# Boolean Algebras and Ketonen Invariants

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# Outline

# Preliminaries

## Remark

*Recall that* $\emptyset^{(\omega)}$ *is defined as the set*

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## Remark

*Note that $\emptyset^{(n)} \equiv_T \emptyset^{(\leq n)}$ and $\emptyset^{(\leq i)} \subseteq \emptyset^{(\leq j)} \subset \emptyset^{(\omega)}$ whenever $i \leq j < \omega$.*

# Feiner Hierarchy

## Definition (Feiner)

*A set $S \subseteq \omega$ is* $(a, b)$ in the Feiner hierarchy *if there exists an index e such that*

1. *The function $\varphi_e^{\emptyset^{(\omega)}}$ is total and is the characteristic function of S, i.e., $\varphi_e^{\emptyset^{(\omega)}}(n) \downarrow = \chi_S(n)$ for all n.*
2. *The computations $\varphi_e^{\emptyset^{(\leq bn+a)}}(n)$ and $\varphi_e^{\emptyset^{(\omega)}}(n)$ are equal; in particular, neither queries any number $\langle k, m \rangle$ with $k > bn + a$.*

## Remark

*In other words, a set $S \subseteq \omega$ is $(a, b)$ in the Feiner hierarchy if membership of n in S can be determined uniformly from the oracle $\emptyset^{(bn+a)}$.*

# Feiner Σ-Hierarchy

## Definition (K)

*A set $S \subseteq \omega$ is $\Sigma^0_{n \mapsto bn+a}$* in the Feiner Σ-hierarchy *if there exists an index e such that*

1. *The set S satisfies $S = W_e^{\emptyset^{(\omega)}}$, i.e., $n \in S$ if and only if $\varphi_e^{\emptyset^{(\omega)}}(n) \downarrow$.*
2. *The computations $\varphi_e^{\emptyset^{(\leq bn+a-1)}}(n)$ and $\varphi_e^{\emptyset^{(\omega)}}(n)$ are equal; in particular, neither queries any number $\langle k, m \rangle$ with $k > bn + a$.*

## Remark

*In other words, a set $S \subseteq \omega$ is $\Sigma^0_{n \mapsto bn+a}$ in the Feiner Σ-hierarchy if membership of n in S is a $\Sigma^0_{bn+a}$ question uniformly in n.*

# Depth Zero Characterization

**Theorem (K)**

*Let $S \subseteq \omega + 1$ be a set with greatest element. Then the following are equivalent:*

1. *The depth zero Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The depth zero Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner $\Sigma$-hierarchy.*

# Proof of Theorem (Sketch)

**Proof of (1), (2) $\implies$ (3).**

Uniformly in $n$ define $\Sigma^0_{2n+3}$ sentences $\varphi_n$ satisfying

$$\mathcal{B}_{u(S)}, \mathcal{B}_{v(S)} \models \varphi_n \qquad \text{if and only if} \qquad n \in S.$$

When defining these formulas, make use of the fact that there are formulas of complexity $\Pi^0_{2\alpha+1}$ (uniformly in $\alpha$) identifying whether an element is an $\alpha$-atom. □

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## Proof of $(3) \implies (1), (2)$.

From an index $e$ witnessing that $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner $\Sigma$-hierarchy, build computable copies of $\mathcal{B}_{u(S)}$ and $\mathcal{B}_{v(S)}$ by building a linear order of the form $\mathcal{L} = \sum_{\tau \in 2^{<\omega}} \mathcal{L}_\tau$.

Relativize and iterate the following technical lemma. □

# Technical Lemma

## Lemma (K)

*There is a procedure, uniform in*

- *a* $\Delta^0_3$ *index for the atomic diagram* $D(\mathcal{A})$ *of a linear order* $\mathcal{A} = (A : \prec) = (\{a_0, a_1, \dots\} : \prec)$ *with distinguished least element* $a_0$ *and*
- *an index for a* $\Sigma^0_3$ *predicate* $\exists n \forall u \exists v R(n, u, v)$

*which yields a* $\Delta^0_1$ *linear order* $\mathcal{L}$ *such that* $\mathcal{L} \cong \sum_{a \in A} \mathcal{L}_a$*, where*

$$\mathcal{L}_{a_n} \cong \begin{cases} 1 + \eta + \omega & \text{if } \forall u \exists v R(n, u, v), \\ \omega & \text{otherwise.} \end{cases}$$

# Ketonen Invariants

**Remark**

*For intuition, it is possible to describe the isomorphism type of a linear order by specifying the location of well-orders and the reverse of well-orders with respect to each other.*

