Research Statement

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1 Introduction

My research lies at the interface of algebra and mathematical logic, focusing on the classical and effective study of algebraic structures. The classical study of algebraic structures is of course quite old. The effective study of algebraic structures is not quite as old: Van der Waerden initiated its study despite lacking the language of computability theory (see [31]), Frölich and Shepherdson formalized this work (see [10]), and Mal′cev (see [27]) and Metakides and Nerode (see [28]) began the exploration of this area in earnest. Both the classical study and the effective study share the same aim of better understanding the algebraic structures that pervade mathematics.

A primary interest of mine, and the dominant subject of this statement, is the classical and effective study of the class of Boolean algebras. Though I have studied a number of other classes of algebraic structures in the effective setting including equivalence structures (see [22]), vector spaces (see [8]), linear orders (see [18], [20], and [21]), and torsion-free abelian groups (see [1] and [7]), Boolean algebras remain the most interesting and elusive class of algebraic structures to me.

The following sections elaborate on my interest and work in Boolean algebras (see [17] and [19]), with useful classical and computability-theoretic background interspersed. In addition, I highlight related results of mine on other classes of algebraic structures.

2 Countable Boolean Algebras (Classically)

Jussi Ketonen's work was a major breakthrough in the classical study of countable Boolean algebras. As part of this work, he found natural algebraic invariants that are isomorphism invariants for countable Boolean algebras (see [23], or [13] or [29] for alternate expositions). These invariants assign to each countable Boolean algebra two countable ordinals: a *rank* and a *depth*. Roughly speaking, the *rank* of a countable Boolean algebra is a measure of size (being the largest ordinal ρ for which there is no $(\rho+1)$-atom but for which there are infinitely many disjoint ρ-atoms) and the *depth* of a countable Boolean algebra is a measure of its complexity (being the smallest ordinal δ for which the ranks of subalgebras of subalgebras of subalgebras (iterated δ many times) of a fixed subalgebra determines the isomorphism type of the fixed subalgebra within the Boolean algebra).

Lutz Heindorf demonstrated that for every countable ordinal α, there is a Boolean algebra $\mathcal{B}$ having depth α (see [29]). The set of depths of subalgebras of these Boolean algebras are downward closed, i.e., for each $\beta < \alpha$, there is a subalgebra $x \in \mathcal{B}$ with depth β. Together with Steffen Lempp, I have furthered this line of inquiry by studying what constraints are possible on the set of depths of subalgebras. For example, we have shown that the set of depths of subalgebras need not be downward closed.

Proposition 2.1 (Kach and Lempp)**.** *For each successor ordinal $\alpha+1$, there is a (rank one) depth $\alpha+2$ Boolean algebra $\mathcal{B}$ having no subalgebra of depth $\alpha+1$.*

Proposition 2.2 (Kach and Lempp)**.** *For each nonzero ordinal γ, there is a (rank ω) depth ω^γ Boolean algebra $\mathcal{B}$ having subalgebras of only depth 0, 1, and ω^γ.*

Proposition 2.3 (Kach and Lempp)**.** *For each nonzero ordinal γ, there is a (rank $\omega+1$) depth ω^γ Boolean algebra $\mathcal{B}$ having subalgebras of only depth 0, 2, and ω^γ.*

Though the depths of subalgebras may not be downward closed, there always are subalgebras with small depth.

Proposition 2.4 (Kach and Lempp). *Every Boolean algebra has a depth zero subalgebra. Every Boolean algebra with nonzero depth has a depth one or depth two subalgebra.*

It seems that there are considerably more constraints on the set of depths of subalgebras. A thorough understanding of these would offer insight into the structure of countable Boolean algebras.

Besides serving as a method to better understand countable Boolean algebras, classical invariants also serve as a tool to answer questions about countable Boolean algebras. For example, it is a consequence of Ershov-Tarksi invariants (an elementary equivalence invariant) that the class of Boolean algebras satisfies Vaught's Conjecture (see [14]). Indeed, quite a bit more is true.

Theorem 2.5 (Camerlo and Gao [3]). *If φ is a (first-order) sentence in the language of Boolean algebras extending the theory of Boolean algebras, then φ has continuum many models if it has more than one model. Moreover, the isomorphism relation restricted to these models is Borel complete in this case.*

Steffen Lempp and I have been working to verify Vaught's Conjecture for Boolean algebras in the infinitary language $L_{\omega_1\omega}$.

