Characterizing the Computable Structures: Boolean Algebras and Linear Orders

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Outline

1. General Background and Notation

2. Shuffle Sums

3. Boolean Algebras of Low Depth
A countable structure (with finite signature) is computable if its universe can be identified with $\omega$ in such a way as to make the functions and relations on it computable.
**Definition**

A countable structure (with finite signature) is computable if its universe can be identified with $\omega$ in such a way as to make the functions and relations on it computable.

**Remark**

Here we will be considering two specific classes of structures: Boolean algebras and linear orders. We view a Boolean algebra as a structure in the signature $\mathcal{B} = (B : +, \cdot, -, 0, 1)$ and a linear order as a structure in the signature $\mathcal{L} = (L : \prec)$. 
The Feiner Hierarchy

**Definition**

For \( n \in \omega \), define \( \emptyset(\leq n) \) to be the set

\[
\emptyset(\leq n) = \{ \langle k, m \rangle : m \in \emptyset(k), k \leq n \}.
\]

**Definition ([1])**

Let \( S \subseteq \omega \) be a set computable in \( \emptyset(\omega) \). Then \( S \) is \((a, b)\) in the Feiner hierarchy if there exists an index \( e \) such that

1. The function \( \varphi_{e}^{\emptyset(\omega)} \) is total and is the characteristic function of \( S \), i.e., \( \varphi_{e}^{\emptyset(\omega)}(n) = \chi_{S}(n) \) for all \( n \).

2. The computations \( \varphi_{e}^{\emptyset(\leq bn+a)}(n) \) and \( \varphi_{e}^{\emptyset(\omega)}(n) \) are identical; in particular, the latter queries \( \emptyset(\omega) \) on no number \( \langle k, m \rangle \) with \( k > bn + a \).
1. General Background and Notation
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3. Boolean Algebras of Low Depth
Definition

Let $S = \{L_x\}$ be a countable set of linear orders. Then the shuffle sum of $S$, denoted $\sigma(S)$, is the linear order obtained by partitioning the rationals $\mathbb{Q}$ into $|S|$ many dense sets $\{Q_x\}$ and replacing each point $q \in Q_x$ with the linear order $L_x$. 
Shuffle Sums

**Definition**

Let \( S = \{ L_x \} \) be a countable set of linear orders. Then the shuffle sum of \( S \), denoted \( \sigma(S) \), is the linear order obtained by partitioning the rationals \( \mathbb{Q} \) into \( |S| \) many dense sets \( \{ Q_x \} \) and replacing each point \( q \in Q_x \) with the linear order \( L_x \).

**Definition**

If \( \mathcal{L} = (L : \prec) \) is a linear order and \( \mathcal{L}_a = (L_a : \prec_a) \) is a linear order for each \( a \in L \), the lexicographic sum of \( \mathcal{L} \) and \( \{ \mathcal{L}_a \}_{a \in L} \) is the linear order with universe \( \{(a, b) : a \in L, b \in L_a\} \) under the lexicographic order induced by \( \prec \) and \( \prec_a \).

**Definition**

The shuffle sum of a set \( S = \{ L_x \}_{x \in \omega} \) is the linear order \( \sum_{a \in \mathbb{Q}} L_a \), where \( L_a \) is the linear order \( L_x \) if \( a \in Q_x \).
A set $S \subseteq \omega + 1$ is a limit infimum set, written \textsc{LimInf} set, if there is a total computable function $g(x, s) : \omega \times \omega \rightarrow \omega$ such that the function

$$f(x) = \liminf_s g(x, s)$$

enumerates $S$. Here we use the convention that $\liminf_s g(x, s) = \omega$ if $\lim_s g(x, s) = \infty$. 
**Definition ([2])**

A set $S \subseteq \omega + 1$ is a limitwise monotonic set relative to $(0')$, written $\text{LimMon}(0')$ set, if there is a total $(0')$-computable function $\tilde{g}(x, t) : \omega \times \omega \rightarrow \omega$ satisfying

$$\tilde{g}(x, t) \leq \tilde{g}(x, t + 1) \quad \text{for all } x \text{ and } t$$

such that the function

$$\tilde{f}(x) = \lim_t \tilde{g}(x, t).$$

enumerates $S$. Again we use the convention that $\lim_t \tilde{g}(x, t) = \omega$ if $\lim_t \tilde{g}(x, t) = \infty$. 
Theorem (K)

For sets $S \subseteq \omega + 1$, the following are equivalent:

1. The shuffle sum $\sigma(S)$ is computable.
2. The set $S$ is a $\text{LIM INF}$ set.
3. The set $S$ is a $\text{LIM MON (0')}$ set.
Proof of (1) implies (2)

Proof (Sketch).

From a computable presentation of $\sigma(S)$, define a computable function $g(x, s)$ as the “sum” of the number of points to the left of $x$ in the block of $x$, one, and the number of points to the right of $x$ in the block of $x$.

