{s+1}^{s+1})_{\leq_{\mathbb{R}}}$ and $\widehat{a}_{s+1}^{s+1} - a_{s+1}^{s+1} < 1/2^{s+1}$.

(2) Suppose that we add a dependency for an approximate basis element b_ℓ^s at stage $s+1$ by making $b_\ell^s = q b_0^{s+1}$ or $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$. In this case, our ordering conditions require us to place some restrictions on the possible choices of q. Recall that we can carry out our general group construction as long as there are infinitely many possible choices for q. At this point, we

place some restrictions on q for the sake of the ordering and later we will place additional restraints on the choice of q for the sake of the diagonalization action. In the end, we only need to verify that there remain infinitely many possible choices for q.

For our declared ordering relations from stage s to continue to hold at stage $s+1$, we need that the image of b_ℓ^s at stage $s+1$ remains constrained to lie in $(a_\ell^s, \widehat{a}_\ell^s)_{\leq_\mathbb{R}}$. Therefore, in the case when $b_\ell^s = q b_0^{s+1}$, we need to choose $q \in (a_\ell^s, \widehat{a}_\ell^s)_{\leq_\mathbb{R}}$ since the image of b_0^{s+1} in $\mathbb{R}$ is 1. There are obviously infinitely many choices of q satisfying this condition.

In the case when $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$, we need to choose q, a_j^{s+1}, and $\widehat{a}_j^{s+1}$ so that

$$(q + a_j^{s+1}, q + \widehat{a}_j^{s+1})_{\leq_\mathbb{R}} \subseteq (a_\ell^s, \widehat{a}_\ell^s)_{\leq_\mathbb{R}}.$$

By choosing a_j^{s+1} and $\widehat{a}_j^{s+1}$ so that the diameter of $(a_j^{s+1}, \widehat{a}_j^{s+1})_{\leq_\mathbb{R}}$ is suitably small, we can guarantee that $\widehat{a}_j^{s+1} - a_j^{s+1} < 1/2^{s+1}$ and that there are infinitely many choices of q for which the relation above holds.

We handle the ordering approximations for all other approximate basis elements b_i^{s+1} for $1 \leq i \leq s$ with $i \neq \ell$ (and $i \neq j$ in the case when $b_\ell^s = b_j^{s+1} + q b_0^{s+1}$) as in the case when our approximate basis is unchanged. For the new approximate basis elements b_ℓ^{s+1} and b_{s+1}^{s+1}, we can assign r_ℓ^{s+1} and r_{s+1}^{s+1} to be any rational multiple of $\sqrt{p_\ell}$ and $\sqrt{p_{s+1}}$ (respectively) and pick a_ℓ^{s+1}, $\widehat{a}_\ell^{s+1}$, a_{s+1}^{s+1}, and $\widehat{a}_{s+1}^{s+1}$ accordingly. (The diagonalization strategy will constrain our choices of these values.)

We verify several properties of this general ordering construction based on the assumptions that each approximate basis element b_i^s eventually reaches a limit and that we choose our intervals and associated rationals in the manner described above.

Lemma 2.6. *For every pair of elements $x, y \in G_s$, if $x \leq_s y$, then $x \leq_{s+1} y$. Hence, we have the implication given in (1).*

Proof. This follows from the definition of $\leq_s$ and the fact that for each $x \in G_s$ the constraining interval for x at stage $s+1$ is contained in the constraining interval for x at stage s. □

By construction, we have $r_0^s = 1$ for all s, so $r_0 := \lim_s r_0^s = 1$ exists. We verify that the other r_i^s also reach limits.

Lemma 2.7. *For each $i \geq 1$, the limit $r_i := \lim_s r_i^s$ exists and is a rational multiple of $\sqrt{p_i}$. Furthermore, once r_i^s reaches its limit, the rational intervals $(a_i^t, \widehat{a}_i^t)_{\leq_\mathbb{R}}$ for $t \geq s$ form a nested sequence converging to r_i.*

Proof. We have $r_i^{s+1} \neq r_i^s$ only when $b_i^{s+1} \neq b_i^s$. Since the latter happens only finitely often, each r_i^s reaches a limit. The remainder of the statement is immediate from the construction. □

Lemma 2.8. *For each pair $x, y \in G_s$, there is a stage $t \geq s$ for which either $x \leq_t y$ or $y \leq_t x$.*

Proof. Since $x \leq_s x$ for all $x \in G_s$, we consider distinct elements $x, y \in G_s$. Let $t \geq s$ be a stage such that x and y have reached their limiting sums and such that for each b_i^t occurring in these sums, the real r_i^t has reached its limit r_i. Because the reals r_i are algebraically independent over $\mathbb{Q}$ and the nested approximations

$(a_i^u, \widehat{a}_i^u)_{\leq_{\mathbb{R}}}$ (for $u \geq t$) converge to r_i, there is a stage at which the interval constraints for x and y are disjoint. At the first such stage, we declare an ordering relation between x and y. □

We define the order $\leq_{\mathcal{G}}$ on $\mathcal{G}$ by $x \leq_{\mathcal{G}} y$ if and only if $x \leq_s y$ for some s. By Lemma 2.6 and Lemma 2.8, this relation is computable and every pair of elements is ordered. By construction, the Δ_2^0-map from G to $\mathbb{R}$ that sends

$$q_0 b_0 + \cdots + q_n b_n \mapsto q_0 + q_1 r_1 + \cdots + q_n r_n$$

is order preserving. Therefore, our ordered group $\mathcal{G}$ is classically isomorphic to the divisible subgroup of $(\mathbb{R}; +)$ with basis $\{1\} \cup \{\sqrt{p_i} : i \geq 1\}$ under the standard ordering.

Part 3. Building C and Diagonalizing. It remains to show how to use this general construction method to build the ordered group $(\mathcal{G}; \leq_{\mathcal{G}})$ together with a noncomputable c.e. set C such that the only C-computable orders on $\mathcal{G}$ are $\leq_{\mathcal{G}}$ and $\leq_{\mathcal{G}}^*$.

The requirements

$$\mathcal{S}_e : C \neq \Phi_e$$

to make C noncomputable are met in the standard finitary manner. The strategy for $\mathcal{S}_e$ chooses a large witness x, keeps x out of C, and waits for $\Phi_e(x)$ to converge to 0. If this convergence never occurs, the requirement is met because $x \notin C$. If the convergence does occur, then $\mathcal{S}_e$ is met by enumerating x into C and restraining C.

The remaining requirements are

$$\mathcal{R}_e : \text{If } \Phi_e^C(x, y) \text{ is an ordering on } \mathcal{G}, \text{ then } \Phi_e^C \text{ is either } \leq_{\mathcal{G}} \text{ or } \leq_{\mathcal{G}}^*.$$

We explain how to meet a single $\mathcal{R}_e$ in a finitary manner, leaving it to the reader to assemble the complete finite injury construction in the usual manner. In particular, note that each approximate basis element b_i^s eventually reaches a limit. To simplify the notation, we let $\leq_e^C$ be the binary relation on $\mathcal{G}$ computed by Φ_e^C. We will assume throughout that $\leq_e^C$ never directly violates any of the Π_1^0 conditions in the definition of a group order. For example, if we see at some stage s that $\leq_e^C$ has violated transitivity, then we can place a finite restraint on C to preserve these computations and win $\mathcal{R}_e$ trivially.

The strategy to meet $\mathcal{R}_e$ is as follows. We wait for a stage s at which $\leq_e^C$ declares either $b_0 \leq_e^C 0_{\mathcal{G}}$ or $b_0 \geq_e^C 0_{\mathcal{G}}$. While we wait, the construction of $\mathcal{G}$ proceeds as in the general description, leaving the approximate basis unchanged at each stage (that is, no dependencies are added) and defining $r_{s+1}^{s+1} = \sqrt{p_{s+1}}$.