Conjecture 2.6. *If φ is a sentence in the infinitary language $L_{\omega_1\omega}$ in the language of Boolean algebras extending the theory of Boolean algebras, then φ has continuum many models if it has uncountably many models.*

The set of such infinitary sentences φ has been partitioned into three classes related to the ranks and depths of the models of φ. For two of these classes, the conjecture has been verified in a strong form (the isomorphism problem is Borel complete if there are uncountably many models). For the remaining class, progress has been made and the isomorphism problem is known to not be Borel complete.

3 Countable Boolean Algebras (Effectively)

Crucial to the effective study of Boolean algebras, and effective algebra more generally, are the notions of computable functions and computable structures.

Definition 3.1. A function $f : \omega^k \to \omega$ or relation $R \subseteq \omega^k$ is *computable* if there is a Turing Machine computing it.

Definition 3.2. A countably infinite algebraic structure (e.g., a linear order, a group, a field) is *computable* if its universe can be identified with the natural numbers $\omega = \{0, 1, 2, \dots\}$ in such a way that the functions and relations become *computable* operations on ω^k (for the appropriate arity k).

Less formally, a function or relation is *computable* if it is intuitively computable, i.e., if it can be computed by a natural algorithm. Similarly, a structure is *computable* if it is intuitively computable. For example, the integers $\mathbb{Z}$ under addition is an example of a computable structure. By identifying the nonnegative integer n with the natural number $2n$ and the negative integer $-n$ with the natural number $2n + 1$, the integers $\mathbb{Z}$ are nicely identified with the natural numbers ω. Moreover, given natural numbers a and b, it is possible to (effectively) recover the integers z_a and z_b they code, compute the integer $z_a + z_b$, and determine the natural number coding the integer $z_a + z_b$.

A major aim of effective algebra is to understand which algebraic objects are computable and, when they are not computable, how much extra computational power is necessary to describe them. My research has centered on this idea, particularly trying to understand how structure imposes constraints on what extra computational information (if any) is necessary. The *degree spectra* of an algebraic object attempts to capture the amount of extra computational information required.

Definition 3.3. The *degree spectra* of an algebraic structure $\mathcal{A}$ is the set of degrees

$$\mathrm{DegSpec}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\},$$

where an algebraic structure $\mathcal{A}$ is $\mathbf{d}$-*computable* if its universe can be identified with ω in such a way that the functions and relations become $\mathbf{d}$-computable operations on ω.

3.1 Computable Boolean Algebras

A fundamental question about the degree spectra of a Boolean algebra is whether it is computable, i.e., whether $\mathbf{0} \in \mathrm{DegSpec}(\mathcal{A})$. Historically, the major line of study for Boolean algebras has been demonstrating if $\mathrm{DegSpec}(\mathcal{A})$ has "almost" computable members, then it also contains $\mathbf{0}$. For example, the following is a well-known open question asked by Downey and Jockusch over fifteen years ago (see [5]). Recall a structure is low$_n$ if it is $\mathbf{a}$-computable for some low$_n$ degree $\mathbf{a}$, i.e., a degree $\mathbf{a}$ satisfying $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$.

Conjecture 3.4 (The Low$_n$ Conjecture)**.** *Every low$_n$ Boolean algebra is computable.*

Though the conjecture is known for $n \in \{1, 2, 3, 4\}$ (see [25], and also [5] and [30]), there seems to be a significant obstacle at $n = 5$. Making use of Ketonen invariants and taking a different approach, I have shown a positive answer to the Low$_n$ Conjecture for the class of depth zero, rank ω Boolean algebras by isolating a necessary and sufficient condition for such a Boolean algebra to be computable.

Theorem 3.5 (Kach [17] and [19])**.** *There is a natural two-to-one map between depth zero, rank ω Boolean algebras and nonempty sets $S \subseteq \omega$.*

Definition 3.6 (Feiner [9])**.** A set $S \subseteq \omega$ is $\Sigma^0_{(2n+3)}$ in the Feiner Hierarchy if there is an index e so that $\varphi_e^{\emptyset^{(2n+2)}}(n)\downarrow$ if and only if $n \in S$.

