Embeddings of Computable Linear Orders

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Outline

Definition

A infinite order type $\mathcal{L}$ is *computable* if it has a *computable presentation*, i.e., if there is a computable binary relation $\prec$ on ω such that $\mathcal{L} \cong (\omega : \prec)$.

If $\mathcal{L}_1 = (L_1 : \prec_1)$ and $\mathcal{L}_2 = (L_2 : \prec_2)$ are computable presentations of computable linear orders, then an embedding $\pi : L_1 \to L_2$ is *computable* if π is computable as a function.

Review of Computable Model Theory

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Theorem (Folklore)

Uniformly in an index for a Δ^0_3 linear order $\mathcal{L}$ with distinguished least element, there is an index for a computable presentation of the linear order $\omega \cdot \mathcal{L}$.

Classical But Not Computable

Theorem (Folklore)

If $\mathcal{L}$ is an infinite order type, then at least one of ω or ω^* classically embeds.

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Theorem (Denisov; Tennenbaum; Lerman)

The order types ω, ω^*, $\omega + \omega^*$, and $\omega + \zeta \cdot \eta + \omega^*$ form a bases for computable presentations of computable linear orders. In other words, if $\mathcal{L} = (L :\prec)$ is any computable presentation of a computable linear order, there is a computable subset of order type one of these.

A Precursor

Theorem (Denisov; Tennenbaum)

There is a computable presentation of the order type $\omega + \omega^*$ *such that neither* ω *nor* ω^* *computably embeds.*

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Proof.

We construct a computable presentation of the order type $\omega + \omega^*$ meeting the following requirements $\mathcal{R}_e$.

$\mathcal{R}_e$: *If W_e is infinite, then $W_e \not\subseteq \omega$ and $W_e \not\subseteq \omega^*$.*

We meet $\mathcal{R}_e$ by putting one element of W_e into ω and one element into ω^*. To facilitate this, we maintain a virtual *fence* separating these points with priority e. Note that if a higher priority fence prevents us from separating points in W_e, we can wait for additional points to be enumerated; if none appear, then we win as $|W_e| < \infty$. $\square$

The Question

Remark

It is natural to ask what can be said about the effectiveness of embeddings of $\mathcal{L}_1$ into $\mathcal{L}_2$, allowing the presentations of $\mathcal{L}_1$ and $\mathcal{L}_2$ to vary to minimize the complexity of the embedding.

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Question

Are there computable linear orders $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1$ classically embeds into $\mathcal{L}_2$ but for no computable presentations of $\mathcal{L}_1$ and $\mathcal{L}_2$ does $\mathcal{L}_1$ computably embed into $\mathcal{L}_2$?

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Remark

Of particular (and natural) interest are the special cases when $\mathcal{L}_1 = \eta$ and $\mathcal{L}_1 = \omega^*$.

Outline

The Goal and the Strategy

Remark

The goal is to produce a computable non-scattered linear order $\mathcal{L}$ such that η does not computably embed into any computable presentation of $\mathcal{L}$.

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The strategy to produce a computable non-scattered linear order that is intrinsically computably scattered will be to encode trees T into linear orders $\mathcal{L}_T$ in such a way that any embedding of η into $\mathcal{L}_T$ gives information about an infinite path through T in a fairly effective manner.

By choosing T simple enough so that $\mathcal{L}_T$ is computable but complex enough so that its paths are complicated, we obtain an appropriate linear order.

The map $T \mapsto \mathcal{L}_T$ depends on the goal.

The Encoding for $T \subseteq 2^{<\omega}$

Definition

If $T \subseteq 2^{<\omega}$ is any tree, define linear orders $\mathcal{L}_{\langle \sigma,\tau \rangle}$ via corecursion by

$$\begin{aligned}\mathcal{L}_{\langle \sigma,\tau \rangle} = \langle \sigma, \tau \rangle &+ \hat{\mathcal{L}}_{\langle \sigma ^\frown 0, \tau ^\frown 0 \rangle} + \hat{\mathcal{L}}_{\langle \sigma ^\frown 0, \tau ^\frown 1 \rangle} \\ &+ \hat{\mathcal{L}}_{\langle \sigma ^\frown 1, \tau ^\frown 0 \rangle} + \hat{\mathcal{L}}_{\langle \sigma ^\frown 1, \tau ^\frown 1 \rangle} + \langle \sigma, \tau \rangle\end{aligned}$$