As the block of $x$ is not computable, we guess that it extends from the point last enumerated to the left of $x$ to the point last enumerated to the right of $x$ (exclusive).

Verify that this approximation works, separating the case when the block size of $x$ is finite from when the block size is infinite.
Proof (Sketch).

From a function $g$ witnessing that $S$ is a LIMINF set, build infinitely many copies of the linear order $g(x, s)$ at all rationals in the set $Q_x$.

When the value of $g(x, s)$ increases, add additional points. When the value of $g(x, s)$ decreases, dissassociate the extra points from the rational in $q \in Q_x$.

Prioritize the disassociated points so that they eventually become permanently associated to some other rational $q' \in Q_{x'}$. 
Proof of (2) if and only if (3)

Proof (Sketch).

From a computable function $g(r, s)$ witnessing that $S$ is a LIM set, define a $(0')$-computable function $\tilde{g}(r, t)$ witnessing that $S$ is a LIM$(0')$ set.

Conversely, from a $(0')$-computable function $\tilde{g}(r, t)$ witnessing that $S$ is a LIM$(0')$ set, define a computable function $g(r, s)$ witnessing that $S$ is a LIM set.
Proof (Sketch).

From a computable function $g(r, s)$ witnessing that $S$ is a \textsc{Liminf} set, define a $(0')$-computable function $\tilde{g}(r, t)$ witnessing that $S$ is a \textsc{Limmon} $(0')$ set.
Proof of (2) if and only if (3)

Proof (Sketch).

From a computable function $g(r, s)$ witnessing that $S$ is a $\text{LIMINF}$ set, define a $(0')$-computable function $\tilde{g}(r, t)$ witnessing that $S$ is a $\text{LIMMON (0')}$ set.

Conversely, from a $(0')$-computable function $\tilde{g}(r, t)$ witnessing that $S$ is a $\text{LIMMON (0')}$ set, define a computable function $g(r, s)$ witnessing that $S$ is a $\text{LIMINF}$ set.
1. General Background and Notation

2. Shuffle Sums

3. Boolean Algebras of Low Depth
Definition ([5])

A measure $\sigma$ is a map from the countable atomless Boolean algebra $\mathcal{F}$ to the countable ordinals satisfying $\sigma(x + y) = \max\{\sigma(x), \sigma(y)\}$.
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By associating the countable atomless Boolean algebra with finite unions of cones of $2^{<\omega}$, a measure can be viewed as a map $\sigma : 2^{<\omega} \to \omega_1$ satisfying $\sigma(\tau) = \max\{\sigma(\tau \upharpoonright 0), \sigma(\tau \upharpoonright 1)\}$. Under this interpretation, a measure can be thought of as a labelled binary tree.
If \( \sigma : \mathcal{F} \rightarrow \omega_1 \) is a measure, define maps \( \Delta^\alpha \sigma \) with domain \( \mathcal{F} \) for \( \alpha < \omega_1 \) recursively by setting \( \Delta^0 \sigma = \sigma \),

\[
\Delta^{\alpha+1} \sigma(x) = \{(\Delta^\alpha \sigma(x_1), \ldots, \Delta^\alpha \sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\},
\]

and \( \Delta^\gamma \sigma(x) \) as the inverse limit of \( \Delta^\beta \sigma(x) \) for \( \beta < \gamma \).

The set \( \Delta^\alpha \sigma(1_\mathcal{B}) \) is the \( \alpha \)th derivative of \( \mathcal{B}_\sigma \).
Theorem (K)

For each set $S \subseteq \omega_1$ satisfying $|S| = 1$, there is exactly one depth zero Boolean algebra with range $S$, namely $B_u(S) = B_v(S)$.

For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are exactly two depth zero Boolean algebras with range $S$, namely $B_u(S)$ and $B_v(S)$.
Theorem (K)

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For each set $S \subseteq \omega_1$ with greatest element satisfying $|S| > 1$, there are exactly two depth zero Boolean algebras with range $S$, namely $B_u(S)$ and $B_v(S)$.

Proof (Sketch).

Show the existence of at least two, then of at most two. For the former, define the Boolean algebras $B_{u(\alpha+1)}$ and $B_{v(\alpha+1)}$ by induction on $\alpha$. For the latter, either use pseudo-indecomposability and primitiveness or appeal directly to the depth zero definition.
Proposition (K)

There are continuum many depth one, rank $\omega$ Boolean algebras with range $\omega + 1$. 
**Proposition (K)**

*There are continuum many depth one, rank \( \omega \) Boolean algebras with range \( \omega + 1 \).*

**Proof (Sketch).**

Code subsets of the positive integers into a Boolean algebra \( B^S \). Have \( \sigma_u(\{0,n\}) \) be a subalgebra of \( B^S \) if and only if \( n \in S \); have \( \sigma_v(\{0,n\}) \) be a subalgebra of \( B^S \) if and only if \( n \notin S \).
The measure $\sigma^S$ when $S = \{1, 3, 5, \ldots \}$
There are continuum many depth $\omega$, rank one measures.