Theorem 3.7 (Kach [17] and [19])**.** *A depth zero, rank ω Boolean algebra is computable if and only if the set it encodes is $\Sigma^0_{(2n+3)}$ in the Feiner Hierarchy.*

Corollary 3.8 (Kach [17] and [19])**.** *If a depth zero, rank ω Boolean algebra is low$_n$ for some $n \in \omega$, then it is computable.*

Though analogous results exist for a variety of other classes of algebraic structures, Theorem 3.7 remains the only algebraic characterization of a class of computable Boolean algebras. Though a major goal is to settle Conjecture 3.4 and understand which countable Boolean algebras are computable, as steps I hope to answer the following conjecture and question.

Conjecture 3.9. *A depth zero Boolean algebra is computable if and only if the set $S \subseteq \omega_1$ it encodes is $\Sigma^0_{(2\alpha+3)}$ in the Feiner Hierarchy.*

Question 3.10. *Which depth ω, rank one Boolean algebras are computable?*

3.2 Jump Degrees

More generally, an aim of effective algebra is to better understand what degree spectra can occur within a class of algebraic structures. As the Low$_n$ Conjecture (for $n \in \{1, 2, 3, 4\}$) demonstrates, there are often severe restrictions. Jump degrees can help capture these restrictions.

Definition 3.11. If $\mathcal{A}$ is any countable structure, α is any computable ordinal, and $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is any degree, then $\mathcal{A}$ has α^{th} *jump degree* $\mathbf{a}$ if the set

$$\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$$

has $\mathbf{a}$ as its least element. The structure $\mathcal{A}$ is said to have *α^{th} jump degree.*

A structure $\mathcal{A}$ has *proper α^{th} jump degree* $\mathbf{a}$ if $\mathcal{A}$ has α^{th} jump degree $\mathbf{a}$ but not β^{th} jump degree for any $\beta < \alpha$. The structure $\mathcal{A}$ is said to have *proper α^{th} jump degree.*

For some classes of algebraic structures, any (proper) jump degree can be realized; for other classes of algebraic structures, this is not the case.

Theorem 3.12 (Andersen, Kach, Melnikov, and Solomon [1])**.** *For each computable ordinal α and degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$, there is a torsion-free abelian group having proper α^{th} jump degree $\mathbf{a}$.*

Theorem 3.13 (Downey and Knight [6], see also [2], [15], and [24])**.** *If a linear order has degree, it must be $\mathbf{0}$. If a linear order has first jump degree, it must be $\mathbf{0}'$. For each computable ordinal $\alpha \geq 2$ and degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$, there is a linear order having proper α^{th} jump degree $\mathbf{a}$.*

Theorem 3.14 (Jockusch and Soare [16])**.** *If a Boolean algebra has n^{th} jump degree (for any $n \in \omega$), it must be $\mathbf{0}^{(n)}$. For each $\mathbf{a} \geq \mathbf{0}^{(\omega)}$, there is a Boolean algebra with proper ω^{th} jump degree $\mathbf{a}$.*

Roughly speaking, these results indicate that it is "easy" to encode information into torsion-free abelian groups but "difficult" to encode information into Boolean algebras. Restrictions on the existence or nonexistence of proper α^{th} jump degrees for Boolean algebras for ordinals $\alpha > \omega$ are not known. Settling the following conjecture is a project I am very much interested in.

Conjecture 3.15. *Fix an ordinal $\alpha > \omega$ and a degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$. Then there is a Boolean algebra having proper α^{th} jump degree $\mathbf{a}$ if and only if α is a limit ordinal.*

3.3 Isomorphisms

In addition to studying the degree spectra of algebraic structures, effective algebra also studies how different presentations of the same structure can be. Various notions of effective categoricity capture this idea.

Definition 3.16. An algebraic structure is *computably categorical* (*Δ^0_n-categorical*, *arithmetically categorical*) if there is a computable isomorphism (Δ^0_n-isomorphism, arithmetic isomorphism) between any two computable presentations $\mathcal{A}$ and $\mathcal{B}$ of the structure.

An algebraic structure is *relatively computably categorical* (*relatively Δ^0_n-categorical*, *relatively arithmetically categorical*) if there is an $(\mathcal{A} \oplus \mathcal{B})$-isomorphism ($\Delta^0_n(\mathcal{A} \oplus \mathcal{B})$-isomorphism, $\Delta^0_n(\mathcal{A} \oplus \mathcal{B})$-isomorphism for some n) between any two presentations $\mathcal{A}$ and $\mathcal{B}$ of the structure.