If $\leq_e^C$ never gives either computation, then $\mathcal{R}_e$ is won trivially. Therefore, we assume that $\leq_e^C$ eventually orders $0_{\mathcal{G}}$ and b_0. We restrain C to preserve this ordering relationship. To simplify the exposition, we assume that $b_0 \geq_e^C 0_{\mathcal{G}}$. By restraining C, we know $\leq_e^C$ is not $\leq_{\mathcal{G}}^*$, so it only remains to ensure that if $\leq_e^C$ is an order, then $\leq_e^C$ is $\leq_{\mathcal{G}}$. (If we have $b_0 \leq_e^C 0_{\mathcal{G}}$, then we can work with $\leq_e^{C^*}$ and show that if $\leq_e^{C^*}$ is an order, then $\leq_e^{C^*}$ is $\leq_{\mathcal{G}}$.)

If $\leq_{\mathcal{G}}$ and $\leq_e^C$ are distinct, then we claim there must eventually be a stage s, an approximate basis element b_j^s (for some $j > 0$), and rationals $q_0 < q_1$ such that

- we have declared $q_0 b_0^s <_s b_j^s <_s q_1 b_0^s$ in G_s, and
- the order $\leq_e^C$ has declared $b_j^s <_e^C q_0 b_0^s$ or $q_1 b_0^s <_e^C b_j^s$.

(If $g \in G_s$ is assigned the sum $q_0 b_0^s$ or $q_1 b_0^s$, then it has reached its limiting sum since $b_0^s = b_0$.) Why must such a situation occur? If not, then in $\leq_e^C$, each $b_i := \lim_s b_i^s$ sits in the same rational cut (relative to b_0) as in $\leq_{\mathcal{G}}$. Since $\mathcal{G}$ is generated by the basis $\{b_i : i \in \omega\}$, the orderings $\leq_{\mathcal{G}}$ and $\leq_e^C$ are generated by their respective orderings of these basis elements. It then follows that the order $\leq_e^C$ is archimedean because $\leq_{\mathcal{G}}$ is archimedean, and furthermore, that $\leq_e^C$ is the same as $\leq_{\mathcal{G}}$. Note that if $\leq_e^C$ states that b_j^s equals some endpoint in the above inequalities, for example $b_j^s = q_0 b_0^s$, then $\leq_e^C$ fails to be an ordering if we continue building $\leq_{\mathcal{G}}$ as stated. Thus, the requirement $\mathcal{R}_e$ is met with no further action beyond restraining C.

Therefore, we assume that we eventually see such an element b_j^s at some stage $s+1$. At this stage, we begin our diagonalization action. We restrain C to preserve the ordering relations that $\leq_e^C$ has given so far. In the general group construction, we leave our approximate basis unchanged (i.e., $b_i^{s+1} := b_i^s$ for all $i \leq s$) and we leave their target images unchanged (i.e., $r_i^{s+1} := r_i^s$ for all $i \leq s$). For the new approximate basis element b_{s+1}^{s+1}, we want to have $b_{s+1}^{s+1} \in (q_0 b_0^s, q_1 b_0^s)_{\leq_{s+1}}$. That is, we want to choose r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ that sits in the interval $(q_0, q_1)_{\leq_{\mathbb{R}}}$ and we want to choose a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ so that $(a_{s+1}^{s+1}, \widehat{a}_{s+1}^{s+1})_{\leq_{\mathbb{R}}} \subseteq (q_0, q_1)_{\leq_{\mathbb{R}}}$. Notice that both b_j^{s+1} and b_{s+1}^{s+1} are then in the interval $(q_0 b_0^s, q_1 b_0^s)_{\leq_{s+1}}$. Furthermore, we want

- if $b_j^s \leq_e^C q_0 b_0^s$, then $b_{s+1}^{s+1} <_{s+1} b_j^{s+1}$, and
- if $q_1 b_0^s \leq_e^C b_j^s$, then $b_j^{s+1} <_{s+1} b_{s+1}^{s+1}$.

We can meet these additional constraints by choosing r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ for which there are appropriate choices of a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ satisfying either $\widehat{a}_{s+1}^{s+1} < a_j^{s+1}$ (in the first case) or $\widehat{a}_j^{s+1} < a_{s+1}^{s+1}$ (in the second case).

Now, we wait for a stage $t > s+1$ at which $\leq_e^C$ declares whether or not b_{s+1}^t is in the interval $(q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$. (We assume $b_j^u = b_j^{s+1}$ while we are waiting. If not, then this strategy would be injured and would start over again. Note that our current requirement is the only one that could cause $b_{s+1}^u \neq b_{s+1}^{s+1}$, so we automatically maintain $b_{s+1}^u = b_{s+1}^{s+1}$ while we wait.) If $\leq_e^C$ never specifies whether or not b_{s+1}^t is in this interval, then $\leq_e^C$ is not an order on $\mathcal{G}$ and $\mathcal{R}_e$ is won. Thus, we assume that $\leq_e^C$ eventually gives us this information. There are two cases to consider.

(1) Suppose at stage t, the order $\leq_e^C$ says that $b_{s+1}^t \notin (q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$. At stage $t+1$, we redefine b_{s+1}^t by adding the dependency $b_{s+1}^t = q b_0^{t+1}$ for a rational q. As described in the construction of the ordering, the rational q must be consistent with our current declared interval for b_{s+1}^t, that is, we must have $q \in (a_{s+1}^t, \widehat{a}_{s+1}^t)_{\leq_{\mathbb{R}}}$. As there are infinitely many such rationals, we can choose one that works in the general group construction (i.e., keeps the assignment of sums one-to-one). Furthermore, by our nesting of interval constraints, we have

$$(a_{s+1}^t, \widehat{a}_{s+1}^t)_{\leq_{\mathbb{R}}} \subseteq (a_{s+1}^{s+1}, \widehat{a}_{s+1}^{s+1})_{\leq_{\mathbb{R}}} \subseteq (q_0, q_1)_{\leq_{\mathbb{R}}}.$$

Therefore, we must have $q_0 < q < q_1$ and any ordering of $\mathcal{G}$ in which b_0 is positive must satisfy $q_0 b_0^t < q b_0^t < q_1 b_0^t$ and hence must satisfy $q_0 b_0^t < b_{s+1}^t < q_1 b_0^t$. As $\leq_e^C$ makes b_0 positive but does not satisfy this ordering relation, we can win $\mathcal{R}_e$ by restraining C one last time to preserve the ordering computation that specifies that $b_{s+1}^t \notin (q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$.

(2) Suppose at stage t, the order $\leq_e^C$ says that $b_{s+1}^t \in (q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$. In this case, there are two subcases.
   (a) Suppose that when we started the diagonalization process at stage $s+1$, we had $b_j^s \leq_e^C q_0 b_0^s$. In this case, we made $b_{s+1}^{s+1} <_{s+1} b_j^{s+1}$ and hence have $b_{s+1}^t <_t b_j^t$. Because we restrained C at stage $s+1$ and because $b_j^t = b_j^s$, we have $b_j^t \leq_e^C q_0 b_0^t$. At stage $t+1$, we restrain C to preserve the fact that $\leq_e^C$ has declared $b_{s+1}^t \in (q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$. Therefore, the order $\leq_e^C$ is committed to making $b_j^t <_e^C b_{s+1}^t$ unless it violates transitivity (and gives us a trivial win).
   We pick a positive rational q and add the dependency relation $b_{s+1}^t = b_j^{t+1} - q b_0^{t+1}$ at stage $t+1$. (For each $i \leq t$ with $i \neq s+1$, we leave $b_i^{t+1} = b_i^t$. In particular, $b_j^{t+1} = b_j^t$.) In the description of the ordering, we explained how to shrink the constraining interval around $r_j^{t+1} = r_j^t$ so that there are infinitely many choices of q for which this added dependency will not disrupt our current ordering facts. (The rational q has to be positive because $b_{s+1}^t <_t b_j^t = b_j^{t+1}$.) We have successfully diagonalized because the relation $b_{s+1}^t = b_j^{t+1} - q b_0^{t+1}$ forces b_{s+1}^t to be less than b_j^{t+1} in any order in which b_0 is positive. But, the order $\leq_e^C$ made b_0 positive and is committed to $b_j^{t+1} = b_j^t <_e^C b_{s+1}^t$. These facts are inconsistent with being an order, so we have won $\mathcal{R}_e$ in this case.
   (b) Suppose that when we started the diagonalization process at stage $s+1$, we had $q_1 b_0^s \leq_e^C b_j^s$. In this case, we made $b_j^{s+1} <_{s+1} b_{s+1}^{s+1}$ and hence have $b_j^t <_t b_{s+1}^t$. Restraining C to preserve the fact that $b_{s+1}^t \in (q_0 b_0^t, q_1 b_0^t)_{\leq_e^C}$ and reasoning as above, the order $\leq_e^C$ is committed to $b_{s+1}^t <_e^C b_j^t$. At stage $t+1$, we pick a positive rational q and add the dependency relation $b_{s+1}^t = b_j^{t+1} + q b_0^{t+1}$. (As above, there are infinitely many choices for an appropriate q.) Because of this added dependency, any order that makes b_0 positive must make b_j^{t+1} less than b_{s+1}^t. However, the order $\leq_e^C$ is committed to $b_{s+1}^t <_e^C b_j^t = b_j^{t+1}$ and hence we have successfully diagonalized to meet $\mathcal{R}_e$.