where

$$\hat{\mathcal{L}}_{\langle \sigma ^\frown i, \tau ^\frown j \rangle} = \begin{cases} \zeta + \mathcal{L}_{\langle \sigma ^\frown i, \tau ^\frown j \rangle} + \zeta & \text{if } \sigma ^\frown i \in T, \\ \zeta & \text{otherwise.} \end{cases}$$

Define $\mathcal{L}_T$ to be the linear order $\mathcal{L}_{\langle \epsilon, \epsilon \rangle}$, where ϵ denotes the empty string.

A Theorem

Theorem

There is a computable, non-scattered, rank two linear order $\mathcal{L}$ that is intrinsically computably scattered.

Proof.

Let $T \subseteq 2^{<\omega}$ be any infinite Δ^0_3 tree with no Δ^0_3 paths. Then $\mathcal{L}_T$ is computable, non-scattered, rank two, and intrinsically computably scattered. $\square$

$\mathcal{L}_T$ is Computable

Claim

If $T \subseteq 2^{<\omega}$ is Δ^0_3, then $\mathcal{L}_T$ is computable.

Remark

Recall that $\mathcal{L}_T$ was effectively defined in terms of the linear orders

$$\hat{\mathcal{L}}_{\langle \sigma ^\frown i, \tau ^\frown j \rangle} = \begin{cases} \zeta + \mathcal{L}_{\langle \sigma ^\frown i, \tau ^\frown j \rangle} + \zeta & \text{if } \sigma ^\frown i \in T, \\ \zeta & \text{otherwise.} \end{cases}$$

Proof.

Let $\exists k \forall m \exists n R(\sigma, k, m, n)$ be a Σ^0_3 predicate for T. Build $\hat{\mathcal{L}}_{\langle \sigma, \tau \rangle}$ by building a sum $\cdots + \hat{\mathcal{L}}_3 + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_2 + \hat{\mathcal{L}}_4 + \ldots$ and attempting to build $\mathcal{L}_{\langle \sigma, \tau \rangle}$ at each $\hat{\mathcal{L}}_k$, adding additional points to $\hat{\mathcal{L}}_k$ only when a new witness n for the next m appears. $\square$

$\mathcal{L}_T$ is Non-Scattered

Claim

If $T \subseteq 2^{<\omega}$ is infinite (i.e., has an infinite path), then $\mathcal{L}_T$ is non-scattered.

Proof.

Let $X \subseteq T$ be an infinite path. Define an embedding $\pi : 2^{<\omega} \to \mathcal{L}_T$ by recursion.

Define $\pi(\epsilon)$ *to be any point in either one of the* $\mathcal{L}_{\langle \epsilon,\epsilon \rangle}$ *copies of* ζ *between* $\mathcal{L}_{\langle x(0),0 \rangle}$ *and* $\mathcal{L}_{\langle x(0),1 \rangle}$.

Define $\pi(\rho^\frown i)$ *to be any element in either one of the* $\mathcal{L}_{\langle x(0)\ldots x(|\rho|),\rho^\frown i \rangle}$ *copies of* ζ *between* $\mathcal{L}_{\langle x(0)\ldots x(|\rho|+1),\rho^\frown i^\frown 0 \rangle}$ *and* $\mathcal{L}_{\langle x(0)\ldots x(|\rho|+1),\rho^\frown i^\frown 1 \rangle}$.

$\square$

$\mathcal{L}_T$ is Intrinsically Computably Scattered

Claim

If $T \subseteq 2^{<\omega}$ is an infinite Δ_3^0 tree with no Δ_3^0 path, then $\mathcal{L}_T$ is intrinsically computably scattered.

Proof.

