# Orders on Computable Torsion-Free Abelian Groups

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# Outline

# Torsion-Free Abelian Groups

## Remark

Disclaimer: Hereout, the word *group* will always refer to a countable torsion-free abelian group. The words *computable group* will always refer to a (fixed) computable presentation.

## Definition

A group $\mathcal{G} = (G : +, 0)$ is *torsion-free* if non-zero multiples of non-zero elements are non-zero, i.e., if

$$(\forall x \in G)(\forall n \in \omega)\,[x \neq 0 \wedge n \neq 0 \implies nx \neq 0]\,.$$

# Rank

**Theorem**

*A countable abelian group is torsion-free if and only if it is a subgroup of $\mathbb{Q}^\omega$.*

**Definition**

The *rank* of a countable torsion-free abelian group $\mathcal{G}$ is the least cardinal $\kappa$ such that $\mathcal{G}$ is a subgroup of $\mathbb{Q}^\kappa$.

# Examples of Torsion-Free Abelian Groups

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The subgroup $\mathcal{H}$ of $\mathbb{Q} \oplus \mathbb{Q}$ (viewed as having generators $b_1$ and $b_2$) generated by $b_1$, $b_2$, and $\frac{b_1+b_2}{2}$

*So elements of $\mathcal{H}$ look like $\beta_1 b_1 + \beta_2 b_2 + \alpha \frac{b_1+b_2}{2}$ for $\beta_1, \beta_2, \alpha \in \mathbb{Z}$.*

has rank two.

## Remark

Note that $\frac{b_1}{2}$ and $\frac{b_2}{2}$ do not belong to $\mathcal{H}$ despite their sum $\frac{b_1+b_2}{2}$ belonging to $\mathcal{H}$. We will often abuse notation and write such things as $\frac{1}{2}b_1 + \frac{1}{2}b_2$ for $\frac{b_1+b_2}{2}$.

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# The Motivating Theorem

## Definition

Fix a group $\mathcal{G} = (G : +, 0)$. A set $B \subset G$ (not containing 0) is a *basis* if it is a maximal linearly independent set (with coefficients in $\mathbb{Z}$).

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## Question

Does this remain true in the effective setting?

In other words, does every computable torsion-free abelian group admit a computable basis?

# Basis Results (I)

## Proposition (Folklore (?))

*Every computable torsion-free abelian group $\mathcal{G}$ has a basis $B \subset G$ computable from $\mathbf{0}'$.*

## Proof.

Enumerate $G$ as $\{a_i\}_{i \in \omega}$. Recursively determine if we should place $a_i \in B$ by checking whether $a_i$ is nonzero and linearly independent (over $\mathbb{Z}$) from $\{a_0, \ldots, a_{i-1}\}$. □

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## Theorem

*The following are equivalent (over* $\mathrm{RCA}_0$*):*

- $\mathrm{ACA}_0$.
- *Every torsion-free abelian group has a basis.*

# Basis Results (I)

**Proof.**

Note that the linear (in)dependence relation can be computed from a basis.

> *Given elements $a_{i_0}, \ldots, a_{i_n}$, write each as a linear combination of the basis elements. Determine linear (in)dependence using linear algebra.*

Thus, it suffices to construct a computable group $\mathcal{G}$ for which the linear (in)dependence relation computes $\mathbf{0}'$.

Let $\mathcal{G}$ be the computable presentation of $\mathbb{Z}^\omega$ with generators $\{g_i\}_{i\in\omega}$. If $i$ enters $K$ at stage $s$, set $g_{2i+1} = s\, g_{2i}$.

Then $i \in K$ if and only if $g_{2i}$ and $g_{2i+1}$ are linearly dependent. $\square$

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**Theorem (Dobritsa (1983))**

*Every computable torsion-free abelian group $\mathcal{G}$ has an isomorphic computable $\mathcal{H}$ admitting a computable basis.*

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## Corollary

*Every computable torsion-free abelian group $\mathcal{G}$ of infinite rank has an isomorphic computable $\mathcal{H}$ for which every basis computes $\mathbf{0}'$.*

## Proof.

Combine Dobritsa's construction with the $\mathrm{ACA}_0$ construction. □

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# The Motivating Question

## Definition

An abelian group $\mathcal{G} = (G : +, 0)$ equipped with a binary relation $\leq$ is *(totally) ordered* if the relation satisfies:

- antisymmetry (if $a \leq b$ and $b \leq a$, then $a = b$),
- transitivity (if $a \leq b$ and $b \leq c$, then $a \leq c$),
- totality ($a \leq b$ or $b \leq a$), and
- translation invariance (if $a \leq b$, then $a + c \leq b + c$).