Combining the requisite parts, we see that the construction of $\mathcal{G}$, the construction of the computable orders, and the construction of the noncomputable, computably enumerable set C yield the theorem.

3. Proof of Theorem 1.5

The construction of $\mathcal{G}$ of isomorphism type $\mathbb{Z}^\omega$ is similar to when the desired isomorphism type is $\mathbb{Q}^\omega$. Again, the proof is divided into several steps. First, we describe our general method of building the computable presentation $\mathcal{G} = (G; +)$. Second, we describe how the computable order $\leq_{\mathcal{G}}$ on $\mathcal{G}$ is constructed. (As before, the order $\leq_{\mathcal{G}}^*$ is the second computable order on $\mathcal{G}$.) Third, we give the construction of C and the diagonalization process that ensures that the only C-computable orders on $\mathcal{G}$ are $\leq_{\mathcal{G}}$ and $\leq_{\mathcal{G}}^*$.

Part 1. General Construction of $\mathcal{G}$. This construction proceeds in a manner similar to Part 1 of Theorem 1.4. As done there, we build G as the union of an increasing sequence of sets $\{G_s\}_{s\in\omega}$, we approximate $+_{\mathcal{G}}$ as the union of an increasing sequence of partial relations $\{+_s\}_{s\in\omega}$, we maintain an approximation $\{b_0^s, b_1^s, \ldots, b_s^s\} \subseteq G_s$ to an initial segment of our eventual basis for $\mathcal{G}$ and every element $g \in G_s$ is assigned a $\mathbb{Q}$-linear sum over this approximate basis of the form

$$g = q_0^s b_0^s + \cdots + q_n^s b_n^s$$

where $n \leq s$, $q_i^s \in \mathbb{Q}$ for $0 \leq i \leq n$, and $q_n^s \neq 0$.

The major difference in the construction of the group is that, at each stage s, we will maintain a positive integer N_i^s for each $i \leq s$. This positive integer will restrain the nonzero coefficients q_i^s allowed in the $\mathbb{Q}$-linear sum for an element $g \in G_s$ by requiring that $d_i^s \mid N_i^s$, where d_i^s is the denominator of q_i^s when expressed in lowest terms. (Throughout this construction, we will use d_q, d_i^s and d_i to denote the denominators of rational numbers q, q_i^s and q_i, respectively, when written in lowest terms.) It will be the case that $N_i := \lim_s N_i^s$ exists and is finite for all i. These properties guarantee that $\mathcal{G}$ is a presentation of the group $\mathbb{Z}^\omega$ since each b_i is not divisible by any $m > N_i$. (Later we will introduce a basis restraint $K \in \omega$ that will prevent us from changing N_i^s too often.)

At a stage $s+1$, we do one of two things: we leave our approximate basis unchanged or we add a dependency relation for an approximate basis element b_ℓ^s (for some $\ell \leq s$). The diagonalization process dictates which happens.

(1) If we leave the basis unchanged, then we proceed as in (1) of Part 1 of Theorem 1.4, adding two new elements b_{s+1}^{s+1} and $q_0^{s+1}b_0^{s+1} + \cdots + q_n^{s+1}b_n^{s+1}$ to G_{s+1} as done there. Before doing so, we set $N_i^{s+1} := N_i^s$ for all $i \leq s$ and $N_{s+1}^{s+1} := 1$. By setting $N_{s+1}^{s+1} = 1$, we are currently building $\mathcal{G}$ so that b_{s+1}^{s+1} has no divisors. We also now require that the tuple $\langle q_0^{s+1}, \ldots, q_n^{s+1}\rangle$ satisfy $d_i^{s+1} \mid N_i^{s+1}$ for all $i \leq n$ with $q_i^{s+1} \neq 0$.

(2) If we redefine the approximate basis element b_ℓ^s (for the sake of diagonalizing) by adding a new dependency relation, then we proceed as in (2) of Part 1 of Theorem 1.4. The difference is that the diagonalization process will tell us either to set $b_\ell^s = qb_k^{s+1}$ for some rational q, or to set $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$ for some integers m_1 and m_2. In either case, the index k will be even and greater than the basis restraint K and $j, k < \ell$. We assign $g \in G_s$ the same sum except we replace each b_i^s by b_i^{s+1} (for $i \leq s$ and $i \neq \ell$) and we replace b_ℓ^s by either qb_k^{s+1} or $m_1 b_j^{s+1} + m_2 b_k^{s+1}$ (as dictated by the diagonalization process). We add three new elements b_ℓ^{s+1}, b_{s+1}^{s+1}, and $q_0^{s+1}b_0^{s+1} + \cdots + q_n^{s+1}b_n^{s+1}$ to G_{s+1}.

Before adding these new elements, we set $N_i^{s+1} := N_i^s$ for $i \leq s$ and $i \notin \{\ell, k, j\}$. We set $N_\ell^{s+1} := 1$ and $N_{s+1}^{s+1} := 1$. For N_k^{s+1}, we set $N_k^{s+1} := N_k^s \cdot N_\ell^s \cdot d_q$ if the dependency was $b_\ell^s = qb_k^{s+1}$ and $N_k^{s+1} := N_k^s \cdot N_\ell^s$ if the dependency was $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$. For N_j^{s+1}, we set $N_j^{s+1} := N_j^s$ if the dependency was $b_\ell^s = qb_k^{s+1}$ and $N_j^{s+1} := N_j^s \cdot N_\ell^s$ if the dependency was $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$. We require that the triple $\langle q_0^{s+1}, \ldots, q_n^{s+1}\rangle$ satisfy $d_i^{s+1} \mid N_i^{s+1}$ for all $i \leq n$ with $q_i^{s+1} \neq 0$.

It remains to check various properties of this construction. We work under the assumption that the limits $b_i := \lim_s b_i^s$ and $N_i := \lim_s N_i^s$ exist for all i. These assumptions follow from the diagonalization process.

Lemma 3.1. *Each $g \in G$ is eventually assigned a sum $q_0^t b_0^t + \cdots + q_n^t b_n^t$ such that at every stage $u \geq t$, the element g is assigned the sum $q_0^u b_0^u + \cdots + q_n^u b_n^u$ with the same coefficients and basis elements. Moreover, the nonzero coefficients in the limiting sum $q_0 b_0 + \cdots + q_n b_n$ satisfy $d_i \mid N_i$ for all $0 \leq i \leq n$.*

Proof. When g enters G, it is assigned a sum. The coefficients and basis elements in this sum only change when the diagonalization process declares a dependency relation for some approximate basis element b_ℓ^s having a nonzero coefficient in g. Since the dependencies are either $b_\ell^s = q b_k^{s+1}$ or $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$ with $j, k < \ell$, this reassignment of sums can only occur finitely often before terminating.