If there were a computable embedding $\pi : \eta \to \mathcal{L}_T$, then we could recover a Δ_3^0 path in T. Specifically, determining whether a set of elements form a maximal block is Π_2^0. Starting with $\rho_0 = \epsilon = \tau_0$, we set $\rho_{s+1} = \rho_s {}^\frown i$ and $\tau_{s+1} = \tau_s {}^\frown j$, where $i, j \in \{0, 1\}$ are such that there exists two maximal blocks of size $\langle \rho^\frown i, \tau^\frown j \rangle$ with the range of π containing at least two points in this interval. $\square$

A Theorem

Theorem

There is a computable, non-scattered, rank two linear order $\mathcal{L}$ that is intrinsically computably scattered.

Proof.

Let $T \subseteq 2^{<\omega}$ be any infinite Δ^0_3 tree with no Δ^0_3 paths. Then $\mathcal{L}_T$ is computable, non-scattered, rank two, and intrinsically computably scattered. $\square$

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The goal is to produce a computable non-well-ordered linear order $\mathcal{L}$ such that ω^* does not computably embed into any computable presentation of $\mathcal{L}$.

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Remark

The strategy to produce a computable non-well-ordered linear order that is intrinsically computably well-ordered will be to encode functions F into linear orders $\mathcal{L}_F$ in such a way that

- Any descending chain in $\mathcal{L}_F$ is (almost) cofinal [downwards] in $\mathcal{L}_F$.
- The linear order $\mathcal{L}_F$ is not computable.
- The linear order $\omega^\omega + \mathcal{L}_F$ is computable.

Again, the map $F \mapsto \mathcal{L}_F$ depends on the goal.

The Encoding for $F : \omega \longrightarrow \omega$

Definition

If $F : \omega \rightarrow \omega$ is a function with infinite support, define the linear order $\mathcal{L}_F$ by

$$\mathcal{L}_F = \cdots + \omega^n \cdot F(n) + \cdots + \omega^2 \cdot F(2) + \omega \cdot F(1) + F(0).$$

A Theorem

Theorem

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered.

Proof.

Let F be any $\Delta^0_{(2n+1)}$-limit infimum function such that $\mathcal{L}_F$ is not computable. Then the linear order $\omega^\omega + \mathcal{L}_F$ is computable, non-well-ordered, scattered, rank $\omega + 1$, and intrinsically computably well-ordered. $\square$

Limit Infimum Functions

Definition

A function $F : \omega \to \omega$ is a *limit infimum function* if there is a total computable function $f : \omega \times \omega \to \omega$ such that

$$F(n) = \liminf_s f(n, s)$$

for all n.

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Definition

A function $F : \omega \to \omega$ is a $\Delta^0_{(2n+1)}$-*limit infimum function* if there is a functional $\varphi_e : \omega \times \omega \to \omega$ such that

$$F(n) = \liminf_s \varphi_e^{0^{(2n)}}(n, s)$$

for all n.

There is an F

Claim

There is a $\Delta^0_{(2n+1)}$-limit infimum function $F : \omega \to \omega$ such that $\mathcal{L}_F$ is not computable.

Proof.

A diagonalization argument that builds a $\{0,1\}$-valued function F. Roughly speaking, the strategy $\mathcal{S}_{i,n}$ (for $n \geq i$) uses $F(2n)$ and $F(2n+1)$ to assure that $\mathcal{L}_F \neq \mathcal{L}_i$.

For example, to assure $\mathcal{L}_F \neq \mathcal{L}_0$, the strategy $\mathcal{S}_{0,0}$ begins setting $f(0,s) = 0$ and $f(1,s) = 1$. If a point appears to the right of a_0, the strategy $\mathcal{S}_{0,0}$ switches to setting $f(0,s) = 1$ and $f(1,s) = 0$. Note that if a_0 is part of the $F(0)$ or $\omega \cdot F(1)$ blocks of $\mathcal{L}_0$, then $\mathcal{L}_F \neq \mathcal{L}_0$. $\square$

$\omega^\omega + \mathcal{L}_F$ is Computable

Claim

If $F : \omega \to \omega$ is a $\Delta^0_{(2n+1)}$-limit infimum function, then $\omega^\omega + \mathcal{L}_F$ is computable.

Proof.