*In a similar manner, it is possible to describe the isomorphism type of a Boolean algebra by specifying the location of $\alpha$-atoms with respect to all other $\beta$-atoms. Ketonen invariants do just this.*

# Interval Algebras

**Definition**

*Let $\mathcal{L}$ be a linear order with least element $x_0$ with the topology generated by basic open sets $[a, b)$. The* interval algebra of $\mathcal{L}$, *denoted $\mathcal{B}_{\mathcal{L}}$, is the Boolean algebra whose universe is the set of clopen subsets of $\mathcal{L}$. The operations of join, meet, and complementation in the Boolean algebra $\mathcal{B}_{\mathcal{L}}$ are given by taking the union, intersection, and complementation of the clopen sets.*

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**Definition**

*If $\mathcal{B}$ is a Boolean algebra, then $\mathcal{L}$ is* a linear order that generates *$\mathcal{B}$ if $\mathcal{B} \cong \mathcal{B}_{\mathcal{L}}$.*

# Cantor-Bendixson Derivatives

## Definition

*Let $\mathcal{L}$ be a linear order with least element $x_0$ under the half-open topology, i.e., the topology with basic open sets $[a, b)$. The* Cantor-Bendixson derivative of $\mathcal{L}$*, denoted $\mathcal{L}'$, is the linear order with universe*

$$\{x_0\} \cup \{x \in \mathcal{L} : x \text{ is not isolated in } \mathcal{L}\}$$

*if $\mathcal{L}$ is infinite and empty universe if $\mathcal{L}$ is finite.*

## Remark

*We emphasize that the given definition of the Cantor-Bendixson derivative differs from the standard definition.*

# Cantor-Bendixson Derivatives

## Definition

*The* $\alpha^{\text{th}}$ Cantor-Bendixson derivative*, denoted* $\mathcal{L}^{(\alpha)}$*, is defined recursively by* $\mathcal{L}^{(0)} = \mathcal{L}$*,* $\mathcal{L}^{(\alpha+1)} = (\mathcal{L}^{(\alpha)})'$*, and* $\mathcal{L}^{(\gamma)} = \cap_{\beta<\gamma}\mathcal{L}^{(\beta)}$ *for limit ordinals* $\gamma$*.*

# Cantor-Bendixson Derivatives

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### Definition

*The* Cantor-Bendixson rank of a linear order $\mathcal{L}$ *is the least ordinal* $\nu$ *such that* $\mathcal{L}^{(\nu)} = \mathcal{L}^{(\nu+1)}$ *if such an ordinal exists. The* perfect kernel of $\mathcal{L}$ *are the points in* $\mathcal{L}^{(\nu)}$.

# Rank Functions

## Definition (Ketonen)

*The* rank function *for a linear order $\mathcal{L}$ of rank $\nu$ is the map $r_{\mathcal{L}} : \mathcal{L}^{(\nu)} \to \omega_1$ given by*

$$r(x) = \min\left\{\beta : x \notin \overline{(\mathcal{L}^{(\beta)} \setminus \mathcal{L}^{(\nu)})}\right\},$$

*i.e., the rank of $x$ is the minimum ordinal $\beta$ such that $x$ is not in the closure of $\mathcal{L}^{(\beta)} \setminus \mathcal{L}^{(\nu)}$.*

## Remark

*For a point $x$ in the perfect kernel of $\mathcal{L}$, the rank function describes the number of Cantor-Bendixson derivatives required until $x$ is no longer a limit of points not in the perfect kernel.*

# Measures

**Definition (Ketonen)**

*A measure $\sigma$ is a map from the countable atomless Boolean algebra $\mathcal{F}$ to the countable ordinals satisfying*

$$\sigma(x+y) = \max\{\sigma(x), \sigma(y)\}$$

*and $\sigma(0) = o$, where $o$ is a special symbol satisfying $o < \alpha$ for all ordinals $\alpha$.*

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## Remark

*By associating the countable atomless Boolean algebra with finite unions of cones of $2^{<\omega}$, a measure can be viewed as a map $\sigma : 2^{<\omega} \to \omega_1$ satisfying $\sigma(\tau) = \max\{\sigma(\tau^\frown 0), \sigma(\tau^\frown 1)\}$. Under this interpretation, a measure can be thought of as a labelled binary tree.*

# Measures from Rank Functions

**Definition (Ketonen)**

*Let $\mathcal{L}$ be a linear order with non-empty perfect kernel, which we identify with $[0,1)\cap\mathbb{Q}$, whose interval algebra we in turn identify with $\mathcal{F}$.*