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## Theorem (Levi (1942))

*An abelian group is orderable if and only if it is torsion-free.*

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# Non-Archimedean Orders on $\mathbb{Q}^\kappa$

## Example

Fixing a basis $\{b_0, b_1\}$ of $\mathbb{Q}^2$, lexicograph order yields an ordering. Under this order, we have

$$b_0 \gg b_1 \gg 0$$

and so, for example, $\frac{1}{2}b_0 > \frac{1}{2}b_0 - 2b_1 > b_1 > 0 > -2b_0 + 18b_1$.

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## Example

Fixing a basis $\{b_i\}_{i\in\omega}$ of $\mathbb{Q}^\omega$, lexicograph order yields an ordering. Under this order, we have

$$b_0 \gg b_1 \gg b_2 \gg \cdots \gg 0$$

and so, for example, $\frac{1}{2}b_0 > b_1 + b_2 > b_1 + 2b_3 > 0 > -b_2 + b_{18} > -b_2$.

# Archimedean Orders on $\mathbb{Q}^\kappa$

## Example

Fixing a basis $\{b_0, b_1\}$ of $\mathbb{Q}^2$ and an irrational $r \in \mathbb{R}$, the order induced by putting $b_0 := 1 \in \mathbb{R}$ and $b_1 := r$ is an ordering on $\mathbb{Q}^2$.

Thus, for example if $r := \sqrt{2} \approx 1.41$, we have $1.4b_0 < b_1 < 1.5b_0$.

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## Example

Fixing a basis $\{b_i\}_{i\in\omega}$ of $\mathbb{Q}^\omega$, the order induced by putting $b_0 := 1 \in \mathbb{R}$ and $b_i := \sqrt{p_i}$ for $i > 0$ is an ordering on $\mathbb{Q}^\omega$.

Under this order, we have $1.4b_0 < b_1 < 1.5b_0$ (as $\sqrt{p_1} = \sqrt{2} \approx 1.41$) and $1.2b_1 < b_2 < 1.3b_1$ (as $\sqrt{p_2}/\sqrt{p_1} = \sqrt{3}/\sqrt{2} \approx 1.22$).

# Orders From a Basis

## Theorem (Solomon (2002))

*Fix a computable torsion-free abelian group $\mathcal{G}$ with rank at least two. Let $B \subseteq G$ be an $X$-computable basis. Then $\mathcal{G}$ has orders in all degrees computing $X$.*

## Proof.

Let $r := X$ (with $r$ irrational). Enumerate $B = \{b_i\}_{i \in \omega}$. The order on $\mathcal{G}$ induced by

$$b_0 = rb_1 \gg b_2 \gg b_3 \gg 0$$

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has degree $X$. □

## Corollary

*Fix a computable torsion-free abelian group $\mathcal{G}$. Then $\mathcal{G}$ has an order of every degree computing $\mathbf{0}'$.*

# Order Results (I)

## Corollary (Low Basis Theorem)

*Every computable torsion-free abelian group has a low order.*

## Proof.

It is a $\Pi^0_1$ property for a relation to be an order. □

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## Theorem (Downey and Kurtz (1986))

*There is a computable torsion-free abelian group admitting no computable order.*

## Theorem (Hatzikiriakou and Simpson (1990))

*The following are equivalent (over* $\mathsf{RCA}_0$*):*

- $\mathsf{WKL}_0$.
- *Every torsion-free abelian group is orderable.*

# More Questions

## Question

Is there, for every $\Pi^0_1$ tree $\mathcal{P}$, a computable torsion-free abelian group whose orders are in one-to-one correspondence with the paths in $\mathcal{P}$?

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## Remark

The immediate answer is NO as $\leq^*$ (where $y \leq^* x$ if and only if $x \leq y$) is an order whenever $\leq$ is an order.

Further, the space of orders on a torsion-free abelian group has size two (if its rank is one) or size continuum (if its rank is greater than one).

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Further, the space of orders on a torsion-free abelian group has size two (if its rank is one) or size continuum (if its rank is greater than one).