Proof (Sketch).
Define a map $\pi$ from the space of uniform Boolean algebras to the space of uniform rank one Boolean algebras. The algebra
$\pi(B)$
is the algebra generated by the (any) characteristic function of a subset of the rationals whose clopen algebra is $B$.
Argue that $\pi(B_u(S))$ and $\pi(B_v(S))$ are (at most) depth $\omega$ for sets $S \subseteq \omega + 1$. 
Proposition (K)

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Argue that $\pi(B_u(S))$ and $\pi(B_v(S))$ are (at most) depth $\omega$ for sets $S \subseteq \omega + 1$. 
The measure $\pi(\sigma_u(2))$
Proposition (K)

If $\sigma$ is a depth zero measure of rank at most $\lambda < \omega_1^{CK}$, then $\sigma$ is computable (i.e., there is a computable measure $\sigma'$ such that $B_\sigma = B_{\sigma'}$) if and only if $\Delta \sigma(1_B)$ is computably enumerable.

Moreover, from an index for either $\sigma$ or $\Delta \sigma(1_B)$, an index for the other can be given uniformly.
Effective Boolean Algebras: The Main Results

Proposition (K)

If $\sigma$ is a depth zero measure of rank at most $\lambda < \omega_1^{\text{CK}}$, then $\sigma$ is computable (i.e., there is a computable measure $\sigma'$ such that $\mathcal{B}_\sigma = \mathcal{B}_{\sigma'}$) if and only if $\Delta\sigma(1_\mathcal{B})$ is computably enumerable.

Moreover, from an index for either $\sigma$ or $\Delta\sigma(1_\mathcal{B})$, an index for the other can be given uniformly.

Theorem (K)

Let $S \subseteq \omega + 1$ be a set with greatest element. Then the following are equivalent:

1. The Boolean algebra $\mathcal{B}_{u(S)}$ is computable.
2. The Boolean algebra $\mathcal{B}_{v(S)}$ is computable.
3. The set $S$ is $(2, 2)$ in the Feiner hierarchy.
Proof of (1), (2) implies (3)

Proof (Sketch).

Uniformly in \( n \), define \( \Sigma_{2n+3}^0 \) sentences \( \varphi_n \) satisfying

\[
\mathcal{B}_u(S), \mathcal{B}_v(S) \models \varphi_n \quad \text{if and only if} \quad n \in S.
\]

When defining these formulas, make use of the fact that there are formulas (uniform in \( \alpha \)) of complexity \( \Pi_{2\alpha+1}^0 \) identifying whether an element is an \( \alpha \)-atom.
Proof of (3) implies (1), (2)

Proof (Sketch).

Assume without loss of generality that $S$ is infinite.

Construct $\mathcal{B}_u(S)$ ($\mathcal{B}_v(S)$, respectively) from an index $e$ witnessing that $S$ is $(2, 2)$ in the Feiner hierarchy. Do so by building a linear order

$$\mathcal{L} = \sum_{\tau \in 2^{<\omega}} \mathcal{L}_\tau$$

and taking its interval algebra.

The linear order $\mathcal{L}_\tau$ depends on $S$ and the value of $\sigma_{u(\omega+1)}(\tau)$ ($\sigma_{v(\omega+1)}(\tau)$, respectively). It is built by iterating the following technical lemma.
Technical Lemma (Statement)

Lemma (K)

Uniformly in

- a $\Delta^0_3$ index for the atomic diagram $D(A)$ of a linear order $A = (A : \prec) = (\{a_0, a_1, \ldots\} : \prec)$ with distinguished first element
- and an index for a $\Sigma^0_3$ predicate $\exists n \forall u \exists v R(n, u, v)$,

there is an index for a $\Delta^0_1$ linear order $B$ such that $B \equiv \sum_{a \in A} L_a$, where $L_{a_n} = 1 + \eta + \omega$ if $\forall u \exists v R(n, u, v)$ and $L_{a_n} = \omega$ otherwise.
Proof (Sketch).

An infinite injury argument using work of Thurber as an outline. Approximate the atomic diagram of $\mathcal{A}$ using the Limit Lemma twice, imposing (without loss of uniformity) constraints on the approximating functions.

Introduce chronological priorities and build each block as the sum of a singleton segment, a dense segment, and a discrete segment. As the approximations change, attach and detach blocks appropriately.
Bibliography

Feiner, Lawrence.
Hierarchies of Boolean algebras.

Hisamiev, N.G.
Criterion for constructivizability of a direct sum of cyclic $p$-groups

Kach, Asher.
Computable shuffle sums of ordinals.
*Archives of Mathematical Logic*, accepted.

Kach, Asher.
Boolean algebras of low Ketonen depth.
In preparation.

Ketonen, Jussi.
The structure of countable Boolean algebras.

Pierce, R.S.
Countable Boolean algebras.