If $F(n) > 0$ for all n, build a computable copy of $\mathcal{L}_F$ by viewing it as the sum

$$+\cdots+\left[\omega^n \cdot (\omega + F(n) - 1)\right] + \cdots + \left[\omega^2 \cdot (\omega + F(2) - 1)\right] \\ +\left[\omega \cdot (\omega + F(1) - 1)\right] + \left[\omega + F(0)\right]$$

and building each summand separately.

For general F, use the Recursion Theorem to assure the garbage either settles down or collects in the copy of ω^ω. $\square$

$\omega^\omega + \mathcal{L}_F$ is Intrinsically Computably Well-Ordered

Claim

If $F : \omega \to \omega$ is any $\Delta^0_{(2n+1)}$-limit infimum function such that $\mathcal{L}_F$ is not computable, then $\omega^\omega + \mathcal{L}_F$ is intrinsically computably well-ordered.

Proof.

If there were a computable embedding $\pi : \omega^* \to \omega^\omega + \mathcal{L}_F$, then the linear order with universe

$$\{x \in \omega^\omega + \mathcal{L}_F : \pi(z) \prec x \text{ for some } z \in \omega^*\}$$

and order inherited from $\omega^\omega + \mathcal{L}_F$ would be computable. But this is $\mathcal{L}_F$, a contradiction. $\square$

A Theorem

Theorem

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ *linear order* $\mathcal{L}$ *that is intrinsically computably well-ordered.*

Proof.

Let F be any $\Delta^0_{(2n+1)}$-limit infimum function such that $\mathcal{L}_F$ is not computable. Then the linear order $\omega^\omega + \mathcal{L}_F$ is computable, non-well-ordered, scattered, rank $\omega + 1$, and intrinsically computably well-ordered. $\square$

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If $T \subseteq \omega^{<\omega}$ is any tree, define linear orders $\mathcal{L}_{\langle\sigma,\tau\rangle}$ via corecursion by

$$\mathcal{L}_{\langle\sigma,\tau\rangle} = \omega + \langle\sigma,\tau\rangle + \zeta + \left(\sum_{\substack{i\in\omega \\ \sigma^\frown i\in T}} \mathcal{L}_{\langle\sigma^\frown i,\tau^\frown 0\rangle} \right)^* + \left(\sum_{\substack{i\in\omega \\ \sigma^\frown i\in T}} \mathcal{L}_{\langle\sigma^\frown i,\tau^\frown 1\rangle} \right) + \zeta + \langle\sigma,\tau\rangle + \omega^*.$$

Let $\mathcal{L}_T$ be the linear order $\mathcal{L}_{\langle\epsilon,\epsilon\rangle}$, where ϵ denotes the empty string.

A Theorem

Theorem

There is a computable, non-scattered linear order $\mathcal{L}$ that is intrinsically hyperarithmetically scattered.

Proof.

Let $T \subseteq \omega^{<\omega}$ be a computable tree with infinite paths but no hyperarithmetic paths. Then $\mathcal{L}_T$ is computable, non-scattered, and intrinsically hyperarithmetically scattered. □

A Theorem

Theorem

If $\mathcal{L}$ is any computable rank one, non-scattered linear order, then there is a computable embedding of η into some computable presentation of $\mathcal{L}$.

Proof.

Any such linear order is (almost) of the form

$$\mathcal{L}_F = \sum_{q \in \mathbb{Q}} F(q)$$

for some function $F : \mathbb{Q} \to \omega \cup \{\omega^*, \zeta, \omega\}$. If it isn't, argue that it might as well be by considering either $\mathcal{L}^* + \mathcal{L}$, $\mathcal{L} + \mathcal{L}^*$, or $\sum_{z \in \zeta} \mathcal{L}$.

Handle the case when F is unbounded on every interval separate from when F is bounded on some interval. $\square$

When F is Bounded on an Interval

Proof.

Demonstrate η computably embeds into *every* computable presentation of $\mathcal{L}$. Add a point in $\mathcal{L}$ into the range of π whenever it is separated on the left and on the right by N points in $\mathcal{L}$ not yet in the range of π. $\square$

Question

If η computably embeds into every computable presentation of a linear order $\mathcal{L}$, must $\mathcal{L}$ be strongly η-like on some interval?