To see that a nonzero coefficient q_i satisfies $d_i \mid N_i$, it suffices to show by induction on s that $d_i^s \mid N_i^s$ whenever $q_i^s \neq 0$. When g first enters G and is assigned a sum, we have $d_i^s \mid N_i^s$ by construction. The coefficient of b_i^s can only change if b_i^s is involved in an added dependency relation.

If $b_\ell^s = q b_i^{s+1}$, the coefficient of b_i^{s+1} in g changes from q_i^s to $q_i^{s+1} = q_i^s + q q_\ell^s$. The denominator d_i^{s+1} divides the product of the denominators d_i^s, d_ℓ^s, and d_q. Since $N_i^{s+1} = N_i^s \cdot N_\ell^s \cdot d_q$, we have that $d_i^{s+1} \mid N_i^{s+1}$ by the induction hypothesis. Notice that this analysis is valid even when $q_i^s = 0$ or $q_\ell^s = 0$.

If $b_\ell^s = m_1 b_i^{s+1} + m_2 b_k^{s+1}$, the coefficient of b_i^{s+1} in g changes from q_i^s to $q_i^{s+1} = q_i^s + m_1 q_\ell^s$. Since $m_1 \in \mathbb{Z}$, the denominator d_i^{s+1} divides the product of the denominators d_i^s and d_ℓ^s. The fact that $d_i^{s+1} \mid N_i^{s+1}$ follows from $N_i^{s+1} = N_i^s \cdot N_\ell^s$ and the inductive hypothesis. Again, note that this analysis is valid even when $q_i^s = 0$ or $q_\ell^s = 0$.

This leaves only the cases when $b_i^s = q b_k^{s+1}$ and $b_i^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$. In these cases, $q_i^{s+1} = 0$ and we have the result at stage $s+1$ trivially. □

Lemma 3.2. *For each rational tuple $\langle q_0, \ldots, q_n \rangle$ with $q_n \neq 0$ and $d_i \mid N_i$ for all $0 \leq i \leq n$ with $q_i \neq 0$, there is an element $g \in G$ such that the limiting sum for g is $q_0 b_0 + \cdots + q_n b_n$.*

Proof. The proof is identical to the proof of Lemma 2.2. □

As in Part 1 of Theorem 1.4, we define the additive function $+_\mathcal{G}$ on $\mathcal{G}$ by setting $x +_\mathcal{G} y = z$ if there is a stage s such that $x +_s y = z$. Since Lemma 2.3 and Lemma 2.4 remain true (with identical proofs) in this setting, this is a well-defined and total function. As a consequence of Lemma 3.2, every element has an additive inverse.

Lemma 3.3. *The group $\mathcal{G}$ is a computable presentation of $\mathbb{Z}^\omega$.*

Proof. By construction, together with the preceding comments, the domain G and addition function $+_\mathcal{G}$ are computable, with the addition function satisfying the axioms for a torsion-free abelian group.

Since the limit $N_i := \lim_s N_i^s$ exists and is finite for all i, viewing $\mathbb{Z}^\omega$ as having basis $\{e_i\}_{i \in \omega}$ with elements $g = \gamma_0 e_0 + \cdots + \gamma_n e_n$ (where $\gamma_0, \ldots, \gamma_n \in \mathbb{Z}$), we have that the map $e_i \mapsto b_i / N_i$ induces an isomorphism. Thus, the isomorphism type of $\mathcal{G}$ is indeed $\mathbb{Z}^\omega$. □

Part 2. Defining the Computable Orders on $\mathcal{G}$. This proceeds in a manner similar to Part 2 of Theorem 1.4. As done there, we approximate the computable ordering $\leq_\mathcal{G}$ as the union of an increasing sequence of binary relations $\{\leq_s\}_{s \in \omega}$.

We again specify the order on $\mathcal{G}$ by building a Δ^0_2-map from G into $\mathbb{R}$. Again, we always map b_0 to the real 1 and assign each b_i^s (for $i \geq 1$) an interval of the form $(a_i^s, \widehat{a}_i^s)_{\leq_\mathbb{R}}$ with $a_i^s, \widehat{a}_i^s \in \mathbb{Q}$ and a real $r_i^s \in (a_i^s, \widehat{a}_i^s)_{\leq_\mathbb{R}}$ that is a rational multiple of $\sqrt{p_i}$. The difference is that we will require $0 \leq a_i^s \leq \widehat{a}_i^s < 1/2^i$ for all *even* i and s.

The reason we do so is that the element b_0 no longer provides a means of measuring the size of arbitrary elements as only integer multiples of b_0 exist. If r_i were large (say, greater than one) for all i, the only elements in $\mathcal{G}$ that would be small (close to zero in size) would be complicated linear combinations of basis elements. By requiring $r_k < 1/2^k$ for all even k, we can measure the size of an arbitrary basis element b_i^s in terms of b_0 through an intermediary basis element b_k^s with even index.

We describe the interaction between the order and changes to the basis approximation.

(1) If we leave the approximate basis unchanged at stage $t+1$, then we proceed as in Case 1 of Part 2 of Theorem 1.4. We let r_{t+1}^{t+1} be a rational multiple of $\sqrt{p_{t+1}}$ such that if $t+1$ is even, then $r_{t+1}^{t+1} < 1/2^{t+1}$. We pick positive rationals a_{t+1}^{t+1} and $\widehat{a}_{t+1}^{t+1}$ such that $a_{t+1}^{t+1} < r_{t+1}^{t+1} < \widehat{a}_{t+1}^{t+1}$ and $\widehat{a}_{t+1}^{t+1} - a_{t+1}^{t+1} < 1/2^{t+1}$. If $t+1$ is even, we also require $\widehat{a}_{t+1}^{t+1} < 1/2^{t+1}$.

(2) If we wish to add the dependency $b_\ell^t = q b_k^{t+1}$ at stage $t+1$, then k is even and ℓ is odd. This dependency relation is consistent with $\leq_t$ provided we can choose q, a_k^{t+1}, and $\widehat{a}_k^{t+1}$ so that $(q a_k^{t+1}, q\widehat{a}_k^{t+1})_{\leq_\mathbb{R}} \subseteq (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}$. It is immediate that there infinitely many such choices for q as we need only ensure $a_k^t \leq a_k^{t+1} \leq \widehat{a}_k^{t+1} \leq \widehat{a}_k^t$ and $\widehat{a}_k^{t+1} - a_k^{t+1} \leq 1/2^{t+1}$.

(3) If we wish to add the dependency $b_\ell^t = m_1 b_j^t - m_2 b_k^t$ at stage $t+1$, then we will be in one of two contexts.

3(a). If we are in a context in which

$$0 < n a_k^t < n\widehat{a}_k^t < a_\ell^t < \widehat{a}_\ell^t < a_j^t < \widehat{a}_j^t < (n+1)a_k^t < (n+1)\widehat{a}_k^t, \tag{2}$$

then we will choose $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leq m_2/n$ and

$$(m_1 a_j^{t+1} - m_2 \widehat{a}_k^{t+1}, m_1 \widehat{a}_j^{t+1} - m_2 a_k^{t+1})_{\leq_\mathbb{R}} \subseteq (a_\ell^t, \widehat{a_\ell}^t)_{\leq_\mathbb{R}}. \tag{3}$$

3(b). If we are in a context in which

$$0 < n a_k^t < n\widehat{a}_k^t < a_j^t < \widehat{a}_j^t < a_\ell^t < \widehat{a}_\ell^t < (n+1)a_k^t < (n+1)\widehat{a}_k^t, \tag{4}$$

then we will choose $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leq m_2(n+1)$ and

$$(m_1 a_k^{t+1} - m_2 \widehat{a}_j^{t+1}, m_1 \widehat{a}_k^{t+1} - m_2 a_j^{t+1})_{\leq_\mathbb{R}} \subseteq (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}. \tag{5}$$

By Lemma 3.5, in each of these contexts, there are infinitely many such choices for m_1 and m_2 satisfying the given conditions.