## Question

Is there a computable torsion-free abelian group with rank at least two whose degrees of orders is not upward closed?

# The Group $\mathbb{Q}^\omega$

**Theorem (Kach, Lange, and Solomon)**

*There is a computable torsion-free abelian group $\mathcal{G}$ of isomorphism type $\mathbb{Q}^\omega$ and a noncomputable c.e. set $C$ such that:*

- *The group $\mathcal{G}$ has exactly two computable orders.*
- *Every $C$-computable order on $\mathcal{G}$ is computable.*

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## Proof.

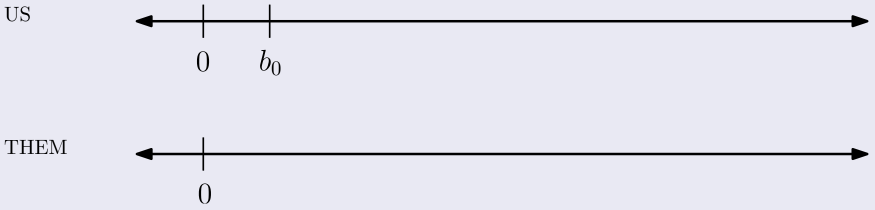
Build the computable presentation $\mathcal{G}$, a computable order $\leq$, and the set $C$ simultaneously via a finite injury construction.

For each $i, e \in \omega$, satisfy the requirements

$$\begin{aligned} \mathcal{P}_i &: \text{That } C \neq \Phi_i. \\ \mathcal{N}_e &: \text{If } \Phi_e^C \text{ is an order on } \mathcal{G}\text{, then } \leq_e^C \text{ is either } \leq \text{ or } \leq^*. \end{aligned}$$

# The Group $\mathbb{Q}^\omega$

## Meeting an $\mathcal{N}_e$ Requirement.



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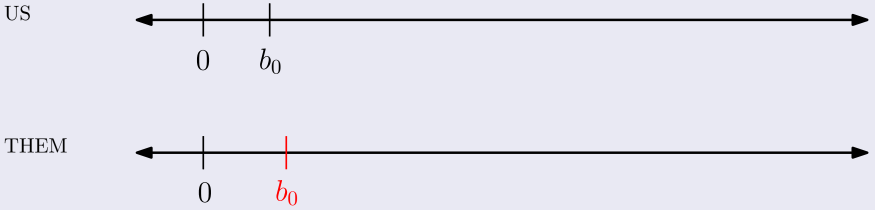

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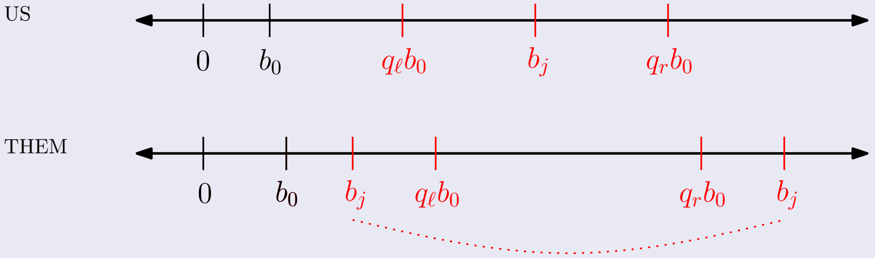

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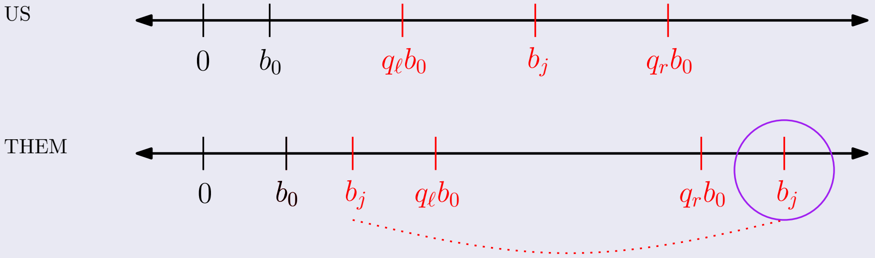