When F is Unbounded on Every Interval

Proof.

For functions $F : \mathbb{Q} \to \omega \cup \{\omega^*, \zeta, \omega\}$ unbounded on every interval, the following are equivalent:

- The linear order $\mathcal{L}_F$ is computable.
- There are limit infimum functions $L : \mathbb{Q} \to \omega$ and $R : \mathbb{Q} \to \omega$ such that $F(q) = L(q)^* + 1 + R(q)$ for all q.
- There are $\mathbf{0}'$-limitwise monotonic functions $L : \mathbb{Q} \to \omega$ and $R : \mathbb{Q} \to \omega$ such that $F(q) = L(q)^* + 1 + R(q)$ for all q.

$\square$

Outline

A Theorem

Theorem

For every computable ordinal α, there is a computable, non-well-ordered linear order $\mathcal{L}$ that is intrinsically $\emptyset^{(\alpha)}$-computably well-ordered.

Proof.

Let $F : \omega \to \omega$ be a $\Delta^0_{(2n+1)}(\emptyset^{(\alpha)})$-limit infimum function such that $\mathcal{L}_F$ is not $\emptyset^{(\alpha)}$ computable. Then $\omega^\alpha \cdot (\omega^\omega + \mathcal{L}_F)$ suffices. □

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Remark

By a result of Harrison, this is best possible.

A Theorem

Theorem

If $\mathcal{L}$ is a computable, rank ω, scattered, non-well-ordered linear order, then there is a computable embedding of ω^* into some computable presentation of $\mathcal{L}$.

Proof.

Demonstrate the ability to build a computable presentation into which ω^* computably embeds if a non-greatest point in $\mathfrak{c}^n(\mathcal{L})$ has no immediate successor in $\mathfrak{c}^n(\mathcal{L})$. Note that ω^* and ζ cannot be the order type of a maximal block in any $\mathfrak{c}^n(\mathcal{L})$ if $\mathcal{L}$ is intrinsically computably well-ordered. $\square$

Lemma

If $\mathcal{L}$ is a Δ^0_3 linear order with distinguished least element having a Δ^0_3 embedding of ω^* and R is any Σ^0_3 predicate, then $R \cdot \mathcal{L}$ has a computable presentation into which ω^* computably embeds.

A Conjecture

Conjecture

There is a computable, non-well-ordered, non-scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered.

Proof.

Define a linear order similar to $\mathcal{L}_T$ for $T \subseteq 2^{<\omega}$ except use linear orders

$$\mathcal{L}_F = \cdots + \omega^n \cdot F(n) + \cdots + \omega \cdot F(1) + F(0) + \omega^\omega$$

as markers rather than finite linear orders $\langle \sigma, \tau \rangle$. As $\mathcal{L}_{F_1} \cong \mathcal{L}_{F_2}$ if and only if $F_1 =^* F_2$, code σ into $\mathcal{L}_F$ by having the support of F be a subset of the multiples of σ. $\square$

Outline

Analagous Results for Other Classes

Theorem

Let X be the class of directed (acyclic) graphs, the class of undirected graphs, the class of commutative rings, the class of two-step nilpotent groups, the class of integral domains, or the class of commutative semigroups.

Then there are computable structures $S_1, S_2 \in X$ such that S_1 classically embeds into S_2 but for no computable presentations of S_1 and S_2 is there a computable embedding.

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Proof.

Show the result for X the class of directed acyclic graphs. The other classes then follow from previous work (Hirschfeldt, Khoussainov, Shore, Slinko). $\square$

Directed (Acyclic) Graphs

Proof.

Let T be an infinite computable tree with no computable paths. Let S_2 be the graph of T after replacing edges by either a directed diamond or a directed hexagon depending on whether the edge represents a string ending in a 0 or a 1. Let S_1 be the graph of exactly one (directed) infinite path. □

Proof.