To explain why appropriate $m_1, m_2 \in \mathbb{N}$ exist for the two contexts above, we rely on the following fact from ordered abelian group theory.

Lemma 3.4. *Let r_1 and r_2 be reals that are linearly independent over $\mathbb{Q}$. For any rational numbers $q_1 < q_2$, there are infinitely many $m_1, m_2 \in \mathbb{N}$ such that $m_1 r_1 - m_2 r_2 \in (q_1, q_2)_{\leq_\mathbb{R}}$. Moreover, one can require m_1 and m_2 to be divisible by any given $n \in \mathbb{N}$.*

Lemma 3.5. *If we are in the context of (2) (respectively (4)), then there are infinitely many choices for m_1 and m_2 that satisfy (3) (respectively (5)).*

Proof. First, suppose we are in the context of (2). We have that b^t_j and b^t_k are currently identified with the rational multiples r^t_j and r^t_k of $\sqrt{p_j}$ and $\sqrt{p_k}$ respectively, so r^t_j and r^t_k are linearly independent over $\mathbb{Q}$. Hence, by Lemma 3.4 (requiring m_2 to be divisible by n), there are infinitely many choices of $m_1, \tilde{m}_2 \in \mathbb{N}$ such that $m_1 r^t_j - \tilde{m}_2 n r^t_k \in (a^t_\ell, \widehat{a}^t_\ell)_{\leq_\mathbb{R}}$. (We let $m_2 := \tilde{m}_2 n$.) We can choose $a^{t+1}_j, \widehat{a}^{t+1}_j, a^{t+1}_k, \widehat{a}^{t+1}_k \in \mathbb{Q}$ with $a^{t+1}_j < r^t_j < \widehat{a}^{t+1}_j$ and $a^{t+1}_k < r^t_k < \widehat{a}^{t+1}_k$ satisfying (3) by shrinking the intervals $(a^t_j, \widehat{a}^t_j)_{\leq_\mathbb{R}}$ and $(a^t_k, \widehat{a}^t_k)_{\leq_\mathbb{R}}$ appropriately.

It remains to see why we must have $m_1 \leq \frac{m_2}{n} = \tilde{m}_2$. Suppose $m_1 > \frac{m_2}{n} = \tilde{m}_2$, so $m_1 - 1 \geq \tilde{m}_2$. Then

$$\begin{aligned} m_1 r^t_j - \tilde{m}_2 n r^t_k &= r^t_j + (m_1 - 1) r^t_j - \tilde{m}_2 n r^t_k \\ &\geq r^t_j + \tilde{m}_2 r^t_j - \tilde{m}_2 n r^t_k \\ &= r^t_j + \tilde{m}_2 (r^t_j - n r^t_k) \\ &> r^t_j \end{aligned}$$

because $r^t_j - n r^t_k > 0$ by (2). We have reached a contradiction since $m_1 r^t_j - \tilde{m}_2 n r^t_k \in (a^t_\ell, \widehat{a}^t_\ell)_{\leq_\mathbb{R}}$ and $r^t_j \in (a^t_j, \widehat{a}^t_j)_{\leq_\mathbb{R}}$ but $\widehat{a}^t_\ell < a^t_j$. So, $m_1 \leq \frac{m_2}{n} = \tilde{m}_2$ as desired.

Now suppose we are in the context of (4). Since r^t_j and r^t_k are linearly independent over $\mathbb{Q}$, by Lemma 3.4 (requiring m_1 to be divisible by $n+1$) there are infinitely many choices of $\tilde{m}_1, m_2 \in \mathbb{N}$ such that $\tilde{m}_1(n+1) r^t_k - m_2 r^t_j \in (a^t_\ell, \widehat{a}^t_\ell)_{\leq_\mathbb{R}}$. (We let $m_1 := \tilde{m}_1(n+1)$.) As before, we can choose $a^{t+1}_j, \widehat{a}^{t+1}_j, a^{t+1}_k, \widehat{a}^{t+1}_k \in \mathbb{Q}$ satisfying (5).

It remains to see why $m_1 = \tilde{m}_1(n+1) \leq m_2(n+1)$. Suppose $m_1 = \tilde{m}_1(n+1) > m_2(n+1)$, so $\tilde{m}_1 - 1 \geq m_2$. Then

$$\begin{aligned} m_1 r^t_k - m_2 r^t_j &= \tilde{m}_1 (n+1) r^t_k - m_2 r^t_j \\ &\geq \tilde{m}_1 (n+1) r^t_k - (\tilde{m}_1 - 1) r^t_j \\ &> \tilde{m}_1 (n+1) r^t_k - (\tilde{m}_1 - 1)(n+1) r^t_k \\ &= (n+1) r^t_k. \end{aligned}$$

The first inequality follows because $\tilde{m}_1 - 1 \geq m_2$ and r^t_j is positive, and the second inequality follows because $r^t_j < (n+1) r^t_k$ by (4). We have reached a contradiction since $m_1 r^t_k - m_2 r^t_j \in (a^t_\ell, \widehat{a}^t_\ell)_{\leq_\mathbb{R}}$ but $\widehat{a}^t_\ell < (n+1) a^t_k$. $\square$

Since none of the differences from Part 2 of Theorem 1.4 impact the verification, again we can define an order $\leq_\mathcal{G}$ on $\mathcal{G}$ by $x \leq_\mathcal{G} y$ if $x \leq_s y$ for some s. As in Part 2 of Theorem 1.4, this relation is computable and a group order on $\mathcal{G}$.

Part 3. Building C and Diagonalizing. This proceeds in a manner slightly different than Part 3 of Theorem 1.4. As done there, we meet the requirements $\mathcal{S}_e$ and $\mathcal{R}_e$. Though we satisfy $\mathcal{S}_e$ in the same manner, we satisfy $\mathcal{R}_e$ in a different manner. Again, we use $\leq^C_e$ to denote the binary relation on $\mathcal{G}$ computed by Φ^C_e, and we assume it never violates the axioms of an ordered group. For $\mathcal{R}_e$, we set the basis restraint $K := e$.

The strategy to satisfy $\mathcal{R}_e$ is as follows. If $\leq_\mathcal{G} \neq \leq^C_e$ and $\leq^*_\mathcal{G} \neq \leq^C_e$, then there must eventually be a stage s, an approximate basis element b^s_j, a nonnegative integer n, and an even index $k > K$ such that:

- we have declared $0 <_s n b^s_k <_s b^s_j <_s (n+1) b^s_k$ in G_s, and

- the order $\leq_e^C$ has declared either (a) $b_k^s >_e^C 0_{\mathcal{G}}$ and either $b_j^s <_e^C nb_k^s$ or $b_j^s >_e^C (n+1)b_k^s$, or (b) $b_k^s <_e^C 0_{\mathcal{G}}$ and either $b_j^s >_e^C nb_k^s$ or $b_j^s <_e^C (n+1)b_k^s$.

In the latter case, we work with the ordering $\leq_e^{C^*}$, transforming the latter case into the former case. We therefore assume that we are in the former case.

When such s, b_j^s, n, and k are found, we say $\mathcal{R}_e$ is *activated*, and we restrain C to preserve the computations ordering b_j^s and nb_k^s and $(n+1)b_k^s$. In Lemma 3.7, we verify that such s, b_j^s, n, and k must exist if $\leq_{\mathcal{G}} \neq \leq_e^C$ and $\leq_{\mathcal{G}}^* \neq \leq_e^C$. As $\mathcal{R}_e$ is trivially satisfied otherwise, we assume such parameters are found at some stage $s+1$.

At stage $s+1$ (without loss of generality, we assume $s+1$ is odd), we order the new approximate basis element b_{s+1}^{s+1} depending on whether $b_j^s <_e^C nb_k^s$ or $b_j^s >_e^C (n+1)b_k^s$.