*The measure $\sigma = \sigma_r$ associated to the rank function $r = r_{\mathcal{L}}$ is the map $\sigma : \mathcal{F} \to \omega_1$ given by*

$$\sigma(x) = \sup\{r(p) : p \in x\}$$

*for non-zero $x \in \mathcal{F}$ and $\sigma(0) = o$.*

# Measure Derivatives

**Definition (Ketonen)**

*If $\sigma : \mathcal{F} \to \omega_1$ is a measure, define maps $\Delta^\alpha\sigma$ with domain $\mathcal{F}$ for $\alpha < \omega_1$ recursively by setting $\Delta^0\sigma = \sigma$,*

$$\Delta^{\alpha+1}\sigma(x) = \{(\Delta^\alpha\sigma(x_1), \dots, \Delta^\alpha\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\},$$

*and $\Delta^\gamma\sigma(x)$ as the inverse limit of $\Delta^\beta\sigma(x)$ for $\beta < \gamma$.*

*The set $\Delta^\alpha\sigma(1_{\mathcal{B}})$ is the $\alpha^{\text{th}}$* derivative of *$\mathcal{B}_\sigma$.*

# Measure Depth

## Remark

*Recall the definitions* $\Delta^0\sigma(x) = \sigma(x)$ *and*
$\Delta^{\alpha+1}\sigma(x) = \{(\Delta^\alpha\sigma(x_1), \ldots, \Delta^\alpha\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\}$.

## Definition (Ketonen)

*The depth of a measure* $\sigma$, *denoted* $\delta = \delta_\sigma$, *is the least ordinal* $\alpha$ *such that*

$$\Delta^\alpha\sigma(x) = \Delta^\alpha\sigma(y) \implies \Delta^{\alpha+1}\sigma(x) = \Delta^{\alpha+1}\sigma(y)$$

*if such an ordinal exists.*

# Measure Depth

## Remark

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*if such an ordinal exists.*

## Theorem (Ketonen)

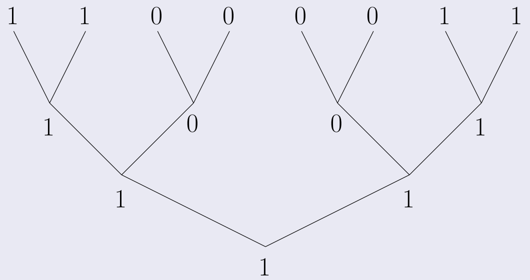
*For every measure* $\sigma$, *the depth* $\delta_\sigma$ *exists and is a countable ordinal. Moreover, the derivative* $\Delta^{\delta+2}\sigma(1)$ *is an isomorphism invariant of* $\mathcal{B}_\sigma$.

# Classical Depth Zero

## Remark

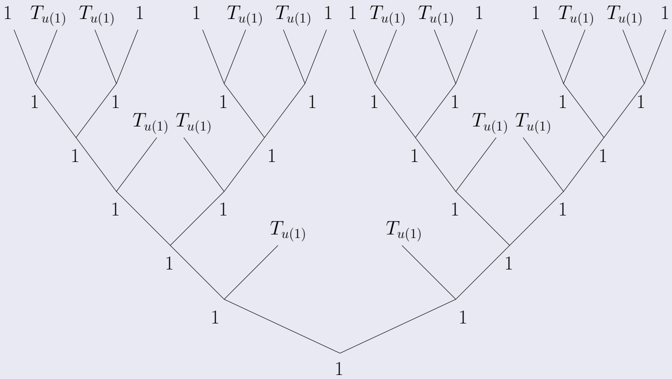
*Recall that a measure $\sigma$ is depth zero if*
$\sigma(x) = \sigma(y) \implies \Delta\sigma(x) = \Delta\sigma(y)$, *i.e.*,
$\sigma(x) = \sigma(y) \implies \{(\sigma(x_0), \ldots, \sigma(x_n))\} = \{(\sigma(y_0), \ldots, \sigma(y_n))\}$.