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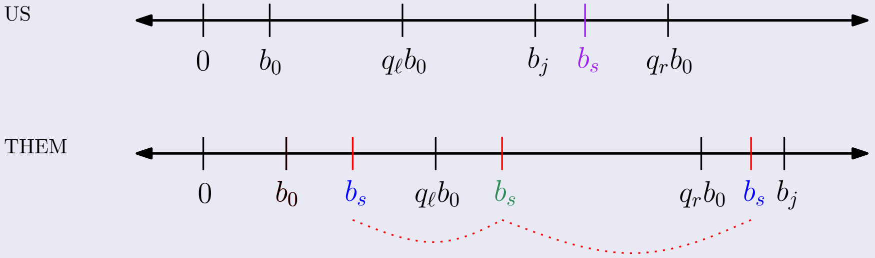

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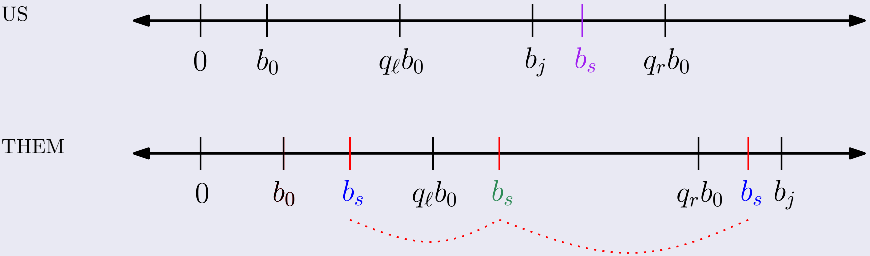


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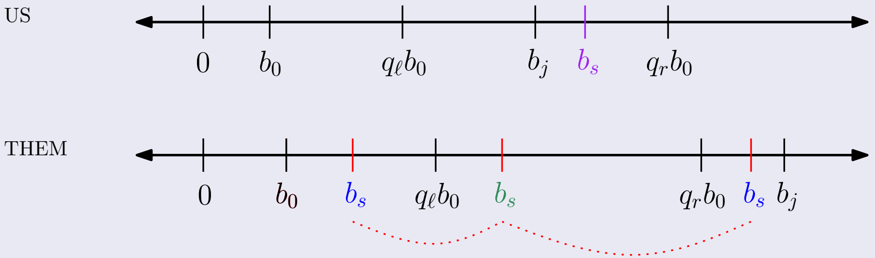


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**Theorem**

*There is a computable torsion-free abelian group $\mathcal{G}$ of isomorphism type $\mathbb{Z}^\omega$ and a noncomputable c.e. set $C$ such that:*

- *The group $\mathcal{G}$ has exactly two computable orders.*
- *Every $C$-computable order on $\mathcal{G}$ is computable.*

*Thus, the set of degrees of orders on $\mathcal{G}$ is not closed upwards.*

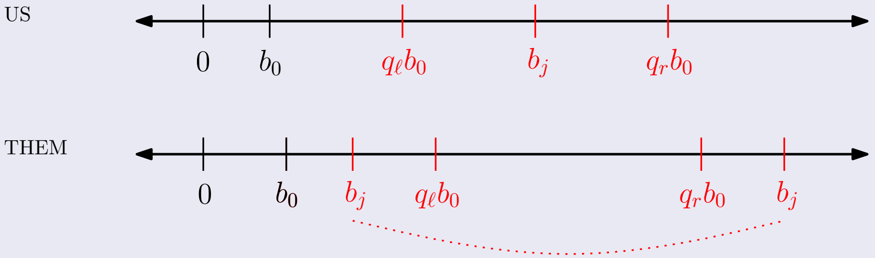
**Proof.**

As before. The major differences are that we can no longer measure *size* using only multiples of $b_0$ and we can no longer create arbitrary rational dependencies. □

# The Group $\mathbb{Z}^\omega$

## Meeting an $\mathcal{N}_e$ Requirement.

Measure size by building the computable order so that the even basis elements $b_{2k}$ satisfy $0 < b_{2k} \leq \frac{1}{2^k}$, identifying $b_0 := 1 \in \mathbb{R}$. Maintain a *basis restraint* $K$ preventing extra divisibility to any basis element $b_k$ with $k < K$.