- If $b_j^s <_e^C nb_k^s$, we order b_{s+1}^{s+1} so that $nb_k^s <_{s+1} b_{s+1}^{s+1} <_{s+1} b_j^s$, that is, we choose r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ and a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ so that $n\widehat{a}_k^s < a_{s+1}^{s+1} < r_{s+1}^{s+1} < \widehat{a}_{s+1}^{s+1} < a_j^s$ and $\widehat{a}_{s+1}^{s+1} - a_{s+1}^{s+1} < 1/2^{s+1}$.
- if $b_j^s >_e^C (n+1)b_k^s$, we order b_{s+1}^{s+1} so that $b_j^s <_{s+1} b_{s+1}^{s+1} <_{s+1} (n+1)b_k^s$, that is, we choose r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ and a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ so that $\widehat{a}_j^s < a_{s+1}^{s+1} < r_{s+1}^{s+1} < \widehat{a}_{s+1}^{s+1} < (n+1)a_k^s$ and $\widehat{a}_{s+1}^{s+1} - a_{s+1}^{s+1} < 1/2^{s+1}$.

We then wait for a stage $t+1$ so that $\leq_e^C$ declares $b_{s+1}^t <_e^C nb_k^s$ or $nb_k^s <_e^C b_{s+1}^t <_e^C (n+1)b_k^s$ or $b_{s+1}^t >_e^C (n+1)b_k^s$. Again, we assume that such a stage $t+1$ is found, else $\mathcal{R}_e$ is trivially satisfied.

- If $\leq_e^C$ declares $b_{s+1}^t <_e^C nb_k^t$ or $b_{s+1}^t >_e^C (n+1)b_k^t$, then we declare $b_{s+1}^t = qb_k^t$ for some $q \in \mathbb{Q}$ with $q \in (n, n+1)_{\leq_{\mathbb{R}}} \cap (a_{s+1}^t, \widehat{a}_{s+1}^t)_{\leq_{\mathbb{R}}}$. This ensures $\leq_e^C$ violates the order axioms as n was a nonnegative integer and $b_k^t >_e^C 0_{\mathcal{G}}$. The fact that there are infinitely many $q \in (n, n+1)_{\leq_{\mathbb{R}}} \cap (a_{s+1}^t, \widehat{a}_{s+1}^t)_{\leq_{\mathbb{R}}}$ allows us to keep the assignments of sums one-to-one in the general group construction.
- If $\leq_e^C$ declares $nb_k^t <_e^C b_{s+1}^t <_e^C (n+1)b_k^t$, then we act depending on whether $b_{s+1}^t <_t b_j^t$ or $b_{s+1}^t >_t b_j^t$.
  - If $b_{s+1}^t <_t b_j^t$, then we declare $b_{s+1}^t = m_1 b_j^t - m_2 b_k^t$ for some positive integers m_1, m_2 with $m_1 \leq m_2/n$ that satisfy our ordering constraints. This ensures $\leq_e^C$ violates the order axioms as this necessitates

$$b_{s+1}^t = m_1 b_j^t - m_2 b_k^t \leq_e^C (m_2/n) b_j^t - m_2 b_k^t <_e^C (m_2/n)(nb_k^t) - m_2 b_k^t = 0_{\mathcal{G}} <_e^C nb_k^t <_e^C b_{s+1}^t.$$

    (The first inequality in the line above holds because we can conclude $b_j^t >_e^C 0_{\mathcal{G}}$ from the first equality since $b_{s+1}^t >_e^C 0_{\mathcal{G}}$ and $b_k^t >_e^C 0_{\mathcal{G}}$.) By Lemma 3.5 there are infinitely many choices for m_1, m_2, so we can choose them to keep the assignments of sums one-to-one in the general group construction.
  - If $b_{s+1}^t >_t b_j^t$, then we declare $b_{s+1}^t = m_1 b_k^t - m_2 b_j^t$ for some positive integers m_1, m_2 with $m_1 \leq m_2(n+1)$ that satisfy our ordering constraints. This ensures $\leq_e^C$ violates the order axioms as this necessitates

$$b_{s+1}^t = m_1 b_k^t - m_2 b_j^t \leq_e^C (m_2(n+1)) b_k^t - m_2 b_j^t <_e^C m_2 b_j^t - m_2 b_j^t = 0_{\mathcal{G}} <_e^C nb_k^t <_e^C b_{s+1}^t.$$

    Again, by Lemma 3.5 there are infinitely many choices for m_1, m_2, so we can choose them to keep the assignments of sums one-to-one in the general group construction.

In all cases, we restrain C to preserve the computations ordering b_{s+1}^t and nb_k^t and $(n+1)b_k^t$ under $\leq_e^C$. This completes our description of building C and diagonalizing.

Having explained how to meet a single $\mathcal{R}_e$ in a finitary manner, we again leave it to the reader to assemble the complete finite injury construction in the usual manner. As before, the finitary nature of $\mathcal{R}_e$ and the action of the construction guarantees that the limits $b_i := \lim_s b_i^s$ and $r_i := \lim_s r_i^s$ exist for all i. We demonstrate $N_i := \lim_s N_i^s$ exists for all i.

Lemma 3.6. *The limit $N_i := \lim_s N_i^s$ exists for all i.*

Proof. For $i = 0$, this is immediate as $N_i^s = 1$ for all s by construction. For $i > 0$, the important observation is that if $N_i^{t+1} \neq N_i^t$, then there was a basis dependency introduced of the form $b_\ell^t = qb_k^t$, $b_\ell^t = m_1 b_j^t - m_2 b_k^t$, or $b_\ell^t = m_1 b_k^t - m_2 b_j^t$ with i being either j, k or ℓ. By construction, in these dependency relations, k is even, ℓ is odd, and j is either even or odd.

First, we claim that if i is odd, then $N_i^s = 1$ for all s after it is defined. For a contradiction, let t be the least stage at which $N_i^{t+1} \neq 1$ for some odd i, and let i be the least odd number such that $N_i^{t+1} \neq 1$. (When N_i^s is initially defined, its value is set to 1. Therefore, we know $N_i^t = 1$ and hence $N_i^{t+1} \neq N_i^t$.) Since i is odd, the index i must be either j or ℓ in the introduced dependency relation. However, since the construction sets $N_\ell^{t+1} = 1$, we cannot have $i = \ell$. Therefore, the new dependency relation is of the form $b_\ell^t = m_1 b_i^t - m_2 b_k^t$, or $b_\ell^t = m_1 b_k^t - m_2 b_i^t$. In either case, $N_i^{t+1} = N_i^t \cdot N_\ell^t$. Since i and ℓ are odd, we have $N_i^t = N_\ell^t = 1$, so $N_i^{t+1} = 1$ for the desired contradiction.

Second, we consider the case when i is even. In this case, the index i is either k or j in the added dependency relation. If $i = k$, then the dependency relation can only be added by a requirement $\mathcal{R}_e$ with $e \leq i$ (because of the basis restraint). These requirements can only cause N_i^s to change finitely often, so $N_i = \lim_s N_i^s$ exists. If $i = j$ in the dependency relation, then $N_i^{t+1} = N_i^t \cdot N_\ell^t$. However, the index ℓ is odd, so $N_\ell^t = 1$, and hence $N_i^{t+1} = N_i^t$ does not actually change. $\square$

We now show that $\mathcal{R}_e$ is activated at some stage s if $\leq_\mathcal{G} \neq \leq_e^C$ and $\leq_\mathcal{G}^* \neq \leq_e^C$.

Lemma 3.7. *If we fail to find a stage s where $\mathcal{R}_e$ is activated for an approximate basis element b_j^s, a nonnegative integer n, and an even index $k > K$, then either $\leq_e^C$ is not an order or $\leq_\mathcal{G} = \leq_e^C$ or $\leq_\mathcal{G}^* = \leq_e^C$.*

Proof. Assume that $\leq_e^C$ is an order on $\mathcal{G}$. First, we claim that if we fail to find a stage s at which $\mathcal{R}_e$ is activated, then either $0_\mathcal{G} <_e^C b_k$ for all even k or $b_k <_e^C 0_\mathcal{G}$ for all even k. (In both cases, we include $k = 0$.)