It is not difficult to classify the computably categorical and the relatively computably categorical Boolean algebras.

Theorem 3.17 (Goncharov [11] and LaRoche [26])**.** *A Boolean algebra is computably categorical (relatively computably categorical) if and only if it has at most finitely many atoms.*

The classification of the Δ^0_n-categorical Boolean algebras is wide open for sufficiently large n. The classification of the arithmetically categorical Boolean algebras is a question that I am actively working on settling.

Conjecture 3.18. *A Boolean algebra $\mathcal{B}$ is arithmetically categorical (relatively arithmetically categorical) if and only if it is finitary.*

If Conjecture 3.18 were true, it would be interesting for two reasons. First, it would provide another pleasing description of the *finitary* algebras. This class can be defined in terms of topology (clopen algebras of compact zero-dimensional metric spaces of finite type), in terms of iterated constructions (finite sums of finite iterations of the T and F operators), in terms of decompositions (primitive algebras with finite diagram), in terms of model theory (countably categorical weak second order theories), and in terms of algebraic invariants (finite depth and finite rank) (see [13]). The existence of a computability-theoretic characterization of the finitary Boolean algebras would futher demonstrate the robustness of this subclass of Boolean algebras. Second, with the following conjecture (for which I have little evidence), it would demonstrate that effective categoricity and relative effective categoricity do not differ within the class of Boolean algebras.

Conjecture 3.19. *For every computable ordinal α, a Boolean algebra is Δ^0_α-categorical if and only if it is relatively Δ^0_α-categorical.*

This would be remarkably different from the behavior for arbitrary algebraic structures.

Theorem 3.20 (Chisholm, Fokina, Goncharov, Harizanov, Knight, and Miller [4])**.** *For every computable ordinal α, there is a Δ^0_α-categorical structure that is not relatively Δ^0_α-categorical.*

Theorem 3.21 (Downey, Kach, Lempp, and Turetsky)**.** *There is a computably categorical structure that is not relatively arithmetically categorical.*

4 Uncountable Boolean Algebras

Though my study of algebraic structures has largely been restricted to the countable setting, recently I have expanded my research to include the uncountable setting. Noam Greenberg and Julia Knight have developed a framework for the effective algebra and computable model theory of algebraic structures of size $\aleph_1$ (see [12]). Within this framework, together with Noam Greenberg, Steffen Lempp, and Daniel Turetsky, I have studied linear orders.

A number of striking differences between the countable setting and the uncountable setting have emerged. Whereas $\mathbf{0}$ and $\mathbf{0}'$ are the only possible degree and first jump degree of a linear order in the countable setting, uncountable linear orders fail to have such constraints.

Theorem 4.1 (Greenberg, Kach, Lempp, and Turetsky)**.** *For every ordinal α and degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$, there is an uncountable linear order $\mathcal{L}$ having proper α^{th} jump degree $\mathbf{d}$.*

In the countable setting, a linear order is computably categorical if and only if it has at most finitely many adjacencies. In the uncountable setting, it would be natural to expect the condition to be at most countably many adjacencies. Perhaps surprisingly, this is not necessary nor sufficient.

Theorem 4.2 (Greenberg, Kach, Lempp, and Turetsky)**.** *An uncountable linear order $\mathcal{L}$ is computably categorical if and only if there is a countable subset X and computably enumerable sets $\{S_n\}_{n\in\omega}$ for which*

- *every cut in $\mathcal{L}$ determined by X is either finite (possibly empty) or has order type η_1*
- *every nonempty cut in $\mathcal{L}$ determined by X appears in exactly one S_n*
- *every cut of size n in $\mathcal{L}$ determined by X appears in S_n*

My intention is to next study the effective algebra of uncountable Boolean algebras. A natural aim would be to discover similarities and differences between the countable setting and the uncountable setting. The following are two questions of particular interest to me.

Question 4.3 (The Low$_n$ Conjecture)**.** *If an uncountable Boolean algebra has a low$_n$ presentation for some n (or $n \in \{1, 2, 3, 4\}$), must it be computable?*

Question 4.4. *For each ordinal α, what are the possible (proper) α^{th} jump degrees of uncountable Boolean algebras?*

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