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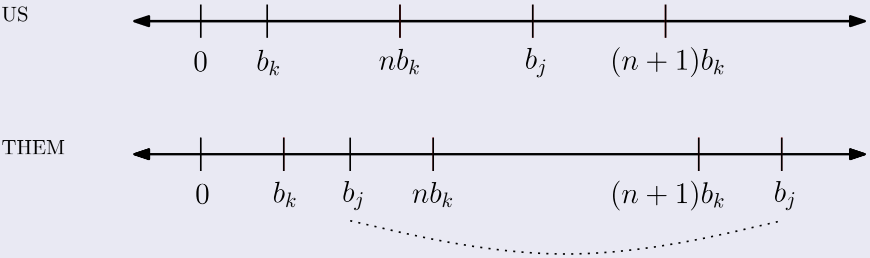

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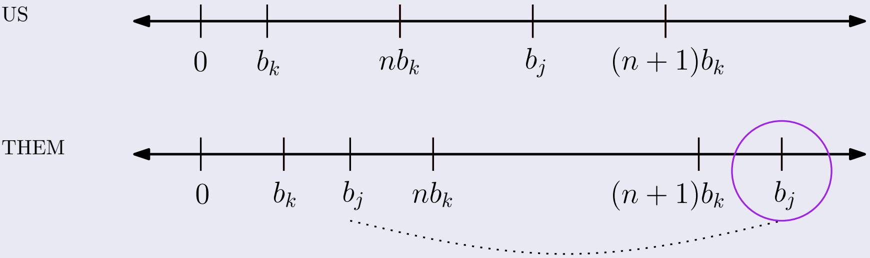

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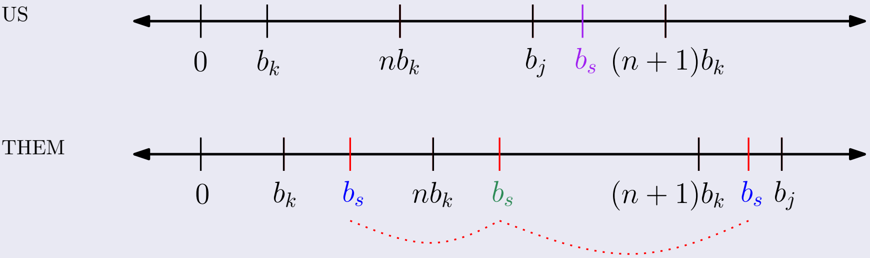

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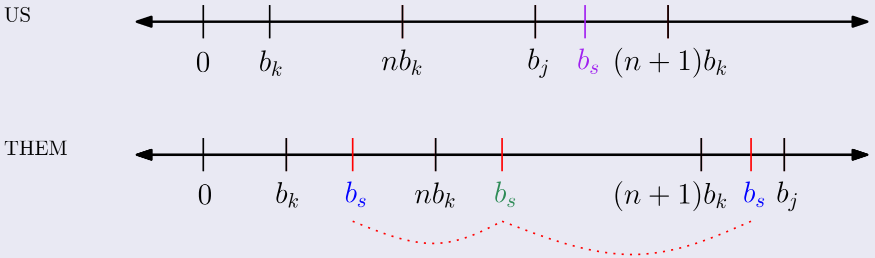


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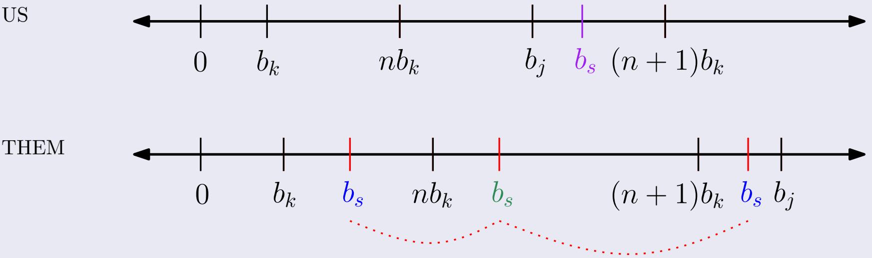


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□

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# The Questions

## Question

Is the choice of $\mathbb{Q}^\omega$ and $\mathbb{Z}^\omega$ important? In other words, is there, for every computable torsion-free abelian group $\mathcal{G}$, an isomorphic computable $\mathcal{H}$ for which the set of degrees of orders on $\mathcal{H}$ is not upward closed?

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What more can be said about the set of degrees of orders for the groups $\mathcal{G}$ constructed?

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What more can be said about the set of degrees of orders for the groups $\mathcal{G}$ constructed?

## Question

Is there a computable torsion-free abelian group whose orders are either computable or compute $\mathbf{0}'$?

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