To prove this claim, suppose that $\mathcal{R}_e$ is never activated and that j_0, j_1 are even with $b_{j_1} <_e^C 0_\mathcal{G} <_e^C b_{j_0}$. Fix a stage s such that $\mathcal{R}_e$ is the highest priority requirement left to act after stage s, $b_{j_1}^s = b_{j_1}$, $b_{j_0}^s = b_{j_0}$ and $b_{j_1} <_e^C 0_\mathcal{G} <_e^C b_{j_0}$ is permanently fixed by stage s. Consider a stage $t \geq s$ and an even index k greater than the basis restraint for $\mathcal{R}_e$ such that $b_k^t = b_k$ has reached its limit and there are $n_0, n_1 \in \omega$ for which

$$0_\mathcal{G} <_t n_0 b_k^t <_t b_{j_0}^t <_t (n_0+1) b_k^t \qquad \text{and} \qquad 0_\mathcal{G} <_t n_1 b_k^t <_t b_{j_1} <_t (n_1+1) b_k^t.$$

Since $\leq_e^C$ is an order, there must be a stage $u \geq t$ at which it declares either $0_\mathcal{G} <_e^C b_k^u$ or $b_k^u < 0_\mathcal{G}$ permanently.

If $0_{\mathcal{G}} <_e^C b_k^u$, then we must eventually see $b_{j_1}^v <_e^C 0_{\mathcal{G}} <_e^C n_1 b_k^v$ for some $v \geq u$. Therefore, $\mathcal{R}_e$ is activated at stage v (with $j = j_1$, $k = k$, and $n = n_1$) for the desired contradiction. Alternately, if $b_k^u <_e^C 0_{\mathcal{G}}$, then we must eventually see $n_0 b_k^v <_e^C 0_{\mathcal{G}} <_e^C b_{j_0}^v$ for some $v \geq u$. Again, $\mathcal{R}_e$ is activated at stage v (with $j = j_0$, $k = k$, and $n = n_0$) for the desired contradiction. This completes the proof of the claim.

To complete the proof of this lemma, assume that $\mathcal{R}_e$ is never activated and $0_{\mathcal{G}} <_e^C b_k$ for all even k. We show that $\leq_e^C = \leq_{\mathcal{G}}$. (It follows by a similar argument that if $b_k <_e^C 0_{\mathcal{G}}$ for all even k, then $\leq_e^C = \leq_{\mathcal{G}}^*$.)

By construction, $(\mathcal{G}; +_{\mathcal{G}}, \leq_{\mathcal{G}})$ can be embedded (as an ordered group) into $(\mathbb{R}; +_{\mathbb{R}}, \leq_{\mathbb{R}})$ by sending each basis element $b_i \in \mathcal{G}$ to $r_i \in \mathbb{R}$. To show that $\leq_{\mathcal{G}} = \leq_e^C$, it suffices to show that the same map is an ordered group embedding of $(\mathcal{G}; +_{\mathcal{G}}, \leq_e^C)$ into $(\mathbb{R}; +_{\mathbb{R}}, \leq_{\mathbb{R}})$.

For each even index k, we fix $n_{0,k} \in \omega$ such that

$$n_{0,k} b_k \leq_{\mathcal{G}} b_0 \leq_{\mathcal{G}} (n_{0,k} + 1) b_k.$$

By the construction, this condition is equivalent to $n_{0,k} r_k \leq_{\mathbb{R}} r_0 \leq_{\mathbb{R}} (n_{0,k} + 1) r_k$. Since k is even, we have $(n_{0,k} + 1) r_k - n_{0,k} r_k = r_k \leq 1/2^k$ and hence

$$\lim_{k \to \infty} n_{0,k} r_k = \lim_{k \to \infty} (n_{0,k} + 1) r_k = r_0 = 1$$

where the limits (and all limits throughout this lemma) are taken over even indices k. More generally, for each index $i \in \omega$ and each even index k, we fix $n_{i,k} \in \omega$ such that

$$n_{i,k} b_k \leq_{\mathcal{G}} b_i \leq_{\mathcal{G}} (n_{i,k} + 1) b_k.$$

As above, this condition is equivalent to $n_{i,k} r_k \leq_{\mathbb{R}} r_i \leq_{\mathbb{R}} (n_{i,k} + 1) r_k$ and we have

$$\lim_{k \to \infty} n_{i,k} r_k = \lim_{k \to \infty} (n_{i,k} + 1) r_k = r_i.$$

Combining these limits, we have

$$\lim_{k \to \infty} \frac{n_{i,k}}{n_{0,k} + 1} = \lim_{k \to \infty} \frac{n_{i,k} r_k}{(n_{0,k} + 1) r_k} = \frac{r_i}{1} = r_i$$

and

$$\lim_{k \to \infty} \frac{n_{i,k} + 1}{n_{0,k}} = \lim_{k \to \infty} \frac{(n_{i,k} + 1) r_k}{n_{0,k} r_k} = \frac{r_i}{1} = r_i.$$

We now translate these results to $(\mathcal{G}, \leq_e^C)$. Because $\mathcal{R}_e$ is never activated and $0_{\mathcal{G}} <_e^C b_k$ for all even k, the inequalities $n_{i,k} b_k \leq_e^C b_i \leq_e^C (n_{i,k} + 1) b_k$ hold for all i and all even k such that k is greater than the basis restraint for $\mathcal{R}_e$. In particular, combining the inequalities $n_{0,k} b_k \leq_e^C b_0 \leq_e^C (n_{0,k} + 1) b_k$ and $n_{i,k} b_k \leq_e^C b_i \leq_e^C (n_{i,k} + 1) b_k$, we have

$$\frac{n_{i,k}}{n_{0,k} + 1} b_0 \leq_e^C b_i \leq_e^C \frac{n_{i,k} + 1}{n_{0,k}} b_0$$

where this inequality is interpreted as representing the corresponding inequality after multiplying through by the denominators so all the coefficients are integers. (Alternately, this inequality can be viewed in the divisible closure of $\mathcal{G}$ using the fact that an order on an abelian group has a unique extension to an order on its divisible closure.) The limits above show that the map sending b_i to r_i defines an embedding of $(\mathcal{G}; +_{\mathcal{G}}, \leq_e^C)$ into $(\mathbb{R}; +_{\mathbb{R}}, \leq_{\mathbb{R}})$ as required. $\square$

We have completed the proof of Theorem 1.5.

4. Remarks and Open Questions

Since both Theorem 1.4 and Theorem 1.5 are typical finite injury constructions, modifications to the constructions are straightforward.

Remark 4.1. Rather than building $\mathcal{G}$ so that there are exactly two computable orders, it is an easy modification to build exactly any even number or an infinite number of computable orders (with no other C-computable orders).

For example, to build $\mathbb{Q}^\omega$ with four computable orders, we double the number of $\mathcal{R}_e$ requirements. We build a computable order $\leq^0_{\mathcal{G}}$ in which $0 \leq^0_{\mathcal{G}} b_0 \leq^0_{\mathcal{G}} b_1$ and a computable order $\leq^1_{\mathcal{G}}$ in which $0 \leq^1_{\mathcal{G}} b_1 \leq^1_{\mathcal{G}} b_0$. For each of these orders, we meet a slightly modified requirement:

$\mathcal{R}^i_e$: If $b_i \leq^C_e b_{1-i}$ and Φ^C_e is an ordering of $\mathcal{G}$, then Φ^C_e is either $\leq^i_{\mathcal{G}}$ or $\leq^{i^*}_{\mathcal{G}}$.

Since these requirements are still finitary (both restraint and injury) in nature, these combine easily to yield the desired result.

Remark 4.2. We note that the computably enumerable set C cannot be complete. The reason is that $\mathbf{0}'$ can compute a basis for any computable torsion-free abelian group $\mathcal{G}$, and hence $\mathcal{G}$ has orders of degree $\mathbf{0}'$.