Let $\omega^\omega + \mathcal{L}_F$ be a computable non-well-ordered intrinsically computably well-ordered linear order. Let S_2 be the graph whose vertices are the elements of $\omega^\omega + \mathcal{L}_F$, with a directed edge connecting vertex i to vertex j if and only if $j < i$ in the linear order. Again, let S_1 be the graph of exactly one (directed) infinite path. □

Trees

Definition

A tree is a partial order $(T : \prec)$ with a least element such that for all $x \in T$, the set $\{y \in T : y \preceq x\}$ is a finite linearly ordered set.

Theorem (Binns, Kjos-Hanssen, Lerman, Schmerl, Solomon)

There are computable trees T_1 and T_2 such that T_1 classically embeds into T_2 but for no computable presentations of T_1 and T_2 is there a computable embedding.

Proof.

Let $T_1 \cong 2^{<\omega}$ and let T_2 be an appropriate perfect binary branching tree. Build T_2 computable so that any function $f : \omega \to \omega$ that dominates the properly $\emptyset''$-computable branching function $b : \omega \to \omega$ satisfies $b \leq_T f \oplus \emptyset'$. $\square$

Boolean Algebras

Theorem

There are no computable Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1$ classically embeds into $\mathcal{B}_2$ but for no computable presentations of $\mathcal{B}_1$ and $\mathcal{B}_2$ is there a computable embedding.

Proof.

If $\mathcal{B}_2$ is superatomic, then $\mathcal{B}_1$ is superatomic; and the result is immediate.

If $\mathcal{B}_2$ is non-superatomic, it suffices to show that the countable atomless Boolean algebra computably embeds into some computable presentation of $\mathcal{B}_2$. Note that it suffices to consider uniform $\mathcal{B}_2$. With α the minimal ordinal in the range of $\sigma_{\mathcal{B}_2}$, note $\mathcal{B}_2 = \mathcal{B}_2 \oplus \mathcal{B}_{\sigma_{u(\{\alpha\})}}$. There is a nice presentation of the latter into which the countable atomless Boolean algebra computably embeds. $\square$

Outline

A Summary of Embedding Results

Theorem

There is a computable, non-scattered, rank two linear order $\mathcal{L}$ that is intrinsically computably scattered.

There is a computable, non-well-ordered, scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered.

There is a computable, non-well-ordered, non-scattered, rank $\omega + 1$ linear order $\mathcal{L}$ that is intrinsically computably well-ordered?

There is a computable, non-scattered, linear order $\mathcal{L}$ that is intrinsically hyperartihmetically scattered.

For many nice classes of algebraic structures X (but not X the class of Boolean algebras), there are computable S_1 and S_2 in X such that S_1 classically embeds into S_2 but for no computable presentations of S_1 and S_2 is there a computable embedding.

Bibliography

Ambos-Spies, Klaus and Cooper, S. Barry and Lempp, Steffen.
Initial Segments of Recursive Linear Orders.
Order, 14(2):101-105, 1997/98.

Harrison, Joseph.
Recursive Pseudo-Well-Orderings.
Transactions of the American Mathematical Society, 131:526-543, 1968.

Binns, Stephen and Kjos-Hanssen, Bjorn and Lerman, Manuel and Schmerl, James H. and Solomon, Reed
Self-Embeddings of Computable Trees.
Notre Dame Journal of Formal Logic, 49:1–37, 2008.

Hirschfeldt, Denis R. and Khoussainov, Bakhadyr and Shore, Richard A. and Slinko, Arkadii M.
Degree Spectra and Computable Dimensions in Algebraic Structures
Annals of Pure and Applied Logic, 115:71-113, 2002.

Kach, Asher M.
Computable Shuffle Sums of Ordinals.
Archive of Mathematical Logic, to appear.

Lerman, Manuel.
On Recursive Linear Orderings.
Logic Year 1979–1980, Lecture Notes in Mathematics, 859:132-142, Springer, Berlin, 1981.

Rosenstein, Joseph G.
Linear Orderings.
Academic Press, New York, 1982.

Watnick, Richard.
A Generalization of Tennenbaum's Theorem on Effectively Finite Recursive Linear Orderings
Journal of Symbolic Logic, 49(2):563-569, 1984.