## Example



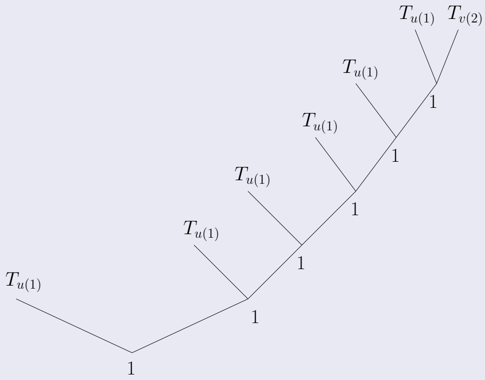
# Depth Zero Measures

## Example (The measure $\sigma_{u(\{0,1\})}$)



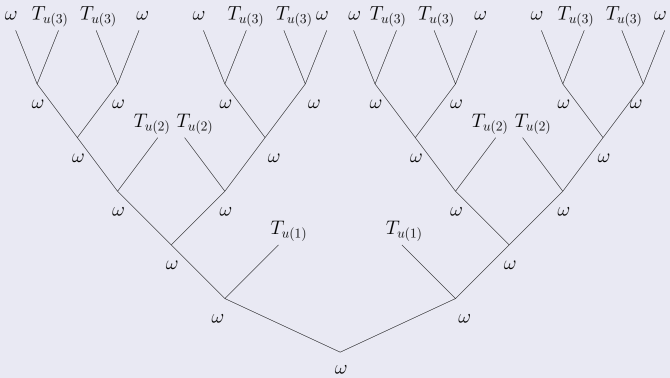
# Depth Zero Measures

## Example (The measure $\sigma_{v(\{0,1\})}$)



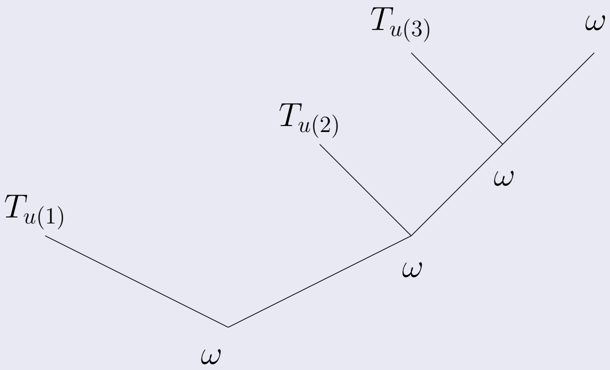
# Depth Zero Measures

## Example (The measure $\sigma_{u(\{0,1,2,\ldots,\omega\})}$)



# Depth Zero Measures

## Example (The measure $\sigma_{V(\{0,1,2,\ldots,\omega\})}$)



# Depth Zero Theorems

## Theorem (K)

*For each set $S \subseteq \omega_1$ satisfying $|S| = 1$, there is exactly one depth zero Boolean algebra with range $S$, namely $B_{u(S)} = B_{v(S)}$.*

*For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are exactly two depth zero Boolean algebras with range $S$, namely $B_{u(S)}$ and $B_{v(S)}$.*

## Theorem (K)

*Let $S \subseteq \omega + 1$ be a set with greatest element. Then the following are equivalent:*

1. *The depth zero Boolean algebra $\mathcal{B}_{u(S)}$ is computable.*
2. *The depth zero Boolean algebra $\mathcal{B}_{v(S)}$ is computable.*
3. *The set $S \setminus \{\omega\}$ is $\Sigma^0_{n \mapsto 2n+3}$ in the Feiner $\Sigma$-hierarchy.*

# Other Classical Results

**Theorem (K)**

*There are continuum many depth one, rank $\omega$ Boolean algebras with range exactly $\omega + 1$.*

**Theorem (K)**

*There are continuum many depth $\omega$, rank one Boolean algebras.*

**Remark**

*The proof of the latter theorem involves defining an explicit map from the space of measures to the space of rank one measures.*

# Other Effectiveness Results

## Proposition

*If $\sigma$ is computable, then $\Delta\sigma(1)$ is computably enumerable.*

## Proposition (K)

*If $\Delta\sigma(1)$ is computably enumerable, then there is a computable measure $\hat{\sigma}$ satisfying $\Delta\sigma(1) = \Delta\hat{\sigma}(1)$.*

*Moreover, if $\sigma$ is depth zero, then $\hat{\sigma}$ can be depth zero.*

## Corollary (K)

*If $\mathcal{B}_\sigma$ is a depth zero Boolean algebra, then $\sigma$ is computable if and only if $\Delta\sigma(1)$ is computably enumerable.*

*Moreover, there is a procedure that, given an index for $\sigma$, gives an index for $\Delta\sigma(1)$, and vice versa.*

# Bibliography


Feiner, Lawrence.
Hierarchies of Boolean algebras.
*Journal of Symbolic Logic*, 35:365-374, 1970.

Kach, Asher.
Boolean Algebras and Ketonen Invariants.
In preparation.

Ketonen, Jussi.
The structure of countable Boolean algebras.
*Annals of Mathematics*, 108(1):41-89, 1978.

Pierce, R.S.
Countable Boolean algebras.
*Handbook of Boolean Algebras*, Vol. 3:775-876, 1989.