We also note that, as long as the construction remains finitary (both restraint and injury), additional requirements on C can be added. Thus lowness requirements can be added, for example, though this would be counter-productive (the weaker C is computationally, the weaker the result).

Remark 4.3. The computable orders in our constructions were always archimedean. If non-archimedean computable orders are desired, it suffices to take $\mathcal{G} \oplus \mathbb{Z}$ (for a copy of $\mathbb{Z}^\omega$) or $\mathcal{G} \oplus \mathbb{Q}$ (for a copy of $\mathbb{Q}^\omega$), where $\mathcal{G}$ is as constructed and every element x can be computably decomposed into $x \in \mathcal{G}$ and $x \in \mathbb{Z}$ (for $\mathbb{Z}^\omega$) or $x \in \mathbb{Q}$ (for $\mathbb{Q}^\omega$). The reason, of course, is that any (C-)computable order on $\mathcal{G} \oplus \mathbb{Z}$ or $\mathcal{G} \oplus \mathbb{Q}$ restricts to a (C-)computable order on $\mathcal{G}$.

Certain natural questions remain open.

Question 4.4. Do Theorems 1.4 and 1.5 remain true when the isomorphism types $\mathbb{Q}^\omega$ and $\mathbb{Z}^\omega$ are replaced by an arbitrary computable isomorphism type of a torsion-free abelian group?

Finally, we wonder what degree spectra for orders on a computable torsion-free abelian group are possible. As noted in Section 1, if the rank of $\mathcal{G}$ is 1, then $\mathcal{G}$ has only two orders both of which are computable. If the rank is finite but greater than 1, then $\mathcal{G}$ has orders of every degree. However, if $\mathcal{G}$ has infinite rank, then the only constraint we know of (beyond Π^0_1 class basis theorems and including the cone above $\mathbf{0}'$, or more generally the cone above the degree of each basis of $\mathcal{G}$) is the following.

Proposition 4.5 (With Daniel Turetsky)**.** *If $\mathcal{G}$ is a computable presentation of a torsion-free abelian group with infinite rank, then deg($\mathbb{X}(\mathcal{G})$) contains infinitely many low degrees.*

Proof. We inductively show deg($\mathbb{X}(\mathcal{G})$) must contain at least n-many low degrees for all n. Fix two linearly independent elements $g, h \in G$ and let T_0 be a computable tree such that $[T_0]$ (the set of infinite paths through T_0) contains exactly the orders $\leq_{\mathcal{G}}$ on $\mathcal{G}$ satisfying

$$0_{\mathcal{G}} <_{\mathcal{G}} g <_{\mathcal{G}} h <_{\mathcal{G}} 4g.$$

Note that the set of orders on $\mathcal{G}$ satisfying this constraint is a Π^0_1 class and hence can be represented in this manner. The Low Basis Theorem applied to T_0 yields a low order of some degree $\mathbf{d}_0$. To get a second order of low degree $\mathbf{d}_1 \neq \mathbf{d}_0$, it suffices (as low over low is low) to build a nonempty $\mathbf{d}_0$-computable subtree T_1 of T_0 having no $\mathbf{d}_0$-computable paths. From this, we obtain a low (low over $\mathbf{d}_0$) order of some degree $\mathbf{d}_1$ not computable from $\mathbf{d}_0$.

The subset T_1 of T_0 is constructed (using an oracle of degree $\mathbf{d}_0$) by killing paths that agree with the e^{th} (candidate) $\mathbf{d}_0$-computable order $\leq_e$ on the relative ordering of g and h for a sufficiently large amount of precision. In particular, to diagonalize against $\leq_e$, we attempt to find positive rationals $q_0 <_{\mathbb{Q}} q_1$ such that $q_1 - q_0 < 2^{-e}$ and $q_0 g <_e h <_e q_1 g$. If and when such rationals are found, we kill initial segments of T_0 that specify $q_0 g <_{\mathcal{G}} h <_{\mathcal{G}} q_1 g$ (if any exist). Notice that $[T_1] \neq \emptyset$ as $\sum_{e=0}^{\infty} 2^{-e} = 2 < 4$.

To get a third order of low degree $\mathbf{d}_2 \notin \{\mathbf{d}_0, \mathbf{d}_1\}$, we repeat this process to construct a $(\mathbf{d}_0 \oplus \mathbf{d}_1)$-computable subtree T_2 of T_1 such that T_2 has no $\mathbf{d}_1$-computable paths. We note that T_2 cannot have any $\mathbf{d}_0$-computable paths as it is a subtree of T_1. The only change we need to make is to require the rationals q_0 and q_1 (being used to diagonalize against the e^{th} (candidate) $\mathbf{d}_1$-computable order $\leq_e$) to satisfy $q_1 - q_0 < 2^{-(e+1)}$. Since $\sum_{e=0}^{\infty} 2^{-(e+1)} = 1 < 2$, we guarantee that $[T_2] \neq \emptyset$. Continuing to repeat this process in the obvious way yields the proposition. $\square$

We close by asking the following general question.

Question 4.6. Describe the possible degree spectra of orders $\mathbb{X}(\mathcal{G})$ on a computable presentation $\mathcal{G}$ of a computable torsion-free abelian group.

Acknowledgements

The first author was partially supported by a grant from the Packard Foundation through a Post-Doctoral Fellowship. The second author was partially supported by NSF DMS-0802961 and NSF DMS-1100604. The authors thanks Daniel Turetsky for allowing them to include Proposition 4.5.

References

[1] T.C. Craven, "The Boolean space of orderings of a field," *Transactions of the American Mathematical Society*, vol. 209, 1975, 225-235.
[2] M.A. Dabkowska, M.K. Dabkowski, V.S. Harizanov and A.A. Togha, "Spaces of orders and their Turing degree spectra", *Annals of Pure and Applied Logic*, vol. 161, 2010, 1134-1143.
[3] V. Dobritsa, "Some constructivizations of abelian groups," *Siberian Journal of Mathematics*, vol. 24, 1983, 167-173.
[4] R. Downey and S.A. Kurtz, "Recursion theory and ordered groups", *Annals of Pure and Applied Logic*, vol. 32, 1986, 137-151.
[5] H.M. Friedman, S.G. Simpson and R.L. Smith, "Countable algebra and set existence axioms," *Annals of Pure and Applied Logic*, vol. 25, 1983, 141-181.
[6] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford, 1963.
[7] K. Hatzikiriakou and S.G. Simpson, "WKL_0 and orderings of countable abelian groups," in *Logic and computation (Pittsburgh, PA, 1987)*, vol. 106 of *Contemporary Mathematics*, 177-180, American Mathematical Society, Providence RI, 1990.
[8] A. Kokorin and V. Kopytov, *Fully ordered groups*, Halsted Press, New York, 1974.

[9] S. Lang, *Algebra*, Revised 3rd Edition, Springer-Verlag, 2002.
[10] F.W. Levi, "Ordered groups," *Proceedings of the Indian Academy of Science, Section A*, vol. 16, 1942, 256-263.
[11] G. Metakides and A. Nerode, "Effective content of field theory", *Annals of Mathematical Logic*, vol. 17, 1979, pp. 289-320.
[12] M. Rabin, "Computable algebra, general theory and theory of computable fields," Transactions of the American Mathematical Society, vol. 95, 1960, 341-360.
[13] S.G. Simpson, *Subsystems of second order arithmetic*, Springer-Verlag, 1998.
[14] R.I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Berlin, 1989.
[15] R. Solomon, "Π^0_1 classes and orderable groups," *Annals of Pure and Applied Logic*, vol. 115, 2002, 279-302.

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: kach@math.uchicago.edu
URL: http://www.math.uchicago.edu/~kach/

Department of Mathematics, Wellesley College, Wellesley, MA 02481, USA
E-mail address: klange2@wellesley.edu
URL: http://palmer.wellesley.edu/~klange

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: solomon@math.uconn.edu
URL: http://www.math.uconn.edu